CONDITIONAL MATCHING PRECLUSION FOR THE STAR GRAPHS

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ABSTRACT. The matching preclusion number of a graph is the minimum number of edges whose deletion results in a graph that has neither perfect matchings nor almost-perfect matchings. For many interconnection networks, the optimal sets are precisely those incident to a single vertex. Recently, the conditional matching preclusion number of a graph was introduced to look for obstruction sets beyond those incident to a single vertex. It is defined to be the minimum number of edges whose deletion results in a graph *with no isolated vertices* that has neither perfect matchings nor almost-perfect matchings. In this paper, we find this number and classify all optimal sets for the star graphs, one of the most popular interconnection networks.

Keywords: Interconnection networks, perfect matching, star graphs

1. INTRODUCTION AND PRELIMINARIES

A *perfect matching* in a graph is a set of edges such that every vertex is incident with exactly one edge in this set. An *almost-perfect matching* in a graph is a set of edges such that every vertex except one is incident with exactly one edge in this set, and the exceptional vertex is incident to none. So if a graph has a perfect matching, then it has an even number of vertices; if a graph has an almost-perfect matching, then it has an odd number of vertices. The *matching preclusion number* of a graph G, denoted by $\mathrm{mp}(G)$, is the minimum number of edges whose deletion leaves the resulting graph without a perfect matching or almost-perfect matching. Any such optimal set is called an *optimal matching preclusion set*. We define $\mathrm{mp}(G) = 0$ if G has neither a perfect matching nor an almost-perfect matching. This concept of matching preclusion was introduced by Brigham et al. in [1] and further studied in [2, 4]. They introduced this concept as a measure of robustness in the event of edge failure in interconnection networks, as well as a theoretical connection to conditional connectivity, "changing and unchanging of invariants" and extremal graph theory. We refer the readers to [1] for details and additional references. In this paper, we will use standard definition for common terms without explicitly defining them [8].

Useful distributed processor architectures offer the advantage of improved connectivity and reliability. An important component of such a distributed system is the system topology, which defines the inter-processor communication architecture. In certain applications every vertex requires a special partner at any given time and the matching preclusion number measures the robustness of this requirement in the event of link failures as indicated in [1]. Hence in these interconnection networks, it is desirable to have the property that the only optimal matching preclusion sets are those whose elements are incident to a single vertex.

Proposition 1.1. *Let G be a graph with an even number of vertices. Then $mp(G) \leq \delta(G)$, where $\delta(G)$ is the minimum degree of G.*

If the inequality in Proposition 1.1 holds as equality, then the set of edges incident to a single vertex is a *trivial optimal matching preclusion set*. As mentioned earlier, it is desirable for an interconnection network to have only trivial optimal matching preclusion sets. It is unlikely that in the event of random link failure, all of them will be at the same vertex. So it is natural to ask what are the next obstruction sets for a graph with link failures to have a perfect matching subject to the condition that the *faulty graph* has no isolated vertices. This motivates the definition given in [3] and further studied in [7]. The *conditional matching preclusion number* of a graph G, denoted by $\mathrm{mp}_1(G)$, is the minimum number of edges whose deletion leaves the resulting graph with no isolated vertices and without a perfect matching or almost-perfect matching. Any such optimal set is called an *optimal conditional matching preclusion set*. We define $\mathrm{mp}_1(G) = 0$ if G has neither a perfect matching nor an almost-perfect matching. We will leave $\mathrm{mp}_1(G)$ undefined if a conditional matching preclusion set does not exist, that is, we cannot delete edges to satisfy both conditions in the definition.

Therefore, the question is: by deleting edges, what are the basic obstructions to a perfect matching or an almost-perfect matching in the resulting graph if no isolated vertices are created? In Proposition 1.1, we see that without the condition of no isolated vertices, an isolated vertex will be the basic obstruction and so deleting all edges incident to G will produce a trivial matching preclusion set. Now for a resulting graph with no isolated vertices, a basic obstruction to a perfect matching will be the existence of a path $u - w - v$ where the degree of u and the degree of v are 1. So to produce such an obstruction set, one can pick any path $u - w - v$ in the original graph and delete all the edges incident to either u or v but not the edges (u, w) and (w, v). We define

$$\nu_e(G) = \min\{d_G(u) + d_G(v) - 2 - y_G(u, v) : u \text{ and } v \text{ are ends of a 2-path}\}$$

where $d_G(\cdot)$ is the degree function and $y_G(u,v) = 1$ if u and v are adjacent and 0 otherwise. (We will suppress G and simply write d and y if it is clear from the context.) So mirroring Proposition 1.1, we have the following easy result.

Proposition 1.2. *Let G be a graph with an even number of vertices. Suppose every vertex in G has degree at least three. Then*

$$mp_1(G) \leq \nu_e(G).$$

We note that the condition "$\delta(G) \geq 3$" is to ensure that the resulting graph (after edges have been deleted) has no isolated vertices. Moreover, this condition is not strictly necessary if we are willing to exclude certain exceptions such as the 4-cycle. For our purposes, Proposition 1.2 suffices.

We call an optimal solution of the form induced by ν_e a *trivial optimal conditional matching preclusion set.* As mentioned earlier, the matching preclusion number measures the robustness of this requirement in the event of link failures, so it is desirable to have the property that the only optimal matching preclusion sets are the trivial ones. Similarly, it is desirable to have the property that the only optimal conditional matching preclusion sets are the trivial ones as well. In [3], the authors introduced this concept and considered the conditional matching preclusion problem for a number of basic networks including the hypercubes, and they were proven to have this desired property. Since the star graphs are superior to the hypercubes in many aspects, it is natural to ask whether they measure up under this parameter. In this paper, we investigate this property for the star graphs. We now define this popular class of interconnection networks. Let $n \geq 3$. The star graph S_n has the $n!$ permutations on $\{1, 2, \ldots, n\}$ as the vertex set. Two vertices $[a_1, a_2, \ldots, a_n]$ and $[b_1, b_2, \ldots, b_n]$ are adjacent if there exists $i \in \{2, 3, \ldots, n\}$ such that $a_1 = b_i$, $b_1 = a_i$ and $a_j = b_j$ if $j \in \{2, 3, \ldots, n\} - \{i\}$. In other words, they are adjacent if one can be obtained from the other by interchanging the symbols in position 1 and position i for some $i = 2, 3, \ldots, n$. Such an edge is called an *i-edge.* So S_n is $(n-1)$-regular and bipartite with the set of even permutations and the set of odd permutations as the partite sets. It is not difficult to see that S_n has girth 6 (that is, the smallest cycle is of length 6), and that S_n is both vertex-transitive and edge-transitive. Figure 1 gives S_4. For convenience, we may write $a_1 a_2 \ldots a_n$ rather than $[a_1, a_2, \ldots, a_n]$ in this paper.

An important property of the star graph is its recursive structure. Such an structure is particularly useful for an inductive argument. Assume $n \geq 4$. Then S_n can be *decomposed* into smaller S_{n-1}'s as follows: Let $2 \leq p \leq n$ be fixed and let H_i be the subgraph of G induced by vertices with i in the pth position for $1 \leq i \leq n$. Then H_i is isomorphic to S_{n-1}. We say S_n is *decomposed along the pth position.* The edges whose end-vertices are in different H_i are the *cross-edges* with respect to the given decomposition.

We note that each vertex is incident to exactly one cross-edge and there are $(n-2)!$ independent[1] cross-edges between two different H_i's. Frequently, for notational convenience, it is assumed that the decomposition is along the nth position.

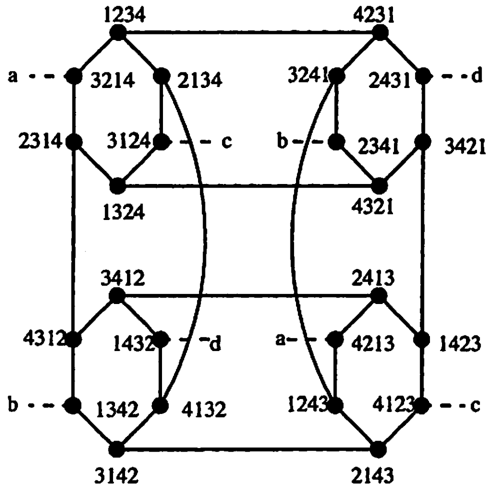

FIGURE 1. S_4

2. THE MAIN RESULT

Before we prove the conditional matching preclusion result for S_n, we need some preliminary results. The first is the result of the matching preclusion problem for S_n.

Theorem 2.1 ([4]). *Let $n \geq 3$. Then $mp(S_n) = \delta(S_n) = n-1$. Moreover, if $n \geq 4$, then the only optimal solutions are the trivial matching preclusion sets.*

We need a number of Hamiltonian results for this paper. The first is Theorem 2.2 which we will now state.

Theorem 2.2 ([5]). *Let $n \geq 4$. Then for every pair of vertices x and y in different partite sets of S_n, there is a Hamiltonian path between x and y in S_n.*

Theorem 2.2 can be generalized in a number of ways. One way is to consider both vertex faults and *edge faults*, that is, consider deleting both vertices and edges. Indeed, we need one version of such stronger result

[1]A set of edges is *independent* if no two of them share an endpoint.

position to decompose S_n such that the number of faulty cross-edges is maximized. We may assume it is along the last position. Let H_i be the subgraph of G induced by vertices with i in the last position for $1 \leq i \leq n$. So H_i is isomorphic to S_{n-1}. Let B be the set of cross-edges. Since B induces a perfect matching, $|F \cap B| \geq 1$. We claim that $|E(H_i) \cap F| \leq 2(n-1) - 5$ for every i. (So we can apply the induction hypothesis for every H_i.) If $|F| = 2n - 5$, then since we chose B to be the set of cross-edges with the most elements of F, $|F \cap B| \geq 2$ since $(2n-5)/(n-1) > 1$ for $n \geq 5$. (This is one of the reasons why $n = 4$ is one of the base cases.) So $|E(H_i) \cap F| \leq 2n - 5 - 2 = 2n - 7 = 2(n-1) - 5$ for every i. If $|F| \leq 2n - 6$, then $|E(H_i) \cap F| \leq 2n - 6 - 1 = 2n - 7 = 2(n-1) - 5$ for every i. Then by the induction hypothesis, $H_i - F$ either has a perfect matching or $H_i - F$ has an isolated vertex, for each i. In order for $H_i - F$ to have an isolated vertex, we must have $|E(H_i) \cap F| \geq n-2$, and hence at most one of these $H_i - F$'s has an isolated vertex. Hence we may assume that exactly one of them has an isolated vertex. For notational convenience, we may assume it is $H_1 - F$. It is clear that it has exactly one isolated vertex as H_1 is an $(n-2)$-regular bipartite graph with girth 6 and we are deleting at most $2n - 7$ edges. Suppose v is such an isolated vertex. Let $e_1, e_2, \ldots, e_{n-2}$ be the edges incident to v in H_1 and e_{n-1} be the cross-edge incident to v. Since $G - F$ has no isolated vertices, e_{n-1} is not a faulty edge. Now by Lemma 2.5, there exist $n-2$ cycles, C_j, $j = 1, 2, \ldots, n-2$, such that C_j contains e_j and e_{n-1} for all j. Moreover, the $n-2$ paths $C_j - \{e_j, e_{n-1}\}$ are pairwise edge disjoint. Since $e_1, e_2, \ldots, e_{n-2}$ are faulty edges, we have $2n-5-(n-2) = n-3$ unidentified faulty edges. So at least one of these $n-2$ paths is *fault-free*, that is, it contains no faulty edges. Again, for notational convenience, we assume that it is $C_1 - \{e_1, e_{n-1}\}$. Now C_1 is a 6-cycle. Then by the proof of Lemma 2.5, it is of the form $v - v_2 - v_3 - v_4 - v_5 - v_6 - v$ where $v, v_6 \in V(H_1)$, v_2, v_3 are vertices of another H_i, say H_2 (for notational simplicity) and v_4, v_5 are vertices of another H_i, say H_3 (for notational simplicity). Indeed, (v, v_6) is the edge e_1. Now H_2 has at most $2n - 5 - (n-2) - 1 = n - 4$ faulty edges, so every vertex in $H_2 - F$ has degree at least 2. Let F_2 be the elements of $F \cap E(H_2)$ together with all the edges in H_2 that are incident to v_2 except the edge (v_2, v_3). Then $H_2 - F_2$ has no isolated vertices and $|F_2| \leq (n-4) + (n-3) = 2n - 7 = 2(n-1) - 5$, so by the induction hypothesis, $H_2 - F_2$ has a perfect matching M_2. Moreover, (v_2, v_3) is an element of M_2. Similarly, we can find a perfect matching in $H_3 - F$ containing the edge (v_4, v_5). Since $|F \cap E(H_1)| \leq 2(n-1) - 5$, if we let $F_1 = (F \cap E(H_1)) - \{(v, v_6)\}$, then $|F_1| \leq 2(n-1) - 6$. Again $H_1 - F_1$ has no isolated vertices as $H_1 - F$ has only one isolated vertex. By the induction hypothesis, $H_1 - F_1$ has a perfect matching M_1. Moreover, (v, v_6) is an element of M_1. Now, for $i \geq 4$, $|F \cap E(H_i)| \leq 2n - 5 - (n-2) - 1 = n - 4$. So by Theorem 2.1,

$H_i - F$ has a perfect matching M_i. Let $M = M_1 \cup M_2 \cup \cdots \cup M_n$. We let $M' = (M - \{(v_2, v_3), (v_4, v_5), (v, v_6)\}) \cup \{(v, v_2), (v_3, v_4), (v_5, v_6)\}$. Clearly, M' is a perfect matching in $G - F$, and we are done. □

Lemma 2.7. $mp_1(S_4) = 4$. *Moreover, every optimal conditional matching preclusion set is trivial.*

Proof. The first statement follows from Theorem 2.6. Let F be a set of edges of size 4 that is a conditional matching preclusion set. We can decompose S_4 along $i = 2, 3, 4$. Since $4/3 > 1$, we may assume that decomposing along $n = 4$ gives us at least 2 faulty cross-edges. If there are more than 2 cross-edges that are faults, then there is at most one fault left. So clearly, $S_4 - F$ has a perfect matching just by using the edges from the four 6-cycles corresponding to the H_i's. Now one of the 6-cycles cannot have perfect matchings, so one of them contains exactly two faulty edges. We may assume that it is H_4. (From Figure 1, one can see that a left-right flip or a top-bottom flip gives an automorphism.) So we have two cases.

We first assume that the two faults in H_4 are incident to the same vertex. This vertex can be any one of the 6 vertices in H_4. The proof of these 6 cases are similar. So we only present the case for 1234. Hence $(1234, 3214), (1234, 2134) \in F$ and therefore $(1234, 4231) \notin F$. We will exhibit 3 perfect matchings in $S_4 - \{(1234, 3214), (1234, 2134)\}$ with the property that they share only one cross-edge, namely, $(1234, 4231)$. Since the other two faults must be cross-edges, at least one of the 3 perfect matchings is fault-free. The matchings given in Figure 2, Figure 3 and Figure 4 have this property.

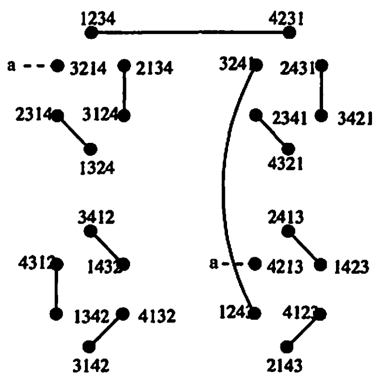

FIGURE 2

The second case is when the two faults are not incident to the same vertex. Moreover, deleting them will destroy all perfect matchings in H_4, a 6-cycle. There are three possibilities and they are similar. So we only present the case when the faults are (1234, 2134) and (2314, 1324). The goal is to show that one of the following is true:

(1) $(1234, 4231), (2314, 4312) \in F$
(2) $(2134, 4132), (1324, 4321) \in F$.

(This will show that F is a trivial conditional matching preclusion set.) Suppose not. Then one of the following must occur.

(1) $(1234, 4231), (2134, 4132) \notin F$
(2) $(1234, 4231), (1324, 4321) \notin F$.
(3) $(2314, 4312), (2134, 4132) \notin F$
(4) $(2314, 4312), (1324, 4321) \notin F$.

However, each case allows a perfect matching. It is clear that Figure 4 is a perfect matching for (2), and Figure 5 is a perfect matching for (3). Now assume we have (1). If neither Figure 3 nor Figure 4 is a perfect matching, then we have identified the two faults that are cross-edges, and Figure 6 will be a perfect matching. The case for (4) is similar. This finishes the proof. □

Theorem 2.8. *Let $n \geq 3$. Then $mp_1(S_n) = 2n - 4$. Moreover, every optimal conditional matching preclusion set is trivial.*

Proof. The first statement is just Theorem 2.6. So we only have to classify all the optimal conditional matching preclusion sets. If $n = 3$, then it is a 6-cycle and the result is clearly true. The case $n = 4$ is given by Lemma 2.7. We proceed with induction. Assume $n \geq 5$. (The first part of the proof is very similar to the proof of Theorem 2.6 but the analysis is tighter.) Let F be an optimal conditional matching preclusion set. Now $|F| = 2n - 4$. We may assume that decomposing S_n along the last position maximizes the number of faulty cross-edges. We define the H_i's as usual. By the choice of the decomposition, $|F \cap B| \geq 2$.

Since the proof is rather lengthy, we start with an outline of the proof. We consider the following condition:

$$(1) \qquad |E(H_i) \cap F| \leq 2(n-1) - 5 \text{ for every } i.$$

The proof can be divided into two steps.

- In Step 1, we show that Condition (1) will lead to a contradiction.
- In Step 2, we consider the possible cases when Condition (1) is violated.

Step 1. We assume Condition (1) is satisfied. Since F is a conditional matching preclusion set, at least one $H_i - F$ does not have a perfect matching. By Theorem 2.6, $F \cap E(H_i)$ is not a conditional matching preclusion set for every i. So $H_i - F$ either has a perfect matching or $H_i - F$ has an

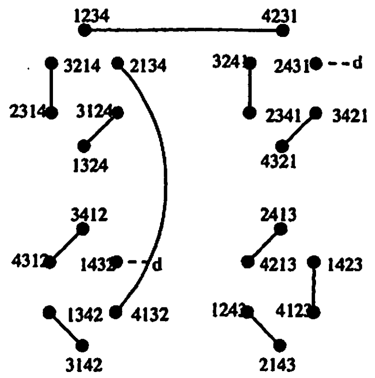

FIGURE 3

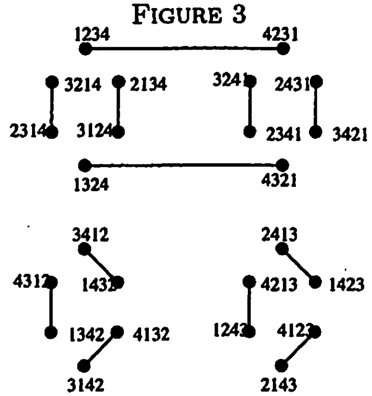

FIGURE 4

isolated vertex, for each i. In order for $H_i - F$ to have an isolated vertex, we must have $|E(H_i) \cap F| \geq n-2$, and hence at most one of these $(H_i - F)$'s has an isolated vertex. Hence we may assume that exactly one of them has an isolated vertex. For notational convenience, we may assume it is $H_1 - F$. It is clear that it has exactly one isolated vertex as H_1 is an $(n-2)$-regular bipartite with girth 6 and we are deleting at most $2n-6$ edges. Suppose v is such an isolated vertex. Let $e_1, e_2, \ldots, e_{n-2}$ be the edges incident to v in H_1 and e_{n-1} be the cross-edge incident to v. Since $G - F$ has no isolated vertices, e_{n-1} is not a faulty edge. Now by Lemma 2.5, there exist $n-2$ cycles, C_j, $j = 1, 2, \ldots, n-2$, such that C_j contains e_j and e_{n-1} for all j. Moreover, the $n-2$ paths $C_j - \{e_j, e_{n-1}\}$ are pairwise edge disjoint. Since $e_1, e_2, \ldots, e_{n-2}$ are faulty edges, we have $2n-4-(n-2) = n-2$ unidentified faulty edges. We consider two cases depending on whether any of these paths is fault-free.

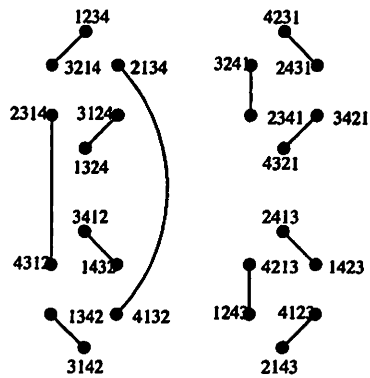

FIGURE 5

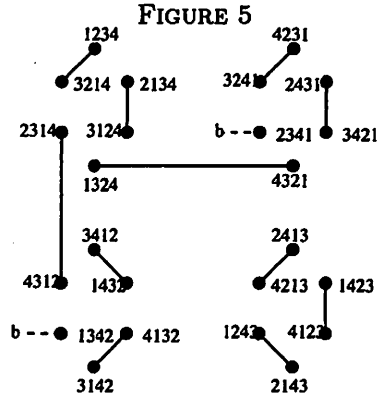

FIGURE 6

Case 1: One of these $n-2$ paths is fault-free. Again, for notational convenience, we assume that it is $C_1 - \{e_1, e_{n-1}\}$. Now C_1 is a 6-cycle. Then by the proof of Lemma 2.5, it is of the form $v - v_2 - v_3 - v_4 - v_5 - v_6 - v$ where $v, v_6 \in V(H_1)$, v_2, v_3 are vertices of another H_i, say H_2 (for notational simplicity) and v_4, v_5 are vertices of another H_i, say H_3 (for notational simplicity). Indeed, (v, v_6) is the edge e_1. Now H_2 has at most $2n-4-(n-2)-2 = n-4$ faulty edges, so every vertex in $H_2 - F$ has degree at least 2. Let F_2 be the elements of $F \cap E(H_2)$ together with all the edges in H_2 that are incident to v_2 except the edge (v_2, v_3), which we know is not a faulty edge. Then $H_2 - F_2$ has no isolated vertices and $|F_2| \leq (n-4)+(n-3) = 2n-7 = 2(n-1)-5$ edges, so by Theorem 2.6, $H_2 - F_2$ has a perfect matching M_2. Moreover, (v_2, v_3) is an element of M_2. Similarly, we can find a perfect matching in $H_3 - F$ containing the edge (v_4, v_5). Recall that $|F \cap E(H_1)| \leq 2(n-1)-5$. Let $F_1 = (F \cap E(H_1)) - \{(v, v_6)\}$

and then $|F_1| \leq 2(n-1)-6$. Again $H_1 - F_1$ has no isolated vertices as $H_1 - F$ has only one isolated vertex. So by Theorem 2.6, $H_1 - F_1$ has a perfect matching M_1. Moreover, (v, v_6) is an element of M_1. Now, for $i \geq 4$, $|F \cap E(H_i)| \leq 2n-4-(n-2)-2 = n-4$, so by Theorem 2.1, $H_i - F$ has a perfect matching M_i. Let $M = M_1 \cup M_2 \cup \cdots \cup M_n$. We let $M' = (M - \{(v_2, v_3), (v_4, v_5), (v, v_6)\}) \cup \{(v, v_2), (v_3, v_4), (v_5, v_6\}$. Clearly, M' is a perfect matching in $G - F$, which is a contradiction.

Case 2: None of these $n-2$ paths is fault-free. Then each path must have exactly one faulty edge. Suppose we consider $C_1 - \{e_1, e_{n-1}\}$, where C_1 is $v - v_2 - v_3 - v_4 - v_5 - v_6 - v$ as before, where $v, v_6 \in V(H_1)$, v_2, v_3 are vertices of another H_i, say H_2, for notational simplicity, and v_4, v_5 are vertices of another H_i, say H_3, for notational simplicity. Indeed, (v, v_6) is the edge e_1. Now H_2 has at most $2n-4-(n-2)-2 = n-4$ faulty edges, so every vertex in $H_2 - F$ has degree at least 2. Let F_2 be the elements of $F \cap E(H_2)$ together with all the edges in H_2 that are incident to v_2 except the edge (v_2, v_3). (That is, if (v_2, v_3) is a faulty edge, we remove it as a faulty edge.) Then $H_2 - F_2$ has no isolated vertices and $|F_2| \leq (n-4)+(n-3) = 2n-7 = 2(n-1)-5$, so by Theorem 2.6, $H_2 - F_2$ has a perfect matching M_2. Moreover, (v_2, v_3) is an element of M_2. Similarly, we can find a perfect matching in $H_3 - F$ containing the edge (v_4, v_5). Recall that $|F \cap E(H_1)| \leq 2(n-1)-5$. Let $F_1 = (F \cap E(H_1)) - \{(v, v_6)\}$ and $|F_1| \leq 2(n-1)-6$. Again $H_1 - F_1$ has no isolated vertices as $H_1 - F$ has only one isolated vertex. By Theorem 2.6, $H_1 - F_1$ has a perfect matching M_1. Moreover, (v, v_6) is an element of M_1. Now, for $i \geq 4$, $|F \cap E(H_i)| \leq 2n-4-(n-2)-2 = n-4$. By Theorem 2.1, $H_i - F$ has a perfect matching M_i. Let $M = M_1 \cup M_2 \cup \cdots \cup M_n$. We let $M' = (M - \{(v_2, v_3), (v_4, v_5), (v, v_6)\}) \cup \{(v, v_2), (v_3, v_4), (v_5, v_6)\}$, which is a perfect matching. So the proof still holds if none of $(v, v_2), (v_3, v_4), (v_5, v_6)$ is a faulty edge. We already know (v, v_2) is not a faulty edge. Note that (v_3, v_4) and (v_5, v_6) are cross-edges. So the argument fails if one of (v_3, v_4) and (v_5, v_6) is a fault. Therefore, if this argument fails for every C_i, we have found $n-2$ faulty edges that are cross-edges. So H_1 must have exactly $n-2$ faulty edges and they are all incident to v. Now recall that e_{n-1} is not a fault. For notational simplicity, we assume the end-vertices of e_{n-1} are v and u_2, and u_2 is a vertex in H_2. Now, we can find non-fault cross-edges $(v, u_2), (w_2, u_3), (w_3, u_4), \ldots, (w_{n-1}, u_n), (w_n, u_1)$ where $u_1 \in V(H_1)$ and $u_i, w_i \in V(H_i)$ for $i = 2, 3, \ldots, n$ such that all these vertices are distinct, v and u_1 belong to different partite sets, and w_i and u_i belong to different partite sets for $i = 2, 3, \ldots, n$. (There are $(n-2)!$ independent cross-edges between two H_i's and half of them are edges between vertices in prescribed partite sets. But there are only $n-2$ cross-edges that are faults. Now $(n-2)!/2 > n-2$ is clearly true for $n \geq 6$. For $n = 5$, we have equality. But this can only occur for one pair of H_i's. Hence we may assume that they don't occur in consecutive H_i's as we may reorder $H_3, H_4, \ldots, H_n$ in the construction.) Now, each of $H_2, H_3, \ldots, H_n$ has no faults and there are exactly $n-2$ faults in H_1, namely, the edges in H_1 that are incident with v. Now, by Theorem 2.2, there is a Hamiltonian path P_i between w_i and u_i in H_i for $i = 2, 3, \ldots, n$. In addition, we will ignore the $n-2$ faults in H_1 to obtain a Hamiltonian path P_1 between v and u_1 in H_1. We note that P_1 contains exactly one fault. Now, $P_1, P_2, \ldots, P_n$ and $(v, u_2), (w_2, u_3), (w_3, u_4), \ldots, (w_{n-1}, u_n), (w_n, u_1)$ form a Hamiltonian cycle in S_n and it contains exactly one fault. Hence $S_n - F$ contains a fault-free perfect matching, a contradiction.

Step 2. What we proved is the following: If Condition (1) is satisfied, then it leads to a contradiction that F is a conditional matching preclusion set. Suppose $|F \cap B| \geq 3$. Then $|E(H_i) \cap F| \leq 2n - 4 - 3 = 2n - 7 = 2(n-1) - 5$ for every i, satisfying Condition (1), which will lead to a contradiction. So $|F \cap B| = 2$. In addition, we may assume Condition (1) is violated (otherwise it leads to a contradiction), that is, one of the H_i's has at least $2n-6$ faulty edges. This can occur only once. For notational simplicity, assume it is H_1. Since $|F \cap B| = 2$, H_1 has exactly $2n-6$ faulty edges. So we have found all the faults. Now each H_i has a perfect matching for $i = 2, 3, \ldots, n$. So $H_1 - F$ has no perfect matching. We consider two cases depending on whether $H_1 - F$ has isolated vertices.

Case 1: $H_1 - F$ has isolated vertices. Let v be an isolated vertex in H_1. Then the cross-edge incident to v is not a faulty edge as $S_n - F$ has no isolated vertices. Let this cross-edge be (v, u_2), and for notational simplicity, assume u_2 is a vertex of H_2. Now let F' be the set obtained from $F \cap E(H_1)$ by deleting edges that are incident to v in H_1, that is, let F' be the elements of F except the $n-2$ edges in H_1 that are incident to v. Then $|F'| = 2n - 6 - (n-2) = n - 4$. Now, we can find non-fault cross-edges $(v, u_2), (w_2, u_3), (w_3, u_4), \ldots, (w_{n-1}, u_n), (w_n, u_1)$ where $u_1 \in V(H_1)$ and $u_i, w_i \in V(H_i)$ for $i = 2, 3, \ldots, n$ such that all these vertices are distinct, v and u_1 belong to different partite sets, and w_i and u_i belong to different partite sets for $i = 2, 3, \ldots, n$. (The reason why these edges exist is similar to the reason given in Case 2 of Step 1. In fact, the argument is somewhat simpler as $|F \cap B| = 2$.) Now, each of $H_2, H_3, \ldots, H_n$ has no faults. By Theorem 2.2, there is a Hamiltonian path P_i between w_i and u_i in H_i for $i = 2, 3, \ldots, n$. By Theorem 2.3, there is a Hamiltonian path P_1 between v and u_1 in $H_1 - F'$. We note that P_1 contains exactly one element of F. Now, $P_1, P_2, \ldots, P_n$ and $(v, u_2), (w_2, u_3), (w_3, u_4), \ldots, (w_{n-1}, u_n), (w_n, u_1)$ form a Hamiltonian cycle in S_n and it contains exactly one fault. Hence $S_n - F$ contains a fault-free perfect matching, a contradiction.

Case 2: $H_1 - F$ has no isolated vertices. Then by the induction hypothesis, $F \cap E(H_1)$ is a trivial conditional matching preclusion set in H_1. So there is a path (of length 2) $z_1 - z_2 - z_3$ such that $F \cap E(H_1) = (N_{H_1}(z_1) \cup N_{H_1}(z_3)) - \{(z_1, z_2), (z_2, z_3)\}$. We want to prove that the two faults in B are exactly the cross-edge incident with z_1 and the cross-edge incident with z_3. By way of contradiction, suppose this is not the case. Then without loss of generality, we may assume the cross-edge incident with z_1 is not a fault. Let the neighbours of z_1 in H_1 other than z_2 be $y_1, y_2, \ldots, y_{n-3}$. If $n \geq 6$, then at least one of the y_i's is incident to a cross-edge that is not a fault. If $n = 5$, then we cannot find such a y_i if both y_1 and y_2 are incident to cross-edges that are faults. But then we have found the two unique cross-edges that are faults. So let the neighbours of z_3 in H_1 other than z_2 be v_1, v_2. Since H_1 is bipartite and of girth 6, the y_i's are distinct from the v_i's. So z_3 and v_1 are incident to cross-edges that are not faults. So relabel if necessary, and we may assume z_1 and y_1 are incident to cross-edges that are not faults. Our objective is to find a perfect matching M_1 in $H_1 - (F \cup \{z_1, y_1\})$. Now by Theorem 2.4, there is a Hamiltonian cycle C_1 in $H_1 - \{z_1, y_1\}$. If C_1 uses (z_2, z_3), then C_1 contains exactly one element of F, and hence C_1 contains a perfect matching in $H_1 - (F \cup \{z_1, y_1\})$. Now suppose C_1 does not use (z_2, z_3). Let C_1 be $z_3 - P_{1,1} - z_2 - P_{1,2} - z_3$, that is, $P_{1,1}$ and $P_{2,2}$ are the two paths obtained from C_1 by deleting z_2 and z_3. Since z_2 and z_3 are in different partite sets, each of $P_{1,1}$ and $P_{1,2}$ has an even number of vertices. So $P_{1,1}$, and $P_{1,2}$ together with (z_2, z_3) contain a perfect matching in $H_1 - (F \cup \{z_1, y_1\})$. Now let (z_1, u_2) and (y_1, w_n) be the cross-edges. We note that u_2, w_n belonging to the same H_i's clearly impossible from the definition of S_n. So we may assume u_2 and w_n belong to different H_i's, say u_2 is in H_2 and w_n is in H_n. Then we can use the usual argument to find non-fault cross-edges $(w_2, u_3), (w_3, u_4), \ldots, (w_{n-1}, u_n)$ where $u_i, w_i \in V(H_i)$ for $i = 2, 3, \ldots, n$, the vertices are all distinct, and u_i and w_i belong to different partite sets for $i = 2, 3, \ldots, n$. Now each of $H_2, H_3, \ldots, H_n$ has no faults. By Theorem 2.2, there is a Hamiltonian path P_i between u_i and w_i in H_i for $i = 2, 3, \ldots, n$. Now, $P_2, \ldots, P_n$ and $(w_2, u_3), (w_3, u_4), \ldots, (w_{n-1}, u_n), (z_1, u_2), (y_1, w_n)$ give a path covering all the vertices in $H_2, H_3, \ldots, H_n$ and $\{z_1, y_1\}$. Because this path has even length, it contains a perfect matching on the vertices of $H_2, H_3, \ldots, H_n$ and $\{z_1, y_1\}$. Together with M_1, this gives a perfect matching of $S_n - F$, a contradiction. Hence the two faults in B are exactly the cross-edge incident with z_1 and the cross-edge incident with z_3. So F is trivial, and we are done. □

3. Conclusion

In this paper, we solved the conditional matching preclusion problem for S_n and classified all the optimal conditional matching preclusion sets. Since this problem is a refinement of the matching preclusion problem, obviously we have to use the result of the matching preclusion problem for S_n (Theorem 2.1). In addition, three Hamiltonian results were used, namely,