The Generalized Pell p-Sequences in Groups

Ömür DEVECI [1], Merve AKDENİZ [2] and Erdal KARADUMAN [3]

[1,2] Kafkas University, Department of Mathematics Faculty of Science and Letters
36100 Kars/ TURKEY

[3] Department of Mathematics, Faculty of Science, Atatürk University, 25240 Erzurum, TURKEY

E- mail: odeveci36@hotmail.com, merveakdeniz55@gmail.com
eduman@atauni.edu.tr

Abstract In this paper, we study the generalized Pell p-sequences modulo m. Also, we define the generalized Pell p-sequences and the basic generalized Pell p-sequences in groups and then we examine these sequences in finite groups. Furthermore, we obtain the periods of the generalized Pell p-sequences and the basic periods of the basic generalized Pell p-sequences in the binary polyhedral groups $\langle n,2,2\rangle$, $\langle 2,n,2\rangle$ and $\langle 2,2,n\rangle$.

2000 Mathematics Subject Classification: 11B50, 20D60, 15A36

Keywords: Generalized Pell p-sequence, period, group, matrix.

1 Introduction and Preliminaries

Many of the obtained numbers by using homogeneous linear recurrence relations and their miscellaneous properties have been studied (see [5,6,15-24,27,29-37,39]). The study of recurrence sequences in groups began with the earlier work of Wall [38] where the ordinary Fibonacci sequences in cyclic groups were investigated. The concept extended to some special linear recurrence sequences by several authors (see [1-3,7-13,25,26,28,38,40]). In this paper, we extend the theory to the generalized Pell p-sequences.

In [23], Kılıç and Taşçı defined the k sequences of the generalized order-k Pell numbers as follows:

for $n>0$ and $1 \leq i \leq k$

This Project was supported by the Commission for the Scientific Research Projects of Kafkas University. The Project number. 2013-FEF-72

$$P_n^i = 2P_{n-1}^i + P_{n-2}^i + \quad + P_{n-k}^i,$$

with initial conditions

$$P_n^i = \begin{cases} 1 & \text{if } n = 1 - i, \\ 0 & \text{otherwise,} \end{cases} \quad \text{for } 1 - k \le n \le 0,$$

where P_n^i is the n^{th} term of the i^{th} sequence.

In [22], Kılıç defined the generalized Pell (p, i) numbers as follows:

for $p(p = 1, 2, \ldots)$, $n > p+1$ and $0 \le \mathrm{i} \le p$,

$$P_p^{(i)}(n) = 2P_p^{(i)}(n-1) + P_p^{(i)}(n-p-1), \tag{1}$$

with initial conditions

$$P_p^{(i)}(1) = \quad = P_p^{(i)}(i) = 0 \text{ and } P_p^{(i)}(i+1) = \quad = P_p^{(i)}(p+1) = 1.$$

Note that if $i = 0$, the initial conditions are $P_p^{(0)}(1) = P_p^{(0)}(2) = \quad = P_p^{(0)}(p+1) = 1$.

In [22], the generalized Pell p-matrix A has been given as:

$$A = \left[a_{ij}\right]_{(p+1)\times(p+1)} = \begin{pmatrix} 2 & 0 & & 0 & 1 \\ 1 & 0 & & 0 & 0 \\ 0 & 1 & & & 0 \\ & & & 0 & \\ 0 & & 0 & 1 & 0 \end{pmatrix}. \tag{2}$$

Also, in [22] Kılıç obtained that

$$A^n = \begin{bmatrix} P_p^{(p)}(n+p+1) & P_p^{(p)}(n+1) & P_p^{(p)}(n+2) & & P_p^{(p)}(n+p) \\ P_p^{(p)}(n+p) & P_p^{(p)}(n) & P_p^{(p)}(n+1) & & P_p^{(p)}(n+p-1) \\ & & & & \\ P_p^{(p)}(n+2) & P_p^{(p)}(n-p+2) & P_p^{(p)}(n-p+3) & & P_p^{(p)}(n+1) \\ P_p^{(p)}(n+1) & P_p^{(p)}(n-p+1) & P_p^{(p)}(n-p+2) & & P_p^{(p)}(n) \end{bmatrix}_{(p+1)\times(p+1)}$$

(3)

and

$$\begin{bmatrix} P_p^{(p)}(n+p+1) \\ P_p^{(p)}(n+p) \\ \vdots \\ P_p^{(p)}(n+2) \\ P_p^{(p)}(n+1) \end{bmatrix} = A \begin{bmatrix} P_p^{(p)}(n+p) \\ P_p^{(p)}(n+p-1) \\ \vdots \\ P_p^{(p)}(n+1) \\ P_p^{(p)}(n) \end{bmatrix}. \tag{4}$$

A sequence is periodic if, after a certain point, it consists only of repetitions of a fixed subsequence. The number of elements in the repeating subsequence is called the period of the sequence. For example, the sequence $a,b,c,d,b,c,d,b,c,d,\ldots$ is periodic after the initial element a and has period 3. A sequence is simply periodic with period k if the first k elements in the sequence form a repeating subsequence. For example, the sequence $a,b,c,d,a,b,c,d,a,b,c,d,\ldots$ is simply periodic with period 4.

2 The Generalized Pell p-Sequences Modulo m

Reducing the generalized Pell p-sequence $\left\{P_p^{(p)}(n)\right\}$ by a modulus m, we can get a repeating sequence, denoted by

$$\left\{P_p^{(p,m)}(n)\right\} = \left\{P_p^{(p,m)}(1), P_p^{(p,m)}(2), \ldots, P_p^{(p,m)}(p), P_p^{(p,m)}(p+1), \ldots, P_p^{(p,m)}(i), \ldots\right\}$$

where $P_p^{(p,m)}(i) = P_p^{(p)}(i) \pmod{m}$. Also, it has the same recurrence relation as in (1).

Theorem 2.1. $\left\{P_p^{(p,m)}(n)\right\}$ is a simply periodic sequence.

Proof. Let $W = \left\{(x_1, x_2, \ldots, x_{p+1}) \mid 0 \le x_i \le m-1\right\}$. Then we have $|W| = m^{p+1}$ being finite, that is, for any $j \ge 0$, there exist $i \ge j$ such that $P_p^{(p,m)}(i+p+1) \equiv P_p^{(p,m)}(j+p+1)$, $P_p^{(p,m)}(i+p) \equiv P_p^{(p,m)}(j+p), \ldots, P_p^{(p,m)}(i+1) \equiv P_p^{(p,m)}(j+1)$. It is easy to see from (1) that $P_p^{(p,m)}(i) \equiv P_p^{(p,m)}(j)$, $P_p^{(p,m)}(i-1) \equiv P_p^{(p,m)}(j-1), \ldots$, $P_p^{(p,m)}(i-j+1) \equiv P_p^{(p,m)}(1)$. Then we get that the $\left\{P_p^{(p,m)}(n)\right\}$ is a simply periodic sequence. □

Let $h_p^p(m)$ denote the smallest periods of $\left\{P_p^{(p,m)}(n)\right\}$.

For a given matrix $N=[n_{ij}]$ with n_{ij}'s being integers, $N \pmod m$ means that every entries of N are reduced modulo m, that is, $N \pmod m = (n_{ij} \pmod m)$. Let $\langle A\rangle_{k^u} = \{A^i \pmod{k^u} \mid i \geq 0\}$ be a cyclic group such that k is a prime and let $|\langle A\rangle_{k^u}|$ denotes the order of $\langle A\rangle_{k^u}$. It is easy to see from (3) that $h_p^p(k^u) = |\langle A\rangle_{k^u}|$.

Example. We have $\{P_2^{(2,3)}(n)\} = \{0,0,1,2,1,0,2,2,1,1,1,0,1,\ldots\}$. So, we get $h_2^2(3)=13$.

Theorem 2.2. Let t be the largest positive integer and let u be a prime such that $h_p^p(u) = h_p^p(u^t)$. Then $h_p^p(u^\alpha) = u^{\alpha-t} \cdot h_p^p(u)$ for every $\alpha \geq t$.

Proof. Let n be a positive integer. Since $A^{h_p^p(u^{n+1})} \equiv I \pmod{u^{n+1}}$, that is, $A^{h_p^p(u^{n+1})} \equiv I \pmod{u^n}$, we get that $h_p^p(u^n)$ divides $h_p^p(u^{n+1})$. On the other hand, writing $A^{h_p^p(u^n)} = I + (a_{ij}^{(n)} \cdot u^n)$, we have

$$A^{h_p^p(u^n)\cdot u} = \left(I + (a_{ij}^{(n)} \cdot u^n)\right)^u = \sum_{i=0}^{u} \binom{u}{i} (a_{ij}^{(n)} \cdot u^n)^i \equiv I \pmod{u^{n+1}},$$

which yields that $h_p^p(u^{n+1})$ divides $h_p^p(u^n) \cdot u$. Therefore, $h_p^p(u^{n+1}) = h_p^p(u^n)$ or $h_p^p(u^{n+1}) = h_p^p(u^n) \cdot u$, and the latter holds if and only if there is an $a_{ij}^{(n)}$ which is not divisible by u. Since $h_p^p(u^t) \neq h_p^p(u^{t+1})$, there is an $a_{ij}^{(t+1)}$ which is not divisible by u, thus, $h_p^p(u^{t+1}) \neq h_p^p(u^{t+2})$. The proof is finished by induction on t. □

It is easy to prove that if $m = \prod_{i=1}^{t} u_i^{k_i}, (t \geq 1)$ where u_i's are distinct primes, then $h_p^p(m) = \operatorname{lcm}[h_p^p(u_i^{k_i})]$.

3 The Generalized Pell p-Sequence and The Basic Generalized Pell p-Sequence in Groups

Let G be a finite j-generator group and let X be the subset of $\underbrace{G\times G\times G\times \quad \times G}_{j}$ such that $(x_0,x_1,\ldots,x_{j-1})\in X$ if and only if G is generated by $x_0,x_1,\ldots,x_{j-1}$. We call $(x_0,x_1,\ldots,x_{j-1})$ a generating j-tuple for G.

Each generating *j-tuple* $(x_0,x_1,\ldots,x_{j-1})\in X$ maps to $|\text{Aut } G|$ distinct elements of X under the action of elements of Aut G. Hence there are

$$d_j(G)=|X|/|\text{Aut } G| \quad (\text{where } |X| \text{ is the number of elements of } X)$$

non-isomorphic generating *j-tuples* for G (see [9]).

The notation $d_j(G)$ was introduced in [14].

Definition 3.1 (Knox [25]). A k-nacci sequence in a finite group is a sequence of group elements $x_0,x_1,x_2,\ldots,x_n,\ldots$ for which, given an initial (seed) set $x_0,x_1,x_2,\ldots,x_{j-1}$ each element is defined by

$$x_n=\begin{cases} x_0x_1\ldots x_{n-1} & \text{for } j\le n<k,\\ x_{n-k}x_{n-k+1}\ldots x_{n-1} & \text{for } n\ge k.\end{cases}$$

We also require that the initial elements of the sequence, $x_0,x_1,x_2,\ldots,x_{j-1}$, generate the group, thus forcing the k-nacci sequence to reflect the structure of the group. The k-nacci sequence of a group G generated by $x_0,x_1,x_2,\ldots,x_{j-1}$ is denoted by $F_k(G;x_0,x_1,\ldots,x_{j-1})$.

In [25], Knox had denoted the period of a k-nacci sequence $F_k(G;x_0,x_1,\ldots,x_{j-1})$ by $P_k(G;x_0,x_1,\ldots,x_{j-1})$.

Definition 3.2 (Deveci and Karaduman [9]). For a j-tuple $(x_0,x_1,\ldots,x_{j-1})\in X$ the basic k-nacci sequence $\overline{F}_k(G:x_0,x_1,\ldots,x_{j-1})$ of the basic period m is a sequence of group elements $b_0,b_1,b_2,\ldots,b_n,\ldots$ for which, given an initial (seed) set $b_0=x_0, b_1=x_1, b_2=x_2,\ldots,b_{j-1}=x_{j-1}$, each element is defined by

$$b_n=\begin{cases} b_0b_1\ldots b_{n-1} & \text{for } j\le n<k,\\ b_{n-k}b_{n-k+1}\ldots b_{n-1} & \text{for } n\ge k.\end{cases}$$

where $m \geq 1$ is the least integer with

$$b_0 = b_m\theta, b_1 = b_{m+1}\theta, b_2 = b_{m+2}\theta, \ldots, b_{k-1} = b_{m+k-1}\theta,$$

for some $\theta \in \operatorname{Aut} G$. Since G is a finite j-generator group and $b_m, b_{m+1}, \ldots, b_{m+j-1}$ generate G, it follows that θ is uniquely determined. The basic k-nacci sequence $\overline{F}_k(G: x_0, x_1, \ldots, x_{j-1})$ is finite containing m element.

In [9], Deveci and Karaduman had denoted the basic period of the basic k-nacci sequence $\overline{F}_k(G: x_0, x_1, \ldots, x_{j-1})$ by $BP_k(G; x_0, x_1, \ldots, x_{j-1})$.

Definition 3.3 (Deveci and Karaduman [12]). A generalized order-k Pell sequence in a finite group is a sequence of group elements $x_0, x_1, \ldots, x_n, \ldots$ for which, given an initial (seed) set $x_0, \ldots, x_{j-1}$, each element is defined by

$$x_n = \begin{cases} x_0 x_1 \ldots (x_{n-1})^2 & \text{for } j \leq n < k, \\ x_{n-k} x_{n-k+1} \ldots (x_{n-1})^2 & \text{for } n \geq k. \end{cases}$$

It is required that the initial elements of the sequence, $x_0, \ldots, x_{j-1}$, generate the group, thus, forcing the generalized order-k Pell sequence to reflect the structure of the group. The generalized order-k Pell sequence of a group generated by $x_0, \ldots, x_{j-1}$ is denoted by $Q_k(G; x_0, x_1, \ldots, x_{j-1})$.

In [12], Deveci and Karaduman had denoted the period of the generalized order-k Pell sequence $Q_k(G; x_0, x_1, \ldots, x_{j-1})$ by $PerQ_k(G; x_0, x_1, \ldots, x_{j-1})$.

Definition 3.4. A generalized Pell p-sequence $(p \geq 2)$ in a finite group is a sequence of group elements $x_0, x_1, \ldots, x_n, \ldots$ for which, given an initial (seed) set $x_0, \ldots, x_{j-1}$, $(p+1 \geq j)$ each element is defined by

$$x_n = \begin{cases} x_0 (x_{n-1})^2 & \text{for } j \leq n < p+1, \\ x_{n-p-1} (x_{n-1})^2 & \text{for } n \geq p+1. \end{cases}$$

It is require that the initial elements of the sequence, $x_0, \ldots, x_{j-1}$, generate the group, thus, forcing the generalized Pell p-sequence to reflect the structure of the group. The generalized Pell p-sequence of a group generated by $x_0, \ldots, x_{j-1}$ is denoted by $Q^{(p)}(G; x_0, x_1, \ldots, x_{j-1})$.

It is important note that the classic generalized Pell p-sequence $(p \geq 2)$ in a cyclic group $C = \langle x \rangle$ is as following

$$x_0 = e, x_1 = e, \ldots, x_{p-1} = e, x_p = x$$

and

$$x_{n+p} = x_{n-1}\left(x_{n+p-1}\right)^2 \text{ for } n \geq 1.$$

Theorem 3.1. A generalized Pell p-sequence in a finite group is simply periodic.

Proof. Let n be the order of G. Since there n^{p+1} distinct $(p+1)$-tuple of elements of G, at least one of the $(p+1)$-tuples appear twice in a generalized Pell p-sequence of the group G. Thus, the subsequence following this $(p+1)$-tuple repeats. Because of the repeating, the generalized Pell p-sequence is periodic.

Since the generalized Pell p-sequence is periodic, there exist natural numbers u and v, with $u > v$, such that

$$x_{u+1} = x_{v+1}, x_{u+2} = x_{v+2}, \ldots, x_{u+p+1} = x_{v+p+1}.$$

By the defining relation of the generalized Pell p-sequence, we know that

$$x_u = \left(x_{u+p+1}\right) \cdot \left(x_{u+p}\right)^{-2} \text{ and } x_v = \left(x_{v+p+1}\right) \cdot \left(x_{v+p}\right)^{-2}.$$

Therefore, $x_u = x_v$, and hence,

$$x_{u-v} = x_{v-v} = x_0, x_{u-v+1} = x_{v-v+1} = x_1, \ldots, x_{u-v+p} = x_{v-v+p} = x_p,$$

which implies that the generalized Pell p-sequence is simply periodic. □

We denote the period of the generalized Pell p-sequence $Q^{(p)}\left(G; x_0, x_1, \ldots, x_{j-1}\right)$ by $PerQ^{(p)}\left(G; x_0, x_1, \ldots, x_{j-1}\right)$.

To examine the concept more fully we study the action of automorphism group Aut G of G on the generalized Pell p-sequence $Q^{(p)}\left(G; x_0, x_1, \ldots, x_{j-1}\right)$, $\left(x_0, x_1, \ldots, x_{j-1}\right) \in X$. Now Aut G consists of all isomorphisms $\theta : G \to G$ and if $\theta \in$ Aut G and $\left(x_0, x_1, \ldots, x_{j-1}\right) \in X$ then $\left(x_0\theta, x_1\theta, \ldots, x_{j-1}\theta\right) \in X$.

For a subset $A \subseteq G$ and $\theta \in$ Aut G the image of A under θ is

$$A\theta=\{a\theta : a\in A\}.$$

Lemma 3.1. Let $(x_0, x_1, \ , x_{j-1})\in X$ and let $\theta\in \text{Aut}\, G$. Then $\left(Q^{(p)}(G; x_0, x_1, ..., x_{j-1})\right)\theta = Q^{(p)}(G; x_0\theta, x_1\theta, ..., x_{j-1}\theta)$.

Proof: Let $Q^{(p)}(G; x_0, x_1, ..., x_{j-1}) = \{a_i\}$. The result is obvious since $\{a_i\}\theta = \{a_i\theta\}$ and

$$a_{i+p}\theta = \left(a_{i-1}\left(a_{i+p-1}\right)^2\right)\theta = a_{i-1}\theta a_{i+p-1}\theta a_{i+p-1}\theta .$$

Suppose ω elements of Aut G map $Q^{(p)}(G; x_0, x_1, ..., x_{j-1})$ into itself. Then there are $|\text{Aut}\, G|/\omega$ distinct generalized Pell p-sequences $Q^{(p)}(G; x_0\theta, x_1\theta, ..., x_{j-1}\theta)$ for $\theta\in\text{Aut}\, G$. □

Definition 3.5. For a j-*tuple* $(x_0, x_1, ..., x_{j-1})\in X$ the basic generalized Pell p-sequence $\overline{Q}^{(p)}(G; x_0, x_1, ..., x_{j-1})$, $(p\geq 2,\ p+1\geq j)$ of the basic period m is a sequence of group elements $a_0, a_1, a_2, ..., a_n, ...$ for which, given an initial (seed) set $a_0 = x_0,\ a_1 = x_1,\ a_2 = x_2, ..., a_{j-1} = x_{j-1}$, each element is defined by

$$a_n = \begin{cases} a_0\left(a_{n-1}\right)^2 & \text{for } j\leq n< p+1, \\ a_{n-p-1}\left(a_{n-1}\right)^2 & \text{for } n\geq p+1 \end{cases}$$

where $m\geq 1$ is the least integer with

$$a_0 = a_m\theta,\ a_1 = a_{m+1}\theta,\ a_2 = a_{m+2}\theta, ..., a_p = a_{m+p}\theta ,$$

for some $\theta\in\text{Aut}\, G$. Since G is a finite j-generator group and $a_m, a_{m+1}, ..., a_{m+j-1}$ generate G, it follows that θ is uniquely determined. The basic generalized Pell p-sequence $\overline{Q}^{(p)}(G; x_0, x_1, ..., x_{j-1})$ is finite containing m element.

We denote the basic period of the basic generalized Pell p-sequence $\overline{Q}^{(p)}(G; x_0, x_1, ..., x_{j-1})$ by $BQ^{(p)}(G; x_0, x_1, ..., x_{j-1})$.

From the definitions, it is clear that the periods of the sequences $Q^{(p)}(G;x_0,x_1,\dots,x_{j-1})$ and $\overline{Q}^{(p)}(G;x_0,x_1,\dots,x_{j-1})$ in a finite group depend on the chosen generating set and the order of the generating elements.

Theorem 3.2. Let G be a finite group and $(x_0,x_1,\dots,x_{j-1})\in X$. If $PerQ^{(p)}(G;x_0,x_1,\dots,x_{j-1})=n$ and $BQ^{(p)}(G;x_0,x_1,\dots,x_{j-1})=m$, then m divides n and there are n/m elements of Aut G which map $Q^{(p)}(G;x_0,x_1,\dots,x_{j-1})$ into itself.

Proof: We have $n=m\cdot\alpha$ where α is order of automorphism $\theta\in \text{Aut } G$ since

$$Q^{(p)}(G;x_0,x_1,\dots,x_{j-1})=\overline{Q}^{(p)}(G;x_0,x_1,\dots,x_{j-1})(G)\cup Q^{(p)}(G;x_0\theta,x_1\theta,\dots,x_{j-1}\theta)(G)\cup$$
$$\cup Q^{(p)}(G;x_0\theta^2,x_1\theta^2,\dots,x_{j-1}\theta^2)(G)\cup\dots$$

and $BQ^{(p)}(G;x_0,x_1,\dots,x_{j-1})=BQ^{(p)}(G;x_0\theta,x_1\theta,\dots,x_{j-1}\theta)$. So we get that $1,\theta,\theta^2,\dots,\theta^{\alpha-1}$ map $Q^{(p)}(G;x_0,x_1,\dots,x_{j-1})$ into itself. □

4. Applications

In this section, we obtain the periods of the generalized Pell p-sequences and the basic periods of the basic generalized Pell p-sequences in the binary polyhedral groups $\langle n,2,2\rangle$, $\langle 2,n,2\rangle$ and $\langle 2,2,n\rangle$ as the applications of the above results.

Definition 4.1. The binary polyhedral group $\langle l,m,n\rangle$, for $l,m,n>1$, is defined by the presentation

$$\langle x,y,z: x^l=y^m=z^n=xyz\rangle.$$

When $l=2$, we obtain for $\langle 2,m,n\rangle$ the presentation

$$\langle y,z: y^m=z^n=(yz)^2\rangle.$$

The binary polyhedral group $\langle l,m,n\rangle$ is finite if and only if the number $k=lmn\left(\frac{1}{l}+\frac{1}{m}+\frac{1}{n}-1\right)=mn+nl+lm-lmn$ is positive. Its order is $4lmn/k$.

For more information on these groups see [4, pp.68-71].

We consider binary polyhedral groups both as 2-generator and as 3-generator groups.

Theorem 4.1. Let G_n be the group defined by the presentation $\langle x, y, z : x^n = y^2 = z^2 = xyz \rangle$.

i. $PerQ^{(2)}(G_n; x, y, z) = \begin{cases} 3n, & n \text{ is even,} \\ 6n, & n \text{ is odd} \end{cases}$

and $BQ^{(2)}(G_n; x, y, z) = \begin{cases} 3n, & n \text{ is even,} \\ 3n, & n \text{ is odd.} \end{cases}$

ii. $PerQ^{(3)}(G_n; x, y, z) = BQ^{(3)}(G_n; x, y, z) = \begin{cases} 4n, & n \text{ is even,} \\ 8n, & n \text{ is odd} \end{cases}$

iii. Let $p \geq 4$.

1. If there is no $m \in [3, p-1]$ such that m is an odd factor of n then,

$$PerQ^{(p)}(G_n; x, y, z) = BQ^{(p)}(G_n; x, y, z) = \begin{cases} n(p+1), & n \text{ is even,} \\ 2n(p+1), & n \text{ is odd.} \end{cases}$$

2. Let t be the biggest odd factor of n in $[3, p-1]$, then two cases occur:

i'. If $t \cdot 3^j \notin [3, p-1]$ for $j \in$, then

$$PerQ^{(p)}(G_n; x, y, z) = BQ^{(p)}(G_n; x, y, z) = \begin{cases} t(n(p+1)), & n \text{ is even,} \\ t(2n(p+1)), & n \text{ is odd.} \end{cases}$$

ii'. If s is the biggest odd number which is in $[3, p-1]$ and $s = t \cdot 3^j$ for $j \in$, then

$$PerQ^{(p)}(G_n; x, y, z) = BQ^{(p)}(G_n; x, y, z) = \begin{cases} s(n(p+1)), & n \text{ is even,} \\ s(2n(p+1)), & n \text{ is odd.} \end{cases}$$

Proof. We first note that $|x| = 2n$, $|y| = 4$ and $|z| = 4$.

i. The sequence $Q^{(2)}(G_n; x, y, z)$ is

$$x, y, z, x^{n+1}, yx^2, z^3, x, yx^4, z, x^{n+1}, yx^6, z^3, x, yx^8, z, \ldots.$$

This sequence can be said to form layers of length six. Using the above, the sequence $Q^{(2)}(G_n;x,y,z)$ becomes:

$$x_0=x,\ x_1=y,\ x_2=z,\ x_3=x^{n+1},\ x_4=yx^2,\ x_5=z^3,$$
$$x_6=x,\ x_7=yx^4,\ x_8=z,\ x_9=x^{n+1},\ x_{10}=yx^6,\ x_{11}=z^3,\ldots,$$
$$x_{6i}=x,\ x_{6i+1}=yx^{4i},\ x_{6i+2}=z,\ x_{6i+3}=x^{n+1},\ x_{6i+4}=yx^{2+4i},\ x_{6i+5}=z,\ldots.$$

So, we need the smallest $i\in$ such that $4i=2nv$ for $v\in$.

If n is even, $i=\frac{n}{2}$. Thus, $PerQ^{(2)}(G_n;x,y,z)=BQ^{(2)}(G_n;x,y,z)=3n$ since $x\theta=x$, $y\theta=y$ and $z\theta=z$ where θ is inner automorphism induced by conjugation by x^n.

If n is odd, $n=i$. Thus, $PerQ^{(2)}(G_n;x,y,z)=6n$ and $BQ^{(2)}(G_n;x,y,z)=3n$ since $x\theta=x^{n+1}$, $y\theta=y$ and $z\theta=z^{-1}$ where θ is a outer automorphism of order 2.

ii. The sequence $Q^{(3)}(G_n;x,y,z)$ is

$$x,y,z,x^{n+1},x^3,yx^6,z^3,x,x^5,yx^{16},z,x^{n+1},x^7,yx^{30},z^3,x,x^9,yx^{48},z,x^{n+1},\ldots.$$

This sequence can be said to form layers of length eight. Using the above, the sequence $Q^{(3)}(G_n;x,y,z)$ becomes:

$$x_0=x,\ x_1=y,\ x_2=z,\ x_3=x^{n+1},\ x_4=x^3,\ x_5=yx^6,\ x_6=z^3,\ x_7=x,$$
$$x_8=x^5,\ x_9=yx^{16},\ x_{10}=z,\ x_{11}=x^{n+1},\ x_{12}=x^7,\ x_{13}=yx^{30},\ x_{14}=z^3,\ x_{18}=x,\ldots,$$
$$x_{8i}=x^{4i+1},\ x_{8i+1}=yx^{8i^2+8i},\ x_{8i+2}=z,\ x_{8i+3}=x^{n+1},$$
$$x_{8i+4}=x^{3+4i},\ x_{8i+5}=yx^{8i^2+16i+6},\ x_{8i+6}=z^3,\ x_{8i+7}=x,\ldots.$$

So, we need the smallest $i\in$ such that $4i=2nv$ for $v\in$.

If n is even, $i=\frac{n}{2}$. Thus, $PerQ^{(3)}(G_n;x,y,z)=4n$.

If n is odd, $n=i$. Thus, $PerQ^{(3)}(G_n;x,y,z)=8n$.

Also, $PerQ^{(3)}(G_n;x,y,z)=BQ^{(3)}(G_n;x,y,z)$ since $x\theta=x$, $y\theta=y$ and $z\theta=z$ where θ is the identity automorphism.

iii. If $p \geq 4$, we have the sequence

$$x_0 = x, x_1 = y, x_2 = z, x_3 = x^{n+1}, x_4 = x^3, x_5 = x^7, \ldots, x_p = x^{2^{p-2}-1} \left(x_\alpha = x^{2^{\alpha-2}-1}, 4 \leq \alpha \leq p\right), \ldots,$$

$$x_{(2p+2)i} = x^{\lambda_1 4i+1}, x_{(2p+2)i+1} = yx^{\lambda_2 4i}, x_{(2p+2)i+2} = z, x_{(2p+2)i+3} = x^{n+1}, x_{(2p+2)i+4} = x^{4i+3},$$

$$x_{(2p+2)i+5} = x^{\lambda_3 4i+7}, \ldots, x_{(2p+2)i+p+1} = x^{\lambda_{p-2} 4i+2^{p-1}-1} \left(x_{(2p+2)i+\alpha} = x^{\lambda_{\alpha-2} 4i+2^{\alpha-2}-1}, 5 \leq \alpha \leq p+1\right),$$

$$x_{(2p+2)i+p+2} = yx^{\lambda_p 4i+2}, x_{(2p+2)i+p+3} = z^3, x_{(2p+2)i+p+4} = x, x_{(2p+2)i+p+5} = x^{4i+1},$$

$$x_{(2p+2)i+p+6} = x^{\lambda_{p+1} 4i+1}, \ldots, x_{(2p+2)i+2p+1} = x^{\lambda_{2p-4} 4i+1} \ldots.$$

where $\lambda_1, \ldots, \lambda_{2p-4} \in$. So we need an i such that $4i = 2nv$ for $v \in$.

1. If there is no $m \in [3, p-1]$ such that m is an odd factor of n, then there are two sub-cases:

First case: If n is even, then $i = \frac{n}{2}$. So, we get $PerQ^{(p)}(G_n; x, y, z) = n(p+1)$.

Second case: If n is odd, then $i = n$. So, we get $PerQ^{(p)}(G_n; x, y, z) = 2n(p+1)$.

2. If t is the biggest odd factor of n in $[3, p-1]$, then two cases occur.

i'. If $t \cdot 3^j \notin [3, p-1]$ for $j \in$, then there are two sub-cases:

First case: If n is even, then $i = t \cdot \frac{n}{2}$. So, we get $PerQ^{(p)}(G_n; x, y, z) = t(n(p+1))$.

Second case: If n is odd, then $i = t \cdot n$. So, we get $PerQ^{(p)}(G_n; x, y, z) = t(2n(p+1))$.

ii'. If s is the biggest odd number which is in $[3, p-1]$ and $s = t \cdot 3^j$ for $j \in$, then there are two sub-cases:

First case: If n is even, then $i = s \cdot \frac{n}{2}$. So, we get $PerQ^{(p)}(G_n; x, y, z) = s(n(p+1))$.

Second case: If n is odd, then $i = s \cdot n$. So, we get $PerQ^{(p)}(G_n; x, y, z) = s(2n(p+1))$.

Also, $PerQ^{(p)}(G_n; x, y, z) = BQ^{(p)}(G_n; x, y, z)$ for $p \geq 4$ since $x\theta = x$, $y\theta = y$ and $z\theta = z$ where θ is the identity automorphism. □

Theorem 4.2. If the group G_n is defined by the presentation $\langle y, z : y^n = z^2 = (yz)^2 \rangle$, then

i. $PerQ^{(2)}(G_n; y, z) = \begin{cases} 3n, & n \text{ is even}, \\ 6n, & n \text{ is odd}. \end{cases}$

iii. Let $p \geq 3$.

1. If there is no $m \in [3, p]$ such that m is an odd factor of n then,

$$PerQ^{(p)}(G_n; y, z) = \begin{cases} n(p+1), & n \text{ is even}, \\ 2n(p+1), & n \text{ is odd}. \end{cases}$$

2. Let t be the biggest odd factor of n in $[3, p]$, then two cases occur:

i'. If $t \cdot 3^j \notin [3, p]$ for $j \in$, then

$$PerQ^{(p)}(G_n; y, z) = \begin{cases} t(n(p+1)), & n \text{ is even}, \\ t(2n(p+1)), & n \text{ is odd}. \end{cases}$$

ii'. If s is the biggest odd number which is in $[3, p]$ and $s = t \cdot 3^j$ for $j \in$, then

$$PerQ^{(p)}(G_n; y, z) = \begin{cases} s(n(p+1)), & n \text{ is even}, \\ s(2n(p+1)), & n \text{ is odd}. \end{cases}$$

Also, $BQ^{(p)}(G_n; y, z) = \begin{cases} \dfrac{PerQ^{(p)}(G_n; y, z)}{2}, & n \equiv 2 \bmod 4, \\ PerQ^{(p)}(G_n; y, z), & \text{otherwise} \end{cases}$ for $p \geq 2$.

If the group G_n is defined by the presentation $\langle x, y, z : x^2 = y^n = z^2 = xyz \rangle$, then

$PerQ^{(p)}(G_n; x, y, z) = \begin{cases} n(p+1), & n \text{ is even}, \\ 2n(p+1), & n \text{ is odd} \end{cases}$ and $BQ^{(p)}(G_n; x, y, z) = \begin{cases} n(p+1), & n \text{ is even}, \\ n(p+1), & n \text{ is odd}. \end{cases}$

Proof. We proceed similarly to the proof of the Theorem 4.1. Firstly, let us consider the 2-generator case. We first note that $|y| = 2n$, $|z| = 4$ and

$|yz|=4$. The sequences $Q^{(2)}(G_n;y,z)$ and $Q^{(p)}(G_n;y,z)\,(p>2)$ are in the following forms, respectively:

$$x_0=y,\ x_1=z,\ x_2=y^{n+1},\ x_3=y^3,\ x_4=zy^6,\ x_5=y,\ x_6=y^5,\ldots,$$

$$x_{6i}=y^{4i+1},\ x_{6i+1}=zy^{8i^2+8i},\ x_{6i+2}=y^{n+1},\ x_{6i+3}=y^{4i+3},\ x_{6i+4}=zy^{8i^2-2},\ x_{6i+5}=y,\ldots$$

and

$$x_0=y,\ x_1=z,\ x_2=y^{n+1},\ x_3=y^3,\ldots,x_{p+1}=y^{2^p-1}\left(x_\alpha=y^{2^{\alpha-1}-1},3\le\alpha\le p+1\right),$$

$$x_{p+2}=zy^{2^{p+1}-2},\ x_{p+3}=y,\ x_{p+4}=y^5,\ldots,x_{2p+1}=y^{2^{p-1}(p-2)+1}\left(x_{p+3+\beta}=y^{2^{\beta+1}\cdot\beta+1},1\le\beta\le p-2\right),\ldots,$$

$$x_{(2p+2)i}=y^{\lambda_1 4i+1},\ x_{(2p+2)i+1}=zy^{\lambda_2 4i},\ x_{(2p+2)i+2}=y^{n+1},\ x_{(2p+2)i+3}=y^{4i+3},$$

$$x_{(2p+2)i+4}=y^{\lambda_3 4i+7},\ldots,x_{(2p+2)i+p+1}=y^{\lambda_p 4i+(2^p-1)}\left(x_{(2p+2)i+\alpha}=y^{\lambda_{\alpha-1}4i+(2^{\alpha-1}-1)},4\le\alpha\le p+1\right),$$

$$x_{(2p+2)i+p+2}=zy^{\lambda_{p+1}4i+(2^{p+1}-2)},\ x_{(2p+2)i+p+3}=y,\ x_{(2p+2)i+p+4}=y^{4i+5},\ x_{(2p+2)i+p+5}=y^{\lambda_{p+2}4i+17},\ldots,$$

$$x_{(2p+2)i+2p+1}=y^{\lambda_{2p-2}4i+(2^{p-1}(p-2)+1)}\left(x_{(2p+2)i+p+4+\beta}=y^{\lambda_{p+\beta+1}4i+(2^{\beta+1}\cdot\beta+1)},1\le\beta\le p-3\right),\ldots.$$

where $\lambda_1,\ldots,\lambda_{2p-2},i\in$. Then we obtain

$BQ^{(p)}(G_n;y,z)=\dfrac{PerQ^{(p)}(G_n;y,z)}{2}$ for $n\equiv 2 \bmod 4$ since $y\theta=y^{1}$ and $z\theta=z^{1}$ where θ is a outer automorphism of order 2,

$PerQ^{(p)}(G_n;y,z)=BQ^{(p)}(G_n;y,z)$ for $n\not\equiv 2 \bmod 4$ since $y\theta=y$ and $z\theta=z$ where θ is the identity automorphism.

Secondly, let us consider the 3-generator case. We first note that $|x|=4$, $|y|=2n$ and $|z|=4$. The sequences $Q^{(2)}(G_n;x,y,z)$ and $Q^{(p)}(G_n;x,y,z)\,(p>2)$ are in the following forms, respectively:

$$x_0=x,\ x_1=y,\ x_2=z,\ldots,$$

$$x_{6i-3}=x^3,\ x_{6i-2}=y^{n+1},\ x_{6i-1}=xy^{4i-1},$$

$$x_{6i}=x,\ x_{6i+1}=y,\ x_{6i+2}=xy^{4i+1},\ldots$$

and

$x_0 = x, x_1 = y, x_2 = z, x_3 = x^3, \dots, x_p = x^3, x_{p+1} = x^3, x_{p+2} = y^{n+1}, x_{p+3} = xy^3, x_{p+4} = x, \dots, x_{2p+1} = x, \dots,$
$x_{(2p+2)i-(p+1)} = x^3, x_{(2p+2)i-p} = y^{n+1}, x_{(2p+2)i-p+1} = xy^{4i-1}, x_{(2p+2)i-p+2} = x, \dots, x_{(2p+2)i-1} = x,$
$x_{(2p+2)i} = x, x_{(2p+2)i+1} = y, x_{(2p+2)i+2} = z, x_{(2p+2)i+3} = x^2, \dots, x_{(2p+2)i+p} = x^2, \dots$
where $i \in$. Then we obtain

$PerQ^{(p)}(G_n; x, y, z) = BQ^{(p)}(G_n; x, y, z)$ if n is even since $x\theta = x$, $y\theta = y$ and $z\theta = z$ where θ is the identity authomorphism,

$BQ^{(p)}(G_n; x, y, z) = \frac{PerQ^{(p)}(G_n; x, y, z)}{2}$ if n is odd since $x\theta = x^{1}$, $y\theta = y^{n+1}$ and $z\theta = z$ where θ is a outer automorphism of order 2. □

Theorem 4.3. If the group G_n is defined by the presentation $\langle y, z : y^2 = z^n = (yz)^2 \rangle$, then

$$PerQ^{(p)}(G_n; y, z) = \begin{cases} n(p+1), & n \text{ is even}, \\ 2n(p+1), & n \text{ is odd} \end{cases}$$

and

$$BQ^{(p)}(G_n; y, z) = \begin{cases} \frac{PerQ^{(p)}(G_n; y, z)}{2}, & n \equiv 2 \bmod 4, \\ PerQ^{(p)}(G_n; y, z), & \text{otherwise.} \end{cases}$$

If the group G_n is defined by the presentation $\langle x, y, z : x^2 = y^2 = z^n = xyz \rangle$, then

$$PerQ^{(p)}(G_n; x, y, z) = \begin{cases} n(p+1), & n \text{ is even}, \\ 2n(p+1), & n \text{ is odd} \end{cases}$$

and

$$BQ^{(p)}(G_n; x, y, z) = \begin{cases} \frac{PerQ^{(p)}(G_n; x, y, z)}{2}, & n \text{ is odd and } p = 2, \\ PerQ^{(p)}(G_n; x, y, z), & \text{otherwise.} \end{cases}$$

Proof. Firstly, let us consider the 2-generator case. We first note that $|y| = 4$, $|z| = 2n$ and $|yz| = 4$.

$BQ^{(p)}(G_n;y,z)=\dfrac{PerQ^{(p)}(G_n;y,z)}{2}$ for $n\equiv 2 \bmod 4$ since $y\theta=y^{-1}$ and $z\theta=z^{n+1}$ where θ is a outer automorphism of order 2,

$PerQ^{(p)}(G_n;y,z)=BQ^{(p)}(G_n;y,z)$ for $n\not\equiv 2 \bmod 4$ since $y\theta=y$ and $z\theta=z$ where θ is identy automorphism.

Secondly, let us consider the 3-generator case. We first note that $|x|=4$, $|y|=4$ and $|z|=2n$.

$BQ^{(2)}(G_n;x,y,z)=\dfrac{PerQ^{(2)}(G_n;x,y,z)}{2}$ if n is odd since $x\theta=x$, $y\theta=y^{-1}$ and $z\theta=z^{n+1}$ where θ is a outer automorphism of order 2,

$BQ^{(p)}(G_n;x,y,z)=PerQ^{(p)}(G_n;x,y,z)$ in other cases since $x\theta=x$, $y\theta=y$ and $z\theta=z$ where θ is the identity automorphism.

The proof is similar to the proof of Theorem 4.1 and is omitted. □

References

[1] H. Aydın and R. Dikici, General Fibonacci sequences in finite groups, *Fibonacci Quart.*, 36(3) (1998), 216-221.

[2] C. M. Campbell, P. P. Campbell, The Fibonacci length of certain centro-polyhedral groups, *J. Appl. Math. Comput.*, 19(1-2) (2005), 231-240.

[3] C. M. Campbell, H. Doostie and E. F. Robertson, Fibonacci length of generating pairs in groups in Applications of Fibonacci Numbers, Vol. 3 Eds. G. E. Bergum et al. Kluwer Academic Publishers, (1990), 27-35.

[4] H. S. M. Coxeter, W. O. J. Moser, *Generator and relations for discrete groups*, 3 rd edition, Springer, Berlin (1972).

[5] P. G. Becker, *k*-regular power series and Mahler-type functional equations, *J. Number Theory*, 49(3) (1994), 269-286.

[6] W. Bosma and C. Kraaikamp, Metrical theory for optimal continued fractions, *J. Number Theory*, 34(3) (1990), 251-270.

[7] O. Deveci, The Pell-Padovan sequences and the Jacobsthal-Padovan sequences in finite groups, *Util. Math.*, in press.

[8] O. Deveci and E. Karaduman, The generalized order-*k* Lucas sequences in Finite groups, *J. Appl. Math.,* (2012), 464580-1-464580-15.

[9] O. Deveci and E. Karaduman, On the basic *k*-nacci sequences in finite groups, *Discrete Dyn. Nat. Soc.*, (2011), 639476-1-639476-13.

[10] O. Deveci and E. Karaduman, The cyclic groups via the Pascal matrices and the generalized Pascal matrices, *Linear Algebra Appl.*, 437(10) (2012), 2538-2545.

[11] O. Deveci, E. Karaduman and C. M. Campbell, On The *k*-nacci sequences in finite binary polyhedral groups. *Algebra Colloq.*, 18 (Spec 1) (2011), 945-954.

[12] O. Deveci and E. Karaduman, The Pell sequences in finite groups, *Util. Math.*, in press.

[13] H. Doostie and M. Hashemi, Fibonacci lengths involving the Wall number $k(n)$, *J. Appl. Math. Comput.* 20 (2006), 171-180.

[14] P. Hall, The Eulerian Functions of a Group, *Q. J. Math.* 7(1) (1936), 134-151.

[15] M.S. El Naschie, Deriving the essential features of the standard model from the general theory of relativity, *Chaos Solitons Fractals*, 24(4) (2005), 941-946.

[16] M.S. El Naschie, Stability analysis of two-slit experiment with quantum particles, *Chaos Solitons Fractals*, 26(2) (2005), 291-294.

[17] S. Falcon and A. Plaza, *k*-Fibonacci sequences modulo m, *Chaos Solitons Fractals*, 41(1) (2009), 497-504.

[18] A.S. Fraenkel and S. T. Klein, Robust universal complete codes for transmission and compression, *Discrete Appl. Math.*, 64(1) (1996), 31-55.

[19] N.D. Gogin and A.A. Myllari, The Fibonacci-Padovan sequence and MacWilliams transform matrices, *Programming and Computer Software, published in Programmirovanie,* 33(2) (2007), 74-79.

[20] D. Kalman, Generalized Fibonacci numbers by matrix methods, *Fibonacci Quart.*, 20(1-2) (1982), 73-76.

[21] G. R. Kaluge, Penggunaan Fibonacci dan Josephus problem dalam algoritma enkripsi transposisi+substitusi, *Makalah IF 3058 Kriptografi-Sem. II Tahun* 2010/2011.

[22]E. Kılıç, The generalized Pell (p,i)-numbers and their Binet formulas, combinatorial representations, sums, *Chaos Solitons Fractals*, 40 (4) (2009), 2047-2063.
[23]E. Kılıç and D.Taşcı, The generalized Binet formula, representation and sums of the generalized order $-k$ Pell numbers, *Taiwanese J. Math.*, 10(6) (2006), 1661-1670.
[24]B.K. Kirchoff, R. Rutishauser, The phyllotaxy of costus (costaceae), *Bot. gaz*, 151(1) (1990), 88-105.
[25]S.W. Knox, Fibonacci sequences in finite groups, *Fibonacci Quart.*, 30(2) (1992), 116-120.
[26]K. Lü and J. Wang, k-step Fibonacci sequence modulo m, *Util. Math.*, 71 (2006), 169-178.
[27]D. M. Mandelbaum, Synchronization of codes by means of Kautz's Fibonacci encoding, *IEEE Transactions on Information Theory*, 18(2) (1972), 281-285.
[28]E. Ozkan, H. Aydin and R. Dikici, 3-step Fibonacci series modulo m, *Appl. Math. Comput.*, 143(1) (2003), 165-172.
[29]R.G.E. Pinch, Recurrent sequences modulo prime Powers, In M. Ganley (ed.) Cryptography and Coding III, IMA Conference Series (ns.) vol.45, Inst. Math. And Its Appl., Oxford university Press 1993, Proceedings, 3rd IMA, Conference on Cryptography and Coding, Cirencester December 1991, 297-310.
[30]A.G. Shannon and A.F. Horadam, Generalized Pell numbers and polinomials, *in Applications of Fibonacci numbers*, Vol. 9 Ed. F. T. Howard. Kluwer Academic Publishers, (2004), 213-224.
[31]V.W. Spinadel, The family of metallic means, *Vis Math.*, 1(3) (1999), http://www.mi.sanu.ac.rs/vismath/spinadel/index.html.
[32]V.W. Spinadel, The metallic means family and forbidden symmetries, *Int. Math. J.*, 2(3) (2002), 279-288.
[33]A.P. Stakhov and B. Rozin, Theory of Binet formulas for Fibonacci and Lucas p-numbers, *Chaos Solitons Fractals*, 27(5) (2006), 1162-1177.
[34]A.P. Stakhov, A.A. Sluchenkova and V. Massingua, Intorduction into Fibonacci coding and cryptography, Kharkov, Osnova, 1999.
[35]W. Stein, Modelling the evolution of Stellar architecture in Vascular plants, *Int. J. Plant Sci.*, 154(2) (1993), 229-263.
[36]D. Taşçı and E. Kılıç, On the order-k generalized Lucas numbers, *Appl. Math. Comput.* 155(3) (2004), 637-641.

[37] S. Vajda, Fibonacci and Lucas numbers, and the Golden section theory and applications, New York, John Wiley&Sons, 1989.
[38] D.D. Wall, Fibonacci series modulo m, *Amer. Math. Monthly*, 67(6) (1960), 525-532.
[39] F. Yilmaz and D. Bozkurt, The generalized order-k Jacobsthal numbers, *Int. J. Contemp. Math. Sci.*, 4(34) (2009), 1685-1694.
[40] H.J. Wilcox, Fibonacci sequences of period n in groups, *Fibonacci Quart.*, 24(4) (1986), 356-361.