A Result on Linear Arboricity of Planar Graphs *

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Abstract

The linear arboricity $la(G)$ of a graph G is the minimum number of linear forests which partition the edges of G. In this paper, it is proved that if G is a planar graph with maximum degree $\Delta \geq 7$ and every 7-cycle of G contains at most two chords, then $la(G) = \lceil \frac{\Delta(G)}{2} \rceil$.

Key words: Planar graph; Linear arboricity; Cycle

1 Introduction

Throughout this paper, we only consider finite, simple and undirected graphs. For a real number x, $\lceil x \rceil$ is the least integer not less than x and $\lfloor x \rfloor$ is the largest integer not larger than x. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$, we use $\Delta(G)$ and $\delta(G)$ to denote the maximum (vertex) degree and the minimum (vertex) degree, respectively. All undefined terminologies and notations follow that of Bondy and Murty [2].

A *linear forest* of a graph G in which each component is a path. A map φ from $E(G)$ to $\{1, 2, \cdots, t\}$ is called a *t-linear coloring* if $(V(G), \varphi^{-1}(\alpha))$ is a linear forest for $1 \leq \alpha \leq t$. The *linear arboricity* $la(G)$ of a graph G defined by Harary [9] is the minimum number t such that G has a t-linear coloring. Akiyama, Exoo and Harary [1] conjectured that $la(G) = \lceil \frac{\Delta(G)+1}{2} \rceil$ for every regular graph G. It is obvious that $la(G) \geq \lceil \frac{\Delta(G)}{2} \rceil$ for any graph G and

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$la(G) \geq \lceil \frac{\Delta(G)+1}{2} \rceil$ for any regular graph G. So the conjecture is equivalent to the following Linear Arboricity Conjecture (for short LAC):

Conjecture 1.1. (LAC) *For any simple graph G,*

$$\lceil \frac{\Delta(G)}{2} \rceil \leq la(G) \leq \lceil \frac{\Delta(G)+1}{2} \rceil. \tag{1}$$

Although Péroche [10] showed that LAC is an NP-hard problem. In fact, the linear arboricity has been determined for many classes of graphs and some corresponding results can be found in [1, 6, 7, 8, 15, 17]. Many results are also obtained for planar graphs, see [3, 4, 11, 13, 16]. Up to now, LAC has already been proved to be true for all planar graphs, see [14, 18]. But determining the planar graphs with linear arboricity $\lceil \frac{\Delta(G)}{2} \rceil$ (or $\lceil \frac{\Delta(G)+1}{2} \rceil$) are still an open problem.

In the following, we only consider the planar graph G with maximum degree $\Delta \geq 7$. Wu [16] et al. proved that if G does not contain 4-, 5-cycles, then $la(G) = \lceil \frac{\Delta(G)}{2} \rceil$. Chen [4] and Wang [13] et al. improved this result and got that if G does not contain chordal i-cycles for some $i \in \{4, 5, 6, 7\}$, then $la(G) = \lceil \frac{\Delta(G)}{2} \rceil$. Here, we generalize this result and get the following result.

Theorem 1. *Let G be a planar graph with maximum degree $\Delta \geq 7$. If every 7-cycle of G contains at most two chords, then $la(G) = \lceil \frac{\Delta(G)}{2} \rceil$.*

We first introduce some more notations and definitions. Let G be a planar graph with face set $F(G)$. For a vertex v of G, the *degree* $d(v)$ is the number of edges incident with v, and for a face f of G, the *degree* $d(f)$ is the length of the boundary walk of f. Let $uv \in E(G)$ and $d(u) = k$, then we call vertex u is a k-neighbor of v. A *k-vertex*, *k^--vertex* or a *k^+-vertex* is a vertex of degree k, at most k or at leat k, respectively. Similarly, we can define a *k-face*, *k^--face* and a *k^+-face*. A k-face with consecutive vertices $v_1, v_2, \cdots, v_k$ along its boundary in some direction (such as the clockwise order) is often said to be a $(d(v_1), d(v_2), \cdots, d(v_k))$-face. Two cycles are said to be *adjacent* if they share at least one edge and two cycles are said to be *intersecting* if they share at least one vertex.

For a t-linear coloring φ and a vertex v of G, we denote by $C^i_\varphi(v)$ the set of colors appears i times at v, where $i = 0, 1, 2$. Then

$$|C^0_\varphi(v)| + |C^1_\varphi(v)| + |C^2_\varphi(v)| = t.$$

Let x be a vertex of G, denote $\varphi(x) = (\varphi(xy_1), \varphi(xy_1), \cdots, \varphi(xy_k))$, where vertices $y_1, y_2, \cdots, y_k$ are distinct neighbors of x. For any two vertices u and v, let $C_\varphi(u,v) = C^2_\varphi(u) \cup C^2_\varphi(v) \cup (C^1_\varphi(u) \cap C^1_\varphi(v))$, i.e., $C_\varphi(u,v)$ is the set of colors that appear two times at u and v. A *monochromatic path* is a path of whose edges receive the same color. For two different edges e_1 and e_2 of G, they are said to be in the *same color component*, denoted by $e_1 \leftrightarrow e_2$ if there is a monochromatic path of G connecting them. Furthermore, if two ends of e_i are known, i.e., $e_i = x_iy_i (i = 1, 2)$, then $x_1y_1 \leftrightarrow x_2y_2$ denotes more accurately that there is a monochromatic path from x_1 to y_2 passing through the edges x_1y_1 and x_2y_2 in G (i.e., y_1 and x_2 are internal vertices in the path). Otherwise, we use $x_1y_1 \not\leftrightarrow x_2y_2$ (or $e_1 \not\leftrightarrow e_2$) to denote that such monochromatic path passing through them does not exist. Note that $x_1y_1 \leftrightarrow x_2y_2$ and $x_1y_1 \leftrightarrow y_2x_2$ are different. $(u, i) \leftrightarrow (v, i)$ denote that u and v have a monochromatic path of color i between them. The number of d-vertices adjacent to a vertex v is denoted by $n_d(v)$ and the number of d-faces incident with a vertex v is denoted by $f_d(v)$.

2 Proof of Theorem 1

In [5], it is proved that $la(G) = \lceil \frac{\Delta(G)}{2} \rceil$ holds for an arbitrary planar graph G with maximum degree $\Delta \geq 9$. It suffices to prove the following result.

(*) *Let G be a planar graph such that $\Delta(G) \leq 8$ and every 7-cycle of G contains at most two chords. Then G has a 4-linear coloring.*

Let $G = (V, E, F)$ be a minimal counterexample to (*) in terms of the number of edges. We first show some known properties.

Lemma 1. [13] *let $uv \in E(G)$ and φ be a 4-linear coloring of $G - uv$. Then the following results hold.*

(1) $|C_\varphi(u,v)| = 4$;

(2) *If there is a color i such that $i \in C^1_\varphi(u) \cap C^1_\varphi(v)$, then $(u,i) \leftrightarrow (v,i)$;*

(3) $d_G(u) + d_G(v) \geq 10$;

(4) *If uv is incident with a 3-cycle $uvwu$ and $d(u) + d(v) = 10$, then $d(w) = 8$;*

(5) *If $d(u) = 7$, $d(v) = 3$ and uv is incident with a 3-cycle, then all neighbors of u except v are 4^+-vertices.*

By Lemma 1, we obtain that

(*a*) $\delta(G) \geq 2$.

(*b*) Any two 4^--vertices of G are not adjacent.

(*c*) Any 3-face is incident with three 5^+-vertices, or at least two 6^+-vertices.

(*d*) Any 7^--vertex has no neighbors of degree 2.

Note that in all figures of the paper, vertices marked • have no edges of G incident with them other than those shown and pair of vertices marked ∘ can be connected to each other.

Lemma 2. [4, 11, 13] *G has no configurations depicted in* Fig. 1.

Proof. The proofs of (1), (3) and (6) can be found in [13], the proof of (2) can be found in [11] and the proofs of (4), (5) and (7) can be found in [4], respectively. □

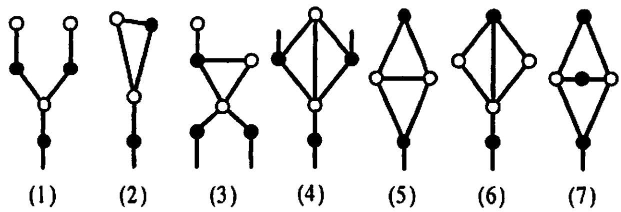

Fig. 1. Reducible configurations of Lemma 2.

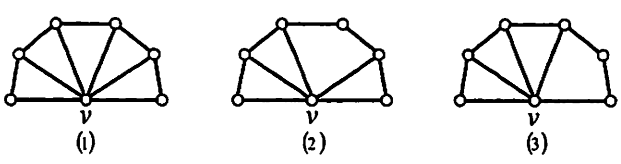

Fig. 2. Special configurations of G depicted in Lemma 3(a).

It is easy to obtain the following lemma, so we omit its proof here.

Lemma 3. *If a planar graph G with 7-cycles contains at most two chords and* $\delta(G) \geq 2$*, then we have*

(*a*) *G has no configurations depicted in* Fig. 2*, where all the vertices showing in* Fig. 2 *are different.*

(*b*) *Every* 6^+*-vertex v is incident with at most* $\lfloor \frac{4d(v)}{5} \rfloor$ *3-faces.*

By the Euler's formula $|V| - |E| + |F| = 2$, we have

$$\sum_{v\in V}(2d(v)-6)+\sum_{f\in F}(d(f)-6)=-12<0. \tag{2}$$

We first define ch to be the initial charge. Let $ch(v) = 2d(v) - 6$ for each $v \in V(G)$ and $ch(f) = d(f) - 6$ for each $f \in F(G)$. Then we will reassign a new charge denoted by $ch'(x)$ to each $x \in V(G) \cup F(G)$ by means of the discharging rules. Since our rules only move charges around, and do not affect the sum, we have

$$\sum_{x\in V(G)\cup F(G)} ch'(x) = \sum_{x\in V(G)\cup F(G)} ch(x) = -12. \tag{3}$$

Now, let us apply the following rules to redistribute the weight that leads a new charge $ch'(x)$.

R1 Each 8-vertex sends 1 to each of its adjacent 2-vertices.

R2 Let f be a 3-face uvw such that $d(u) \le d(v) \le d(w)$.

R2.1 If $d(u) \le 3$, then f receives $\frac{3}{2}$ from each of v and w.

R2.2 If $d(u) = 4$, then f receives $\frac{1}{2}$ from u and $\frac{5}{4}$ from each of its incident 6^+-vertices.

R2.3 Suppose $d(u) = d(v) = 5$. If some of u and v is incident with five 3-faces, say the vertex is u, then f receives $\frac{4}{5}$ from u, $\frac{6}{5}$ from v , and 1 from w. Otherwise, f receives $\frac{7}{8}$ from u, $\frac{7}{8}$ from v , and $\frac{5}{4}$ from w.

R2.4 If $d(u) = 5, d(v) \ge 6$ and $d(w) \ge 6$, then f receives $\frac{1}{2}$ from u, $\frac{5}{4}$ from v and $\frac{5}{4}$ from w.

R2.5 If $d(u) \ge 6$, then f receives 1 from each of its incident vertices.

R3 Let f be a 4-face.

R3.1 If f is incident with two 3^--vertices, then each 7^+-vertex incident with f sends 1 to f.

R3.2 If f is incident with a 3^--vertex and a 4-vertex (or 5-vertex), then each incident 7^+-vertex of f sends $\frac{3}{4}$, and the 4-vertex (or 5-vertex) sends $\frac{1}{2}$ to f.

R3.3 If f is incident with a 3^--vertex and three 6^+-vertices, then each incident 6^+-vertex of f sends $\frac{2}{3}$ to f.

R3.4 If f is incident with four 4^+-vertices, then f receives $\frac{1}{2}$ from each of its incident 4^+-vertices.

R4 Let f be a 5-face. If f is incident with two 3^--vertices, then f receives $\frac{1}{3}$ from each of its incident 7^+-vertices. Otherwise, f receives $\frac{1}{4}$ from each of its incident 4^+-vertices.

In the following, we will show that $ch'(x) \geq 0$ for each $x \in V(G) \cup F(G)$, a contradiction to (3), this completes the proof.

Let $f \in F(G)$. Clearly, $ch'(f) = ch(f) = d(f) - 6 \geq 0$ if $d(f) \geq 6$. If $d(f) = 5$, then f is incident with at most two 3^--vertices by Lemma 1 and then we can obtain that $ch'(f) \geq ch(f) + \min\{3 \times \frac{1}{3}, 4 \times \frac{1}{4}, 5 \times \frac{1}{4}\} = 0$ by R4. If $d(f) = 4$, the $ch'(f) \geq ch(f) + \min\{2 \times 1, 2 \times \frac{3}{4} + \frac{1}{2}, 3 \times \frac{2}{3}, 4 \times \frac{1}{2}\} = 0$ by R3. Suppose that $d(f) = 3$. Then f is not a $(3,7,7)$-face, $(4,6,6)$-face, $(4,6,7)$-face, $(5,5,5)$-face, $(5,5,6)$-face and $(5,5,7)$-face by Lemma 1. Thus $ch'(f) \geq ch(f) + \min\{2 \times \frac{3}{2}, 2 \times \frac{5}{4} + \frac{1}{2}, \frac{4}{5} + \frac{6}{5} + 1, 2 \times \frac{7}{8} + \frac{5}{4}, 3 \times 1\} = 0$ by R2.

Let $v \in V(G)$. If $d(v) = 2$, then $ch'(v) \geq ch(v) + 2 \times 1 = 0$ by R1. If $d(v) = 3$, then $ch'(v) = ch(v) = 0$. If $d(v) = 4$, then $ch'(v) \geq ch(v) - 4 \times \frac{1}{2} = 0$ by R2, R3 and R4. Suppose that $d(v) = 5$. If $f_3(v) = 5$, then $ch'(v) \geq ch(v) - 5 \times \frac{4}{5} = 0$ by R2. Otherwise, $ch'(v) \geq ch(v) - (f_3(v) \times \frac{7}{8} + (5 - f_3(v)) \times \frac{1}{2}) = \frac{12 - 3f_3(v)}{8} \geq 0$ by R2 and R3. Suppose that $d(v) = 6$. Then each neighbor of v is a 4^+-vertex and $f_3(v) \leq 4$ by Lemma 1 and Lemma 3(b). If $f_3(v) = 4$, then $ch'(v) \geq ch(v) - \max\{4 \times \frac{5}{4} + 2 \times \frac{1}{2}, 4 \times \frac{5}{4} + \frac{1}{2} + \frac{1}{4}\} = 0$ by R2, R3 and R4. Otherwise, $ch'(v) \geq ch(v) - (f_3(v) \times \frac{5}{4} + (6 - f_3(v)) \times \frac{2}{3}) = \frac{24 - 7f_3(v)}{12} > 0$ by R2 and R3.

Suppose that $d(v) = 7$. Then each neighbor of v is a 3^+-vertex and $f_3(v) \leq 5$ by Lemma 1 and Lemma 3(b). We use $f_3^*(v)$ to denote the number of 3-faces incident with v, each of which is incident with a 3-vertex. Then $f_3^*(v) \leq 2$ by Lemma 1. Suppose that $f_3^*(v) = 0$, that is to say, each 3-face incident with v is only incident with 4^+-vertices. If $f_3(v) = 5$, Then $ch'(v) \geq ch(v) - 5 \times \frac{5}{4} - 2 \times \frac{1}{2} = \frac{3}{4} > 0$ by R2 and R3. Otherwise, $ch'(v) \geq ch(v) - (f_3(v) \times \frac{5}{4} + (7 - f_3(v)) \times 1) = \frac{4 - f_3(v)}{4} \geq 0$ by R2 and R3. Suppose that $f_3^*(v) = 1$. Then v is adjacent to only one 3-vertex, and it follows that $ch'(v) \geq ch(v) - (\frac{3}{2} + (f_3(v) - 1) \times \frac{5}{4} + (7 - f_3(v)) \times \frac{3}{4}) = \frac{5 - f_3(v)}{2} \geq 0$ by R2 and R3. Suppose that $f_3^*(v) = 2$. If $f_3(v) \leq 4$, then $ch'(v) \geq ch(v) - (2 \times \frac{3}{2} + (f_3(v) - 2) \times \frac{5}{4} + (7 - f_3(v)) \times \frac{3}{4}) = \frac{9 - 2f_3(v)}{4} > 0$ by R2 and R3. Otherwise, v is incident with a face f such that $d(f) \geq 4$ and f is incident with at least four 4^+-vertices, or $d(f) = 3$ and all vertices incident with f are 6^+-vertices. So $ch'(v) \geq ch(v) - (2 \times \frac{3}{2} + \max\{3 \times \frac{5}{4} + \frac{3}{4} + \frac{1}{2}, 2 \times \frac{5}{4} + 1 + 2 \times \frac{3}{4}\}) = 0$ by R2 and R3.

Let v be a 8-vertex. Then v is adjacent to at most two 2-vertices by Lemma 1 and $f_3(v) \leq 6$ by Lemma 3(b).

Case 1. v is not adjacent to any 2-vertex.

Suppose $f_3(v) \leq 4$. Then $ch'(v) \geq ch(v) - (f_3(v) \times \frac{3}{2} + (8 - f_3(v)) \times 1) = \frac{4-f_3(v)}{2} \geq 0$ by R2 and R3. Suppose $f_3(v) = 5$. If $f_4(v) = 3$, then $ch'(v) \geq ch(v) - max\{5\times\frac{3}{2}+1+\frac{2}{3}+\frac{1}{2}, 5\times\frac{3}{2}+3\times\frac{2}{3}\} = \frac{1}{3} > 0$ by R2 and R3. If $f_4(v) \leq 2$, then $ch'(v) \geq ch(v) - (5\times\frac{3}{2}+f_4(v)\times 1+(8-5-f_4(v))\times\frac{1}{3}) = \frac{9-4f_4(v)}{6} > 0$ by R2, R3 and R4. Suppose $f_3(v) = 6$, then $f_5(v) \leq 2$ and it follows that $ch'(v) \geq ch(v) - (6 \times \frac{3}{2} + 2 \times \frac{1}{3}) = \frac{1}{3} > 0$ by R2 and R4.

Case 2. v is adjacent to exactly one 2-vertex, say u.

Subcase 2.1. uv is incident with a 3-face.

Suppose that each neighbor of v except u and the 8-neighbor is a 4^+-vertex, i.e., $n_3(v) = 0$. If $f_3(v) = 6$, then $ch'(v) \geq ch(v)-1-(\frac{3}{2}+5\times\frac{5}{4}+\frac{1}{3}) = \frac{11}{12} > 0$ by R2 and R4. Otherwise, $ch'(v) \geq ch(v) - 1 - (\frac{3}{2} + (f_3(v) - 1) \times \frac{5}{4} + (8 - f_3(v)) \times \frac{3}{4}) = \frac{11-2f_3(v)}{4} > 0$ by R2 and R3.

Let $n_3(v) \geq 1$. Suppose $f_3(v) = 6$. Then v is incident with two 5^+-faces and is adjacent to at most three 3-vertices, that is, $n_3(v) \leq 3$ by Lemma 2. If $n_3(v) = 3$, then v is incident with a 3-face incident with all 6^+-vertices, and it follows that $ch'(v) \geq ch(v)-1-(4\times\frac{3}{2}+\frac{5}{4}+1+\frac{1}{3}) = \frac{5}{12} > 0$ by R2, R3 and R4. Otherwise, $ch'(v) \geq ch(v)-1-(3\times\frac{3}{2}+3\times\frac{5}{4}+2\times\frac{1}{3}) = \frac{1}{12} > 0$ by R2 and R4.

Suppose $f_3(v) = 5$. If $f_4(v) = 3$, then $ch'(v) \geq ch(v) - 1 - (\frac{3}{2} + 4 \times \frac{5}{4} + \frac{3}{4} + 2 \times \frac{1}{2}) = \frac{3}{4} > 0$ by R2 and R3. If $f_4(v) = 2$, then $n_3(v) \leq 3$. Assume $n_3(v) = 3$, then $ch'(v) \geq ch(v)-1-(4\times\frac{3}{2}+\frac{5}{4}+1+\frac{1}{2}) = \frac{1}{4} > 0$ by R2 and R3. Assume $n_3(v) = 2$, then $ch'(v) \geq ch(v)-1-max\{3\times\frac{3}{2}+2\times\frac{5}{4}+1+\frac{3}{4}+\frac{1}{4}, 3\times\frac{3}{2}+2\times\frac{5}{4}+\frac{3}{4}+\frac{2}{3}+\frac{1}{3}, 3\times\frac{3}{2}+2\times\frac{5}{4}+\frac{3}{4}+\frac{1}{2}+\frac{1}{3}, 3\times\frac{3}{2}+2\times\frac{5}{4}+1+\frac{1}{2}\} = 0$ by R2, R3 and R4. Otherwise, $ch'(v) \geq ch(v)-1-(2\times\frac{3}{2}+3\times\frac{5}{4}+2\times\frac{3}{4}+\frac{1}{3}) = \frac{5}{12} > 0$ by R2, R3 and R4. If $f_4(v) = 1$, then $n_3(v) \leq 3$. Assume $n_3(v) = 3$, then $ch'(v) \geq ch(v) - 1 - (4 \times \frac{3}{2} + \frac{5}{4} + \frac{3}{4} + 2 \times \frac{1}{3}) = \frac{1}{3} > 0$ by R2, R3 and R4. Assume $n_3(v) = 2$, then $ch'(v) \geq ch(v) - 1 - max\{3 \times \frac{3}{2} + 2 \times \frac{5}{4} + \frac{3}{4} + \frac{1}{3} + \frac{1}{4}, 3\times\frac{3}{2}+2\times\frac{5}{4}+\frac{3}{4}+2\times\frac{1}{3}\} = \frac{7}{12} > 0$ by R2, R3 and R4. Assume $n_3(v) = 1$, then $ch'(v) \geq ch(v) - 1 - (2 \times \frac{3}{2} + 3 \times \frac{5}{4} + \frac{3}{4} + 2 \times \frac{1}{3}) = \frac{5}{6} > 0$ by R2, R3 and R4. If $f_4(v) = 0$, then $ch'(v) \geq ch(v) - 1 - (5 \times \frac{3}{2} + 3 \times \frac{1}{3}) = \frac{1}{2} > 0$ by R2 and R4.

Suppose $f_3(v) = 4$. If $f_4(v) = 3$, then $n_3(v) \leq 3$. Assume $n_3(v) = 3$, then $ch'(v) \geq ch(v)-1-max\{4\times\frac{3}{2}+1+2\times\frac{2}{3}+\frac{1}{3}, 4\times\frac{3}{2}+1+2\times\frac{2}{3}+\frac{1}{4}, 4\times\frac{3}{2}+3\times\frac{2}{3}+\frac{1}{3}, 3\times\frac{3}{2}+\frac{5}{4}+1+\frac{3}{4}+\frac{2}{3}+\frac{1}{3}, 3\times\frac{3}{2}+\frac{5}{4}+2\times\frac{3}{4}+\frac{2}{3}+\frac{1}{3}\} = \frac{1}{3} > 0$ by R2, R3 and R4. Assume $n_3(v) = 2$, then $ch'(v) \geq ch(v) - 1 - max\{3 \times \frac{3}{2} + \frac{5}{4} + 1 + \frac{3}{4} + \frac{2}{3} + \frac{1}{4}, 3 \times \frac{3}{2} + \frac{5}{4} + 2 \times \frac{2}{3} + \frac{3}{4} + \frac{1}{4}, 2 \times \frac{3}{2} + 2 \times \frac{5}{4} + 2 \times \frac{3}{4} + 1 + \frac{1}{4}, 2 \times \frac{3}{2} + 2 \times \frac{5}{4} + 3 \times \frac{3}{4} + \frac{1}{3}\} = \frac{7}{12} > 0$ by R2, R3 and R4. Assume $n_3(v) = 1$, then

$ch'(v) \geq ch(v)-1-(2\times\frac{3}{2}+2\times\frac{5}{4}+4\times\frac{3}{4}) = \frac{1}{2} > 0$ by R2 and R3. If $f_4(v) \leq 2$, then $ch'(v) \geq ch(v)-1-(4\times\frac{3}{2}+f_4(v)\times 1+(8-4-f_4(v))\times\frac{1}{3}) = \frac{5-2f_4(v)}{3} > 0$ by R2, R3 and R4.

Suppose $f_3(v) = 3$. If $f_4(v) = 5$, then $n_3(v) \leq 4$. Assume $n_3(v) = 4$, then $ch'(v) \geq ch(v)-1-max\{3\times\frac{3}{2}+2\times 1+2\times\frac{2}{3}+\frac{3}{4}, 3\times\frac{3}{2}+2\times 1+3\times\frac{2}{3}\} = \frac{5}{12} > 0$ by R2 and R3. Assume $n_3(v) = 3$, then $ch'(v) \geq ch(v)-1-max\{3\times\frac{3}{2}+1+2\times\frac{3}{4}+2\times\frac{2}{3}, 3\times\frac{3}{2}+2\times\frac{3}{4}+3\times\frac{2}{3}, 2\times\frac{3}{2}+\frac{5}{4}+1+3\times\frac{3}{4}+\frac{2}{3}\} = \frac{2}{3} > 0$ by R2 and R3. Assume $n_3(v) \leq 2$, then $ch'(v) \geq ch(v) - 1 - (3\times\frac{3}{2} + 1 + 4\times\frac{3}{4}) = \frac{1}{2} > 0$ by R2 and R3. If $f_4(v) \leq 4$, then it follows that $ch'(v) \geq ch(v) - 1 - (3\times\frac{3}{2} + f_4(v)\times 1 + (8-3-f_4(v))\times\frac{1}{3}) = \frac{17-4f_4(v)}{6} > 0$ by R2, R3 and R4.

Suppose $f_3(v) \leq 2$, then $ch'(v) \geq ch(v)-1-(f_3(v)\times\frac{3}{2}+(8-f_3(v))\times 1) = \frac{2-f_3(v)}{2} \geq 0$ by R2 and R3.

Subcase 2.2. Two faces incident with uv are 4^+-faces.

Note that $f_3(v) \leq 4$ by Lemmas 2 and 3. Suppose $f_3(v) = 4$. If $f_4(v) = 4$, then $ch'(v) \geq ch(v)-1-max\{\frac{3}{2}+3\times\frac{5}{4}+3\times\frac{3}{4}+\frac{1}{2}, 4\times\frac{5}{4}+1+2\times\frac{3}{4}+\frac{1}{2}\} > 0$ by R2 and R3. If $f_4(v) = 3$, then $ch'(v) \geq ch(v)-1-max\{2\times\frac{3}{2}+2\times\frac{5}{4}+3\times 1+\frac{1}{4}, 2\times\frac{3}{2}+2\times\frac{5}{4}+2\times 1+\frac{3}{4}+\frac{1}{3}, 2\times\frac{3}{2}+2\times\frac{5}{4}+2\times 1+\frac{1}{2}+\frac{1}{3}, 2\times\frac{3}{2}+2\times\frac{5}{4}+1+2\times\frac{3}{4}+\frac{1}{3}\} = \frac{1}{4} > 0$ by R2, R3 and R4. If $f_4(v) \leq 2$, then $ch'(v) \geq ch(v)-1-(4\times\frac{3}{2}+f_4(v)\times 1+(8-4-f_4(v))\times\frac{1}{3}) = \frac{5-2f_4(v)}{3} > 0$ by R2, R3 and R4.

Suppose $f_3(v) = 3$. If $f_4(v) = 5$, then $ch'(v) \geq ch(v)-1-max\{3\times\frac{3}{2}+3\times 1+2\times\frac{2}{3}, 3\times\frac{3}{2}+2\times 1+3\times\frac{2}{3}, 2\times\frac{3}{2}+\frac{5}{4}+3\times 1+\frac{3}{4}+\frac{1}{2}, 2\times\frac{3}{2}+\frac{5}{4}+3\times 1+2\times\frac{3}{4}, 2\times\frac{3}{2}+\frac{5}{4}+3\times 1+\frac{3}{4}+\frac{2}{3}\} = \frac{1}{6} > 0$ by R2 and R3. If $f_4(v) \leq 4$, then $ch'(v) \geq ch(v)-1-(3\times\frac{3}{2}+f_4(v)\times 1+(8-3-f_4(v))\times\frac{1}{3}) = \frac{17-4f_4(v)}{6} > 0$ by R2, R3 and R4.

Suppose $f_3(v) \leq 2$. Then $ch'(v) \geq ch(v)-1-(f_3(v)\times\frac{3}{2}+(8-f_3(v))\times 1) = \frac{2-f_3(v)}{2} \geq 0$ by R2 and R3.

Case 3. v is adjacent to two 2-vertices.

Then $f_3(v) \leq 4$ and if u is a neighbor of v such that uv is incident with a 3-face, then $d(u) \geq 4$ by Lemma 2(3).

Suppose $f_3(v) = 4$. If $f_4(v) = 4$, then $ch'(v) \geq ch(v)-2-(4\times\frac{5}{4}+4\times\frac{3}{4}) = 0$ by R2 and R3. If $f_4(v) = 3$, then $ch'(v) \geq ch(v)-2-max\{4\times\frac{5}{4}+1+2\times\frac{3}{4}+\frac{1}{4}, 4\times\frac{5}{4}+1+\frac{3}{4}+\frac{1}{2}+\frac{1}{3}\} = \frac{1}{4} > 0$ by R2, R3 and R4. If $f_4(v) \leq 2$, then $ch'(v) \geq ch(v)-2-(4\times\frac{5}{4}+f_4(v)\times 1+(8-4-f_4(v))\times\frac{1}{3}) = \frac{5-2f_4(v)}{3} > 0$ by R2, R3 and R4.

Suppose $f_3(v) = 3$. If $f_4(v) = 5$, then $ch'(v) \geq ch(v) - 2 - max\{3 \times \frac{5}{4} + 2 \times 1 + 2 \times \frac{3}{4} + \frac{1}{2}, 3 \times \frac{5}{4} + 1 + 3 \times \frac{3}{4} + \frac{1}{3}, 3 \times \frac{5}{4} + 4 \times \frac{3}{4} + \frac{2}{3}, 3 \times \frac{5}{4} + 1 + 2 \times \frac{3}{4} + \frac{2}{3} + \frac{1}{2}, 3 \times \frac{5}{4} + 1 + 2 \times \frac{3}{4} + 2 \times \frac{2}{3}\} = \frac{1}{4} > 0$ by R2, R3 and R4. If $f_4(v) = 4$, then $ch'(v) \geq ch(v) - 2 - max\{3 \times \frac{5}{4} + 1 + 3 \times \frac{3}{4} + \frac{1}{3}, 3 \times \frac{5}{4} + 1 + 2 \times \frac{3}{4} + \frac{1}{2} + \frac{1}{4}\} = \frac{2}{3} > 0$ by R2, R3 and R4. If $f_4(v) \leq 3$, then $ch'(v) \geq ch(v) - 2 - (3 \times \frac{5}{4} + f_4(v) \times 1 + (8 - 3 - f_4(v)) \times \frac{1}{3}) = \frac{31 - 8f_4(v)}{12} > 0$ by R2, R3 and R4.

Suppose $f_3(v) = 2$. If $f_4(v) = 6$, then $ch'(v) \geq ch(v) - 2 - max\{2 \times \frac{5}{4} + 4 \times 1 + 2 \times \frac{3}{4}, 2 \times \frac{5}{4} + 3 \times 1 + 2 \times \frac{3}{4} + \frac{2}{3}, 2 \times \frac{5}{4} + 2 \times 1 + 4 \times \frac{3}{4}, 2 \times \frac{5}{4} + 1 + 4 \times \frac{3}{4} + \frac{2}{3}, 2 \times \frac{5}{4} + 1 + 2 \times \frac{3}{4} + 3 \times \frac{2}{3}\} = 0$ by R2 and R3. If $f_4(v) \leq 5$, then $ch'(v) \geq ch(v) - 2 - (2 \times \frac{5}{4} + f_4(v) \times 1 + (8 - 2 - f_4(v)) \times \frac{1}{3}) = \frac{21 - 4f_4(v)}{6} > 0$ by R2, R3 and R4.

Suppose $f_3(v) = 1$. If $f_4(v) = 7$, then $ch'(v) \geq ch(v) - 2 - max\{\frac{5}{4} + 5 \times 1 + 2 \times \frac{3}{4}, \frac{5}{4} + 3 \times 1 + 4 \times \frac{3}{4}, \frac{5}{4} + 2 \times 1 + 4 \times \frac{3}{4} + \frac{2}{3}, \frac{5}{4} + 1 + 6 \times \frac{3}{4}\} = \frac{1}{4} > 0$ by R2 and R3. If $f_4(v) \leq 6$, then $ch'(v) \geq ch(v) - 2 - (\frac{5}{4} + f_4(v) \times 1 + (8 - 1 - f_4(v)) \times \frac{1}{3}) = \frac{53 - 8f_4(v)}{12} > 0$ by R2, R3 and R4.

Suppose $f_3(v) = 0$. Then $ch'(v) \geq ch(v) - 2 - 8 \times 1 = 0$ by R3.

Hence we complete the proof of (*), that is, Theorem 1 is true.

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