ON REMOVABLE SERIES CLASSES IN CRITICALLY CONNECTED BINARY MATROIDS

Y. M. BORSE

ABSTRACT. Let M be a simple connected binary matroid with corank at least two such that M has no connected hyperplane. Seymour proved that M has a non-trivial series class. We improve this result by proving that M has at least two disjoint non-trivial series classes L_1 and L_2 such that both $M\backslash L_1$ and $M\backslash L_2$ are connected. Our result extends the corresponding result of Kriesell regarding critically 2-connected graphs.

Keywords: binary matroid, connected hyperplane, series class

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1. INTRODUCTION

All graphs and matroids considered here are supposed to be simple. Given a connected graph G with n vertices, let $r^*(G) = |E(G)| - n + 1$, where $E(G)$ is the set of edges of G. An *ear* P in a graph G is a maximal path all of whose internal vertices have degree two in G. For an ear P of a graph G, let $G - P$ be the graph obtained from G by deleting edges and internal vertices of P. For a matroid M, let $E(M)$, $r(M)$ and $r^*(M)$, respectively denote the ground set, the rank and the corank of M. A matroid M is *cosimple* if it has no cocircuit of size less than 3. A matroid M is *connected* if every two distinct elements of M are contained in some common circuit. A *series class* of M is a maximal subset L of $E(M)$ such that any two elements of L form a 2-cocircuit of M. Thus a series class is the matroid extension of an ear of a graph. A series class L is *non-trivial* if $|L| \geq 2$. For notations and definitions, we follow Oxley [6].

A 2-connected graph G is *critically* (resp. *minimally*) 2-connected if $G - x$ is not 2-connected for every vertex (resp. edge) x of G. A connected matroid M is *minimally connected* if $M \setminus e$ is not connected for every $e \in E(M)$. A cocircuit C^* of a connected matroid is *non-separating* if $M\backslash C^*$ is connected. A connected matroid M is *critically connected* if it does not have a non-separating cocircuit. Thus a critically connected matroid has no connected hyperplane. It follows that the cycle matroid of a critically

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(resp. minimally) 2-connected simple graph is critically (resp. minimally) connected.

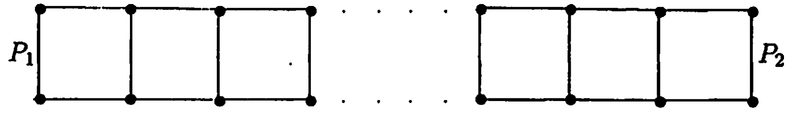
It follows from Oxley ([6], Exercise 2, pp. 338) that a minimally connected matroid M with $r^*(M) \geq 2$ has at least $r^*(M)+1$ non-trivial series classes P such that $M \setminus P$ is connected; see also [1]. This result extends the corresponding result for minimally 2-connected graphs due to Zhang and Guo [8]. Kriesell [3] proved that a critically 2-connected graph different from a cycle has at least two non-trivial removable ears.

Theorem 1.1 ([3]). *Let G be a critically 2-connected graph with $r^*(G) \geq 2$. Then G has at least two non-trivial ears P_1, P_2 such that $G - P_i$ is 2-connected for $i = 1, 2$.*

Kriesell used this result to prove a structure theorem ([3], Theorem 1) for the class of critically 2-connected graphs. Theorem 1.1 also improves a result of Nebeský [5] which states that a critically 2-connected graph different from a cycle has a non-trivial ear. Kelmans [2] and, independently, Seymour (*see* [7]) extended Nebeský's result to binary matroids by proving that every simple and cosimple binary matroid has a non-separating cocircuit. In other words, they proved that a connected binary matroid with corank at least two has a non-trivial series class. We improve this result and extend Theorem 1.1 to binary matroids as follows.

Theorem 1.2. *Let M be a critically connected binary matroid with $r^*(M) \geq 2$. Then M has at least two non-trivial series classes L_1, L_2 such that $M \setminus L_i$ is connected for $i = 1, 2$.*

The condition of binary matroid in Theorem 1.2 is necessary because a uniform matroid $U_{r,n}$ with $n - 1 > r > 2$ is critically connected but has no non-trivial series class. Further, we show that the lower bound in Theorem 1.2 is best possible. Let P be a path with at least 4 vertices. Let G be the cartesian product of P with the complete graph K_2. Then G is a ladder graph obtained from two copies of P by joining the corresponding vertices of these two paths by a 1-factor as shown in the following figure. Obviously, G is a critically 2-connected graph with exactly two non-trivial ears P_1 and P_2. Let $M(G)$ be the cycle matroid of G and let L_i be the series class of $M(G)$ corresponding to the ear P_i of G for $i = 1, 2$. Then $M(G)$ is a critically connected matroid with corank at least two. Further, L_1 and L_2 are the only non-trivial series classes in $M(G)$ such that $M(G) \setminus L_i$ is connected for $i = 1, 2$.



2. Proof

In this section, we prove Theorem 1.2. The following two results are well known.

Lemma 2.1 ([6], pp. 304). *Let M be a binary matroid. Then*
(i) $|C \cap C^*|$ is even for any circuit C and cocircuit C^* of M;
(ii) for any two distinct circuits C_1 and C_2, $C_1 \Delta C_2$ is union of disjoint circuits of M.

Lemma 2.2 ([4]). *Let M be a connected simple and cosimple binary matroid. Then, for every element e,*
(i) there are at least two non-separating cocircuits containing e;
(ii) there are at least two non-separating circuits avoiding e.

Lemma 2.3. *Let M be a matroid and let A be a nonempty subset of $E(M)$ such that $|C \cap A| \neq 1$ for each circuit C of M. Then A contains a cocircuit of M.*

Proof. Assume that A dose not contain a cocircuit of M. Then A is independent in M^* and can be extended to a basis B^* of M^*. Let $B = E(M) - B^*$. Then B is a basis of M. Let $a \in A$. Then $B \cup \{a\}$ contains a circuit C with $a \in C$. Obviously, $C - \{a\} \subset E(M) - B^* \subset E(M) - A$. Thus $C \cap A = \{a\}$, a contradiction. □

Lemma 2.4. *Let M be a critically connected binary matroid with $r^*(M) = 2$. Then there are at least two non-trivial series classes L_1 and L_2 such that $M \backslash L_i$ is connected for $i = 1, 2$.*

Proof. Consider the dual matroid M^* of M. It is connected, binary and has rank 2. Let N be the simplified matroid of M^*. Then N is a triangle. Let $E(N) = \{e_1, e_2, e_3\}$. Let P_i be the parallel class of M^* corresponding to the element e_i of N for $i = 1, 2, 3$. Then P_1, P_2 and P_3 are series classes of M. Since N/e_i is connected, M^*/P_i is connected and hence $(M^*/P_i)^* = M \backslash P_i$ is also connected for $i = 1, 2, 3$. As M is simple, M^* is cosimple. Therefore at least two of the three parallel classes of M^* are non-trivial. Thus M has two non-trivial series classes such that deletion of any one of them leaves a connected matroid. □

Lemma 2.5. *Let M be a critically connected binary matroid with $r^*(M) > 2$ and let L be a non-trivial series class in M such that $M \backslash L$ is not connected. Suppose B is a component of $M \backslash L$ and C is a circuit in M with $C \cap L \neq \phi$. Then the restriction of M to the set $B \cup C$ is a critically connected matroid.*

Proof. Since L is a series class of M, $L \subset C$. Therefore C intersects each component of $M \backslash L$. Let N be the restriction of M to the set $B \cup C$. Then

N is a connected matroid. Suppose N is not critically connected. Then N has a cocircuit Q such that $N \backslash Q$ is connected. Let $P = C - B$. Then $L \subset P$. As C intersects a component of $M \backslash L$ other than B and $|L| \geq 2$, $|P| \geq 3$. It is easy to see that P is a series class of N. Hence P is not a subset of Q. If $P \cap Q \neq \phi$, then each element of $P - Q$ is a coloop of $N \backslash Q$, a contradiction. Thus $P \cap Q = \phi$.

Let Z be a circuit in $N \backslash Q$ containing an element of P. Then $P \subset Z$ and hence $L \subset Z$. Therefore Z intersects all components of $M \backslash L$. This implies that $M \backslash Q$ is connected. Since M is critically connected, it has no non-separating cocircuit. Hence Q is not a cocircuit of M. Further, Q does not contain any cocircuit of M because it is already a cocircuit of N. By Lemma 2.3, M has a circuit C_1 with $|C_1 \cap Q| = 1$. By Lemma 2.1, C_1 is not a circuit of N. Therefore $L \cap C_1 \neq \phi$ and $C_1 \neq C$. As L is a series class of M, $L \subset C_1$. By Lemma 2.1, the circuit C of N contains an even number of elements of Q. Hence $C \Delta C_1$ contains an odd number of elements of Q. These elements of Q are distributed in disjoint circuits in B that are contained in $C \Delta C_1$. Hence one of such circuits in B receives an odd number of elements of Q, which is a contradiction by Lemma 2.1. □

Lemma 2.6. *Let M be a critically connected binary matroid with $r^*(M) \geq 2$ and let L be a non-trivial series class in M such that $M \backslash L$ is connected. Suppose $L \cup x$ is circuit for some element x of M. Then $M \backslash x$ is critically connected.*

Proof. Let e_1 and e_2 be two elements of $M \backslash x$. Since M is connected, there is a circuit Z of M containing both e_1 and e_2. If $x \notin Z$, then Z is a circuit of $M \backslash x$. Suppose $x \in Z$. Then, by orthogonality, $Z = L \cup \{x\}$ or Z is disjoint from L. If $Z \cap L = \phi$, then $(Z - x) \cup L$ is a circuit in $M \backslash x$ containing both e_1 and e_2. Suppose $Z = L \cup \{x\}$. Then e_1 and e_2 belong to L. Since x is not in the series class L of M, there is a circuit C_x in M containing x that is disjoint from L. Clearly, $(C_x - x) \cup L$ is a circuit in $M \backslash x$. Thus there is a circuit in $M \backslash x$ containing both e_1 and e_2. Hence $M \backslash x$ is connected.

Suppose $M \backslash x$ has a non-separating cocircuit Q. Since L is a non-trivial series class, $|L| \geq 2$. If $|L| = 2$, then L becomes a non-separating 2-cocircuit of M, a contradiction to the fact that M is critically connected. Therefore $|L| \geq 3$. If Q intersects L, then $L - Q \neq \phi$ and the elements of $L - Q$ are coloops in $M \backslash Q$ and hence in $(M \backslash x) \backslash Q$, a contradiction. Thus $Q \cap L = \phi$. Since $(M \backslash x) \backslash Q$ is connected and $L \cup x$ is a circuit, $M \backslash Q$ is also connected. Hence Q is not a cocircuit of M because M is critically connected. As Q is a cocircuit of $M \backslash x$, no subset of Q can be a cocircuit of M. By Lemma 2.3, there is a circuit C in M which contains exactly one element of Q. By Lemma 2.1, C is not a circuit of $M \backslash x$. Hence $x \in C$. Therefore $(C - x) \cup L$ is a circuit of $M \backslash x$ containing only one element of Q, which is a contradiction by Lemma 2.1. Thus $M \backslash x$ is critically connected. □

Proof of Theorem 1.1. We prove the result by induction on $r^*(M)$. By Lemma 2.4, the result follows when $r^*(M) = 2$. Suppose $r^*(M) > 2$. Assume that the result holds for all critically connected matroids N with $2 \leq r^*(N) < r^*(M)$. By Lemma 2.2, M is not cosimple and hence it has a non-trivial series class. We prove the result by considering two cases.

Case (i). M has a non-trivial series class L such that $M \backslash L$ is not connected.

Then $M \backslash L$ has at least two components and further, each component is non-trivial. Let C be a circuit in M with $C \cap L \neq \phi$. Then $L \subset C$ and further, C intersects all components of $M \backslash L$. Let B be a component of $M \backslash L$ and let N be the restriction of M to the set $B \cup C$. By Lemma 2.5, N is critically connected. Also, $2 \leq r^*(N) < r^*(M)$. By induction hypothesis, N has at least two non-trivial series class L_1 and L_2 such that both $N \backslash L_1$ and $N \backslash L_2$ are connected. Note that $C - B$ is a series class of N containing L. We may assume that L_1 is disjoint from this series class. Therefore $L_1 \cap L = \phi$. It follows that L_1 is a series class of M. As $N \backslash L_1$ is connected, there is a circuit C_1 in $N \backslash L_1$ containing an element of L. Then C_1 is a circuit in M. Hence it intersects all components of $M \backslash L$. Therefore C_1 and all the components of $M \backslash L$ which are different from B are contained in a single component of $M \backslash L_1$. It implies that $M \backslash L_1$ is connected. Thus each component of $M \backslash L$ contains a non-trivial series class of M whose deletion from M leaves a connected matroid. Since $M \backslash L$ has at least two components, the result follows.

Case (ii). $M \backslash P$ is connected for every non-trivial series class P of M.

By Lemma 2.2, M has a non-trivial series class L. Therefore $M \backslash L$ is connected by assumption. We prove the result by finding a non-trivial series class L_1 of M that is disjoint from L by considering two subcases.

Subcase (i). $L \cup x$ is not a circuit for any element x of M.

Let M' be the matroid obtained from M by replacing L by an element $'a'$. Then M' is simple. Suppose M' is cosimple. By Lemma 2.2, there is a non-separating cocircuit Q in M' avoiding a. It follows that Q is a non-separating cocircuit of M, a contradiction. Hence M' is not cosimple and thus has a non-trivial series class L_1. Note that $a \notin L_1$. Clearly, L_1 is also a non-trivial series class of M which is disjoint from L.

Subcase (ii). $L \cup \{x\}$ is a circuit for some element x of M.

Obviously, $2 \leq r^*(M \backslash x) < r^*(M)$. Since $M \backslash L$ is connected, $M \backslash x$ is critically connected by Lemma 2.6. By induction, $M \backslash x$ has at least two non-trivial series classes L_1 and L_2 such that both $M \backslash x \backslash L_1$ and $M \backslash x \backslash L_2$ are connected. It is easy to see that L is contained in a series class of $M \backslash x$. Since L_1 and L_2 are disjoint, without loss of generality we may assume that L_1 is disjoint from L. Suppose L_1 is not a series class of M. Then we get two elements z_1 and z_2 in L_1 and a circuit Z in M such that $z_1 \in Z$

but $z_2 \notin Z$. Then Z is not a circuit of $M\backslash x$ and hence $x \in Z$. Further, $Z \neq L \cup \{x\}$ because L is disjoint from L_1. Therefore, by orthogonality, Z is disjoint from L. Hence $(Z - x) \cup L$ is a circuit of $M\backslash x$ containing z_1 and avoiding z_2, which is a contradiction to the fact that L_1 is a series class of $M\backslash x$. Therefore L_1 is a non-trivial series class of M also.

Thus, in both subcases, we have proved that M has two disjoint non-trivial series classes L and L_1. By assumption, both $M\backslash L$ and $M\backslash L_1$ are connected. □

REFERENCES

[1] Y. M. Borse and B. N. Waphare, On removable series classes in connected matroids, *Australas. J. Combin.* 40(2008), 247-252.

[2] A. K. Kelmans, The concept of a vertex in a matroid, the non-separating circuits and a new criterion for graph planarity. In: *Algebraic Methods for Graph Theory, vol. 1, Colloq. Math. Soc. János Bolyai (Szeged, Hungary, 1978)*, vol. 25, North-Holland, Amsterdam, 1981, pp. 345-388.

[3] M. Kriesell, Almost all 3-connected graphs contain a contractible set of k vertices, *J. Combin. Theory Ser. B* 83(2001), 305-319.

[4] J. McNulty and H. Wu, Connected hyperplanes in binary matroids, *J. Combin. Theory Ser. B* 79(2000), 87-97.

[5] L. Nebeský, On induced subgraphs of a block, *J. Graph Theory* 1(1977), 69-74.

[6] J. G. Oxley, *Matroid Theory*, Oxford University Press, New York, 1992.

[7] J. G. Oxley, Cocircuit coverings and packings for binary matroids, *Math. Proc. Cambridge Philos. Soc.* 83(1978), 347-351.

[8] Fuji Zhang and Xiaofeng Guo, Reducible chains in several types of 2-connected graphs, *Discrete Math.* 105(1992), 285-291.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF PUNE, PUNE 411 007, INDIA.
E-mail address: ymborse11@gmail.com