Optimal orientations of $P_3 \times K_5$ and $C_8 \times K_3$

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Abstract. For a graph G, let $\mathcal{D}(G)$ be the set of all strong orientations of G. *Orientation number* of G, denoted by $\vec{d}(G)$, is defined as $\min\{\, d(D) \,|\, D \in \mathcal{D}(G)\,\}$, where $d(D)$ denotes the diameter of the digraph D. In this paper, we prove that $\vec{d}(P_3 \times K_5) = 4$ and $\vec{d}(C_8 \times K_3) = 6$, where $\times$ is the tensor product of graphs.

1 Introduction

Let G be a simple undirected graph with vertex set $V(G)$ and edge set $E(G)$. For $v \in V(G)$, the *eccentricity*, denoted by $e_G(v)$, of v is defined as $e_G(v) = \max\,\{d_G(v,x) \,|\, x \in V(G)\,\}$, where $d_G(v,x)$ denotes the distance from v to x in G. The *diameter* of G, denoted by $d(G)$, is defined as $d(G) = \max\{\, e_G(v) \,|\, v \in V(G)\,\}$.

Let D be a digraph with vertex set $V(D)$ and arc set $A(D)$ which has neither loops nor multiple arcs (that is, arcs with same tail and same head). For $v \in V(D)$, the notions $e_D(v)$ and $d(D)$ are defined as in the undirected graph. For $x, y \in V(D)$, we write $x \rightarrow y$ or $y \leftarrow x$ if $(x,y) \in A(D)$. For sets $X, Y \subseteq V(D)$, $X \rightarrow Y$ denotes $\{(x,y) \in A(D) : x \in X$ and $y \in Y\}$. For distinct vertices $v_1, v_2, \ldots, v_k$, $v_1 \rightarrow v_2 \rightarrow \ldots \rightarrow v_k$ represents the directed path in D with arcs $v_1 \rightarrow v_2, v_2 \rightarrow v_3, \ldots, v_{k-1} \rightarrow v_k$. For subsets $V_1, V_2, \ldots, V_k$ of V, we write $V_1 \rightarrow V_2 \rightarrow \ldots \rightarrow V_k$ for the set of all directed paths of length $k-1$ whose ith vertex is in V_i, $1 \leq i \leq k$.

For graphs G and H, the *tensor product*, $G \times H$, of G and H is the graph with vertex set $V(G) \times V(H)$ and $E(G \times H) = \{\, (u,v)(x,y) : ux \in$

$E(G)$ and $vy \in E(H)$ }. For $x \in V(G)$, the *H-layer* H_x is the subset $\{(x,y) : y \in V(H)\}$ of vertices of $G \times H$, and similarly, for $y \in V(H)$, the *G-layer* G_y of $G \times H$ is $\{(x,y) : x \in V(G)\}$.

An *orientation* of a graph G is a digraph D obtained from G by assigning a direction to each of its edges. By abuse of notation, by D we mean an orientation of G and also the digraph arising out of the orientation of G.

A vertex v is *reachable* from a vertex u of a digraph D if there is a directed path in D from u to v. An orientation D of G is *strong* if any pair of vertices in D are mutually reachable in D. Robbins' celebrated one-way street theorem [5] states that a connected graph G has a strong orientation if and only if G is 2-edge-connected. For a 2-edge-connected graph G, let $\mathcal{D}(G)$ denote the set of all strong orientations of G. The *orientation number* of G, denoted by $\vec{d}(G)$, is defined as $\min\{ d(D) \mid D \in \mathcal{D}(G) \}$. In [2], $\vec{d}(G) - d(G)$ is defined as $\rho(G)$. Any orientation D in $\mathcal{D}(G)$ with $d(D) = \vec{d}(G)$ is called an *optimal orientation* of G. For results on orientations of graphs, see a survey by Koh and Tay [2].

Let P_n, C_n and K_n denote the path, cycle and complete graph of order n, respectively. Notations and terminology not defined here can be seen in [1].

Except for few pairs (r,s), we have evaluated $\rho(G \times H)$ for combinations of graphs including $K_r \times K_s$, $P_r \times K_s$ and $C_r \times K_s$, see [3] and [4]. We have proved, in [4], that $\rho(P_3 \times K_5) \leq 1$ and $\rho(C_8 \times K_3) \leq 2$. In this paper, we prove $\rho(P_3 \times K_5) = 1$ and $\rho(C_8 \times K_3) = 2$.

2 Proof

First we show that $\rho(P_3 \times K_5) = 1$, a particular case left out in [4].

Theorem 2.1. $\rho(P_3 \times K_5) = 1$.

Proof. Let $V(P_3) = \{0,1,2\}$, $E(P_3) = \{\{0,1\},\{1,2\}\}$ and $V(K_5) = \{0,1,2,3,4\}$. As $d(P_3 \times K_5) = 3$ and $\vec{d}(P_3 \times K_5) \leq 4$, see [4], to complete the proof, it is enough to show that $\vec{d}(P_3 \times K_5) \neq 3$. If possible assume that there is an orientation D of $P_3 \times K_5$ so that $d(D) = 3$.
Claim. For $i \in \{0,2\}$ and $j \in \{0,1,2,3,4\}$, $d_D^+((i,j)) = 2 = d_D^-((i,j))$.

By the nature of the graph $P_3 \times K_5$, it is enough to verify this claim for the vertex $(0,0)$. If $d_D^+((0,0)) = 1$, then, by symmetry, we assume that

$N^+_D((0,0)) = \{(1,1)\}$. Now $d_D((0,0),(2,1)) > 3$, a contradiction. Hence $d^+_D((0,0)) \neq 1$. Similarly, $d^-_D((0,0)) \neq 1$ (can be obtained by considering the converse digraph of D) and therefore $d^+_D((0,0)) = 2 = d^-_D((0,0))$.

By symmetry, we assume that $N^+_D((0,0)) = \{(1,1),(1,2)\}$ and $N^-_D((0,0)) = \{(1,3),(1,4)\}$. As $d(D) = 3$, $d_D((0,0),(0,j)) = 2$, for every $j \in \{1,2,3,4\}$ and $d_D((0,0),(2,j)) = 2$, for every $j \in \{0,1,2,3,4\}$. Consequently, $(1,1) \to \{(0,2),(2,2)\}$, $(1,2) \to \{(0,1),(2,1)\}$ and either $(1,1) \to (2,0)$ or $(1,2) \to (2,0)$; by symmetry, we assume that $(1,1) \to (2,0)$. Again, since $d_D((0,j),(0,0)) = 2$, for every $j \in \{1,2,3,4\}$ and $d_D((2,j),(0,0)) = 2$, for every $j \in \{0,1,2,3,4\}$, we have $(0,4) \to (1,3)$, $(0,3) \to (1,4)$, $(2,4) \to (1,3)$ and $(2,3) \to (1,4)$. As $d^-_D((2,0)) = 2$, we have to consider three cases: $(1,2) \to (2,0)$, $(1,3) \to (2,0)$ and $(1,4) \to (2,0)$. But the cases $(1,3) \to (2,0)$ and $(1,4) \to (2,0)$ are similar.

Case 1. $(1,2) \to (2,0)$.

As $(2,0) \leftarrow \{(1,1),(1,2)\}$, $(2,0) \to \{(1,3),(1,4)\}$. Now $d_D((2,0),(0,4)) \neq 2$, a contradiction.

Case 2. $(1,3) \to (2,0)$.

As $(2,0) \leftarrow \{(1,1),(1,3)\}$, $(2,0) \to \{(1,2),(1,4)\}$. Since $d_D((2,0),(0,j)) = 2$, $j \in \{2,4\}$, $(1,2) \to (0,4)$ and $(1,4) \to (0,2)$. As $d^+_D((0,2)) = 2$, $(0,2) \to \{(1,0),(1,3)\}$. Since $d_D((0,1),(2,0)) = 2$, $(0,1) \to (1,3)$. Now $d_D((0,1),(0,4)) = 2$ implies that $(0,1) \to (1,0) \to (0,4)$. Finally, $d_D((0,4),(0,1)) \neq 2$, a contradiction. ∎

Next, we shall show that $\rho(C_8 \times K_3) = 2$, again a particular case left out in [4].

Theorem 2.2. $\rho(C_8 \times K_3) = 2$.

Proof. Let $V(K_3) = \{0,1,2\}$, $V(C_8) = \{0,1,2,3,4,5,6,7\}$ and $E(C_8) = \{\{i,(i+1)(mod\ 8)\} : i \in V(C_8)\}$. As $d(C_8 \times K_3) = 4$ and $\vec{d}(C_8 \times K_3) \leq 6$, see [4], to complete the proof, it is enough to show that $\vec{d}(C_8 \times K_3) \notin \{4,5\}$. If possible assume that there is an orientation D of $C_8 \times K_3$ so that $d(D) \leq 5$. Note that $C_8 \times K_3$ is a bipartite graph with bipartition $X = \{0,2,4,6\} \times \{0,1,2\}$ and $Y = \{1,3,5,7\} \times \{0,1,2\}$. Hence, for $x_1, x_2 \in X$ and $y_1, y_2 \in Y$, $d_D((x_1,x_2))$, $d_D((y_1,y_2)) \in \{2,4\}$ and $d_D((x_1,y_1)) \in \{1,3,5\}$.

Claim 1. For $(i,j) \in V(C_8 \times K_3)$, $d^+_D((i,j)) = 2 = d^-_D((i,j))$.

It is not difficult to see that $C_8 \times K_3$ is vertex-transitive; hence it is enough to verify Claim 1 for the vertex $(0,0)$. Suppose $d^+_D((0,0)) =$

1 and, by symmetry, we assume that $(1,1)$ is the unique out-neighbour of $(0,0)$. Now $d_D((0,0),(6,j)) = 4$, for every $j \in V(K_3)$. $d_D((0,0),(6,0)) = 4$ implies that $(1,1) \to (0,2) \to (7,1) \to (6,0)$. $d_D((0,0),(6,1)) = 4$ implies that $(0,2) \to (7,0) \to (6,1)$.

If $(7,1) \to (6,2)$, then $d_D((5,0),(7,1)) \geq 6$, a contradiction. Hence, $(6,2) \to (7,1)$. $d_D((0,0),(6,2)) = 4$ implies that $(7,0) \to (6,2)$.

Suppose $(1,0) \to (0,2)$. Then $d_D((0,2),(2,1)) = 4$ implies that $(7,0) \to (0,1)$, and hence $d_D((5,0),(7,0)) \geq 6$, again a contradiction. Thus we conclude that $(0,2) \to (1,0)$.

$d_D((6,0),(0,2)) = 4$ implies that $(6,0) \to (7,2)$. $d_D((6,1),(0,2)) = 4$ implies that $(6,1) \to (7,2)$. $d_D((7,2),(5,1)) = 4$ implies that $(7,2) \to (0,1) \to (7,0)$ and $(6,2) \to (5,1)$. $d_D((7,2),(5,2)) = 4$ implies that $(6,1) \to (5,2)$. $d_D((7,0),(1,0)) = 4$ implies that $(0,1) \to (1,0)$. $d_D((7,0),(1,2)) = 4$ implies that $(0,1) \to (1,2)$. Now $d_D((2,1),(0,1)) \geq 6$, a contradiction. This completes the proof of Claim 1.

Claim 2. There exists a vertex (i,j) such that two out-neighbours of (i,j) are in the same K_3-layer.

Suppose all vertices have their two out-neighbours in different K_3-layers. By Claim 1, there exist exactly two paths $(0,0) \to (1,i_1) \to (2,i_2) \to (3,i_3) \to (4,i_4)$ and $(0,0) \to (7,j_1) \to (6,j_2) \to (5,j_3) \to (4,j_4)$ of length 4 from the vertex $(0,0)$ to vertices in $\{(4,0),(4,1),(4,2)\}$, where $i_1, i_2, i_3, i_4, j_1, j_2, j_3, j_4 \in \{0,1,2\}$. If $k \in \{0,1,2\} \backslash \{i_4, j_4\}$, then $d_D((0,0),(4,k)) \geq 6$, a contradiction.

Claim 3. Let $(i,j) \in V(C_8 \times K_3)$.
(i) If $(i,j) \to \{(i+1,j'),(i+1,j'')\}$ for $0 \leq j' < j'' \leq 2$ and $j', j'' \neq j$, then $(i+1,j') \to (i,j'') \to (i-1,j') \to (i,j)$ and $(i+1,j'') \to (i,j') \to (i-1,j'') \to (i,j)$.
(ii) If $(i,j) \to \{(i-1,j'),(i-1,j'')\}$ for $0 \leq j' < j'' \leq 2$ and $j', j'' \neq j$, then $(i-1,j') \to (i,j'') \to (i+1,j') \to (i,j)$ and $(i-1,j'') \to (i,j') \to (i+1,j'') \to (i,j)$.

As $C_8 \times K_3$ is vertex-transitive, it is enough to verify Claim 3 for the vertex $(0,0)$. Suppose that $(0,0) \to \{(1,1),(1,2)\}$. By Claim 1, $\{(7,1),(7,2)\} \to (0,0)$. $d_D((0,0),(6,0)) = 4$ implies that at least one of $(1,1) \to (0,2)$ or $(1,2) \to (0,1)$ must be true. Suppose exactly one of them is true. We may assume that $(1,1) \to (0,2)$ and $(0,1) \to (1,2)$. $d_D((0,0),(6,0)) = 4$ implies that $(0,2) \to (7,1) \to (6,0)$. $d_D((0,0),(6,1)) = 4$ implies that $(0,2) \to (7,0) \to (6,1)$. By Claim 1, $(0,2) \leftarrow (1,0)$ and $(7,1) \leftarrow (6,2)$. $d_D((0,0),(6,2)) = 4$ implies that $(7,0) \rightarrow (6,2)$. By Claim 1, $(7,0) \leftarrow (0,1)$. Again by Claim 1, $(0,1) \leftarrow \{(1,0),(7,2)\}$. Now $d_D((7,0),(1,0)) \geq 6$, a contradiction.

Thus, $(1,1) \rightarrow (0,2)$ and $(1,2) \rightarrow (0,1)$. By considering the converse digraph, we have also that $(0,1) \rightarrow (7,2)$ and $(0,2) \rightarrow (7,1)$. An analogous argument will complete the proof of Claim 3.

By Claim 2, as $C_8 \times K_3$ is vertex-transitive, we may assume that $(0,0) \rightarrow \{(1,1),(1,2)\}$. By Claim 3, $(1,1) \rightarrow (0,2) \rightarrow (7,1) \rightarrow (0,0)$ and $(1,2) \rightarrow (0,1) \rightarrow (7,2) \rightarrow (0,0)$. $d_D((0,0),(6,0)) = 4$ implies that either $(7,1) \rightarrow (6,0)$ or $(7,2) \rightarrow (6,0)$. But these two cases are similar, so we may assume that $(7,1) \rightarrow (6,0)$. By Claim 1, $(6,2) \rightarrow (7,1)$. $d_D((0,0),(6,2)) = 4$ implies that $(7,0) \rightarrow (6,2)$. $d_D((0,0),(6,1)) = 4$ implies that either $(7,0) \rightarrow (6,1)$ or $(7,2) \rightarrow (6,1)$. By Claim 3, if $(7,0) \rightarrow (6,1)$, then $(6,1) \rightarrow (7,2) \rightarrow (0,1)$. But $(0,1) \rightarrow (7,2)$ earlier, a contradiction. Thus, $(7,2) \rightarrow (6,1) \rightarrow (7,0)$. By Claim 1, $(6,0) \rightarrow (7,2)$.

Let a K_3-layer be called an *X-layer* if each vertex in the K_3-layer has its two out-neighbours in different K_3-layers, and called a *Y-layer* otherwise. For $i = 0,1,\ldots,7$, call the K_3-layer containing the vertex $(i,1)$ as the i-th K_3-layer.

By Claim 1, the 6-th K_3-layer is an X-layer. By Claim 1, either $(0,1) \rightarrow (7,0) \rightarrow (0,2)$ or $(0,2) \rightarrow (7,0) \rightarrow (0,1)$. In either case, exactly one of $(0,1)$ or $(0,2)$ will have its two in-neighbours in the 1-st K_3-layer. By symmetry, since the 7-th and 6-th K_3-layers are X-layers, the 1-st and 2-nd K_3-layers are also X-layers. Thus, if the i-th K_3-layer is a Y-layer, then the $(i+j)$-th K_3-layers for $j = 2,1,-1,-2$, will be X-layers. Hence at most one of the 3-rd, 4-th and 5-th K_3-layers can be a Y-layer.

Suppose the 4-th K_3-layer is a Y-layer. There exists a vertex $(4,j)$ such that its out-neighbours are in different K_3-layers. Then at least one of the vertices in the 0-th K_3-layer cannot be reached by $(4,j)$ in less than 5 steps. Suppose the 3-rd K_3-layer (the argument is similar for the 5-th K_3-layer) is a Y-layer. Let $(4,j)$ be the vertex such that its out-neighbour in the 3-rd K_3-layer has its out-neighbours in different K_3-layers. Then at least one of the vertices in the 0-th K_3-layer cannot be reached by $(4,j)$ in less than 5 steps.

Thus, $\vec{d}(C_8 \times K_3) \geq 6$ and the theorem is proved. ∎

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