On a characterization for a graphic sequence to be potentially $K_{r+1} - E(G)$-graphic*

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Abstract: Let G be a subgraph of the complete graph K_{r+1} on $r+1$ vertices and $K_{r+1}-E(G)$ be the graph obtained from K_{r+1} by deleting all edges of G. A non-increasing sequence $\pi = (d_1, d_2, \ldots, d_n)$ of nonnegative integers is said to be potentially $K_{r+1} - E(G)$-graphic if it is realizable by a graph on n vertices containing $K_{r+1} - E(G)$ as a subgraph. In this paper, we give characterizations for $\pi = (d_1, d_2, \ldots, d_n)$ to be potentially $K_{r+1} - E(G)$-graphic for $G = 3K_2, K_3, P_3, K_{1,3}$ and $K_2 \cup P_2$, which are analogous to Erdős-Gallai characterization using a system of inequalities. These characterizations partially answer one problem due to Lai and Hu [10].

Keywords: graph, degree sequence, potentially $K_{r+1} - E(G)$-graphic sequence.

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1. Introduction

The set of all sequences $\pi = (d_1, d_2, \ldots, d_n)$ of non-negative, non-increasing integers with $d_1 \leq n-1$ is denoted by NS_n. A sequence $\pi \in NS_n$

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is said to be *graphic* if it is the degree sequence of a simple graph G on n vertices, and such a graph G is called a *realization* of π. The set of all graphic sequences in NS_n is denoted by GS_n. If a sequence π consists of the terms $d_1, \ldots, d_t$ having multiplicities $m_1, \ldots, m_t$, we may write $\pi = (d_1^{m_1}, \ldots, d_t^{m_t})$. Given any two graphs G and H, $G \cup H$ is the disjoint union of G and H. For $V_1 \subseteq V(G)$, $G[V_1]$ is the induced subgraph of V_1 in G. The following well-known result due to Erdős and Gallai [3] which gave a characterization for π to be graphic.

Theorem 1.1 [3] Let $\pi = (d_1, d_2, \ldots, d_n) \in NS_n$, where $\sum_{i=1}^{n} d_i$ is even. Then π is graphic if and only if

$$\sum_{i=1}^{t} d_i \leq t(t-1) + \sum_{i=t+1}^{n} min\{t, d_i\}$$

for each t, $1 \leq t \leq n$.

For a given graph G, a sequence $\pi \in NS_n$ is said to be *potentially G-graphic* if there exists a realization of π containing G as a subgraph. Rao [12] and Kézdy and Lehel [8] independently gave a characterization for a sequence $\pi \in NS_n$ to be potentially K_{r+1}-graphic. This is a generalization of Erdős-Gallai characterization (Theorem 1.1) for π to be graphic (which corresponds to $r = 0$).

Theorem 1.2 [12,8] Let $n \geq r+1$ and $\pi = (d_1, d_2, \ldots, d_n) \in NS_n$, where $d_{r+1} \geq r$ and $\sum_{i=1}^{n} d_i$ is even. Then π is potentially K_{r+1}-graphic if and only if

$$\sum_{i=1}^{s} d_i + \sum_{i=1}^{t} d_{r+1+i}$$
$$\leq (s+t)(s+t-1) + \sum_{i=s+1}^{r+1} min\{s+t, d_i - r + s\} + \sum_{i=r+t+2}^{n} min\{s+t, d_i\} \qquad (a)$$

for any s and t, $0 \leq s \leq r+1$ and $0 \leq t \leq n-r-1$.

Motivated by the above characterization, Lai in [10] further proposed the following

Problem 1.1 Find a characterization for a sequence $\pi \in NS_n$ to be potentially $K_{r+1} - E(G)$-graphic, where G is a subgraph of K_{r+1}.

If G is a subgraph of K_{r+1} with one edge, then $G = K_2$. For this case, Lai [9] and Eschen and Niu [4] independently characterized the potentially $K_4 - E(K_2)$-graphic sequences. M.X. Yin and J.H. Yin [17] characterized the potentially $K_5 - E(K_2)$-graphic sequences. J.H. Yin and Li [14] gave two sufficient conditions for a graphic sequence π to be potentially $K_{r+1} - E(K_2)$-graphic. Recently, J.H. Yin and Wang [15] presented the following characterization for a sequence $\pi \in NS_n$ to be potentially $K_{r+1}-E(K_2)$-graphic which is analogous to Erdős-Gallai characterization using a system of inequalities.

Theorem 1.3 [15] Let $n \geq r+1$ and $\pi = (d_1, d_2, \ldots, d_n) \in NS_n$, where $\sum_{i=1}^{n} d_i$ is even. Then π is potentially $K_{r+1}-E(K_2)$-graphic if and only if π satisfies one of the following conditions:

(1) $d_{r+1} \geq r$ and (a) holds for any s and t, $0 \leq s \leq r+1$ and $0 \leq t \leq n-r-1$.

(2) $d_{r-1} \geq r$, $d_{r+1} \geq r-1$ and

$$\begin{aligned}
&\sum_{i=1}^{p}(d_i - r) + \sum_{i=r}^{r-1+p'}(d_i - r + 1) + \sum_{i=r+2}^{q+r+1} d_i \\
&\leq 2(p+p')q + q(q-1) + \sum_{i=p+1}^{r-1} min\{q, d_i - r\} \qquad (b)\\
&\quad + \sum_{i=r+p'}^{r+1} min\{q, d_i - r + 1\} + \sum_{i=q+r+2}^{n} min\{p+p'+q, d_i\}
\end{aligned}$$

for any p, p' and q, $0 \leq p \leq r-1$, $0 \leq p' \leq 2$ and $0 \leq q \leq n-r-1$.

If G is a subgraph of K_{r+1} with two edges, then $G = P_2$ (a path of length 2) or $G = 2K_2$ (the disjoint union of 2 copies of K_2). For this case, Chen and Li [2] characterized the potentially $K_4 - E(P_2)$-graphic sequences and M.X. Yin et al. [18] characterized the potentially $K_5 - E(P_2)$-graphic sequences. Hu and Lai [7] characterized the potentially $K_5 - E(2K_2)$-graphic sequences and Liu and Lai [11] characterized the potentially $K_6 - E(2K_2)$-graphic sequences. Recently, Wang and J.H. Yin [13] further obtained characterizations for a sequence $\pi \in NS_n$ to be potentially $K_{r+1} - E(G)$-graphic for $G = P_2$ and $2K_2$, which are analogous to Erdős-Gallai characterization using a system of inequalities.

Theorem 1.4 [13] Let $n \geq r+1$ and $\pi = (d_1, d_2, \ldots, d_n) \in NS_n$, where $\sum_{i=1}^{n} d_i$ is even. Let $d'_{r-1} \geq d'_r \geq d'_{r+1}$ be the rearrangement in non-increasing order of $d_{r-1} - r + 1$, $d_r - r + 1$ and $d_{r+1} - r + 2$. Then π is potentially $K_{r+1} - E(P_2)$-graphic if and only if π satisfies one of the following conditions:

(1) $d_{r+1} \geq r$ and (a) holds for any s and t, $0 \leq s \leq r+1$ and $0 \leq t \leq n-r-1$.

(2) $d_{r-1} \geq r$, $d_{r+1} \geq r-1$ and (b) holds for any p, p' and q, $0 \leq p \leq r-1$, $0 \leq p' \leq 2$ and $0 \leq q \leq n-r-1$.

(3) $d_{r-2} \geq r$, $d_r \geq r-1$, $d_{r+1} \geq r-2$ and

$$\begin{aligned}&\sum_{i=1}^{p}(d_i - r) + \sum_{i=r-1}^{r-2+p'} d'_i + \sum_{i=r+2}^{q+r+1} d_i \\ &\leq 2(p+p')q + q(q-1) + \sum_{i=p+1}^{r-2} min\{q, d_i - r\} \\ &\quad + \sum_{i=r-1+p'}^{r+1} min\{q, d'_i\} + \sum_{i=q+r+2}^{n} min\{p+p'+q, d_i\}\end{aligned} \qquad (c)$$

for any p, p' and q, $0 \leq p \leq r-2$, $0 \leq p' \leq 3$ and $0 \leq q \leq n-r-1$.

Theorem 1.5 [13] Let $n \geq r+1$ and $\pi = (d_1, d_2, \ldots, d_n) \in NS_n$, where $\sum_{i=1}^{n} d_i$ is even. Then π is potentially $K_{r+1} - E(2K_2)$-graphic if and only if π satisfies one of the following conditions:

(1) $d_{r+1} \geq r$ and (a) holds for any s and t, $0 \leq s \leq r+1$ and $0 \leq t \leq n-r-1$.

(2) $d_{r-1} \geq r$, $d_{r+1} \geq r-1$ and (b) holds for any p, p' and q, $0 \leq p \leq r-1$, $0 \leq p' \leq 2$ and $0 \leq q \leq n-r-1$.

(3) $d_{r-3} \geq r$, $d_{r+1} \geq r-1$ and

$$\begin{aligned}&\sum_{i=1}^{p}(d_i - r) + \sum_{i=r-2}^{r-3+p'}(d_i - r + 1) + \sum_{i=r+2}^{q+r+1} d_i \\ &\leq 2(p+p')q + q(q-1) + \sum_{i=p+1}^{r-3} min\{q, d_i - r\} \\ &\quad + \sum_{i=r-2+p'}^{r+1} min\{q, d_i - r + 1\} + \sum_{i=q+r+2}^{n} min\{p+p'+q, d_i\}\end{aligned} \qquad (d)$$

for any p, p' and q, $0 \leq p \leq r-3$, $0 \leq p' \leq 4$ and $0 \leq q \leq n-r-1$.

If G is a subgraph of K_{r+1} with three edges, then G is one of $3K_2, K_3, P_3$, $K_{1,3}$ and $K_2 \cup P_2$. For this case, Hu and Lai [7] characterized the potentially $K_5 - E(G)$-graphic sequences for $G = K_3, P_3, K_{1,3}$ and $K_2 \cup P_2$. Chen [1] characterized the potentially $K_6 - E(3K_2)$-graphic sequences. M.X. Yin and J.H. Yin [16] characterized the potentially $K_6 - E(K_3)$-graphic sequences. The purpose of this paper is to give characterizations for a sequence $\pi \in NS_n$ to be potentially $K_{r+1} - E(G)$-graphic for $G = 3K_2, K_3, P_3, K_{1,3}$ and $K_2 \cup P_2$, which are also analogous to Erdős-Gallai characterization using a system of inequalities. That is, we establish the following five theorems.

Theorem 1.6 Let $n \geq r+1$ and $\pi = (d_1, d_2, \ldots, d_n) \in NS_n$, where $\sum_{i=1}^{n} d_i$ is even. Then π is potentially $K_{r+1} - E(3K_2)$-graphic if and only if π satisfies one of the following conditions:

(1) $d_{r+1} \geq r$ and (a) holds for any s and t, $0 \leq s \leq r+1$ and $0 \leq t \leq n-r-1$.

(2) $d_{r-1} \geq r$, $d_{r+1} \geq r-1$ and (b) holds for any p, p' and q, $0 \leq p \leq r-1$, $0 \leq p' \leq 2$ and $0 \leq q \leq n-r-1$.

(3) $d_{r-3} \geq r$, $d_{r+1} \geq r-1$ and (d) holds for any p, p' and q, $0 \leq p \leq r-3$, $0 \leq p' \leq 4$ and $0 \leq q \leq n-r-1$.

(4) $d_{r-5} \geq r$, $d_{r+1} \geq r-1$ and

$$\begin{aligned}&\sum_{i=1}^{p}(d_i - r) + \sum_{i=r-4}^{r-5+p'}(d_i - r + 1) + \sum_{i=r+2}^{q+r+1} d_i \\ &\leq 2(p+p')q + q(q-1) + \sum_{i=p+1}^{r-5} min\{q, d_i - r\} \\ &\quad + \sum_{i=r-4+p'}^{r+1} min\{q, d_i - r + 1\} + \sum_{i=q+r+2}^{n} min\{p+p'+q, d_i\}\end{aligned} \qquad (e)$$

for any p, p' and q, $0 \leq p \leq r-5$, $0 \leq p' \leq 6$ and $0 \leq q \leq n-r-1$.

Theorem 1.7 Let $n \geq r+1$ and $\pi = (d_1, d_2, \ldots, d_n) \in NS_n$, where $\sum_{i=1}^{n} d_i$ is even. Then π is potentially $K_{r+1} - E(K_3)$-graphic if and only if π satisfies one of the following conditions:

(1) $d_{r+1} \geq r$ and (a) holds for any s and t, $0 \leq s \leq r+1$ and $0 \leq t \leq n-r-1$.

(2) $d_{r-1} \geq r$, $d_{r+1} \geq r-1$ and (b) holds for any p, p' and q, $0 \leq p \leq r-1$, $0 \leq p' \leq 2$ and $0 \leq q \leq n-r-1$.

(3) $d_{r-2} \geq r$, $d_r \geq r-1$, $d_{r+1} \geq r-2$ and (c) holds for any p, p' and q, $0 \leq p \leq r-2$, $0 \leq p' \leq 3$ and $0 \leq q \leq n-r-1$.

(4) $d_{r-2} \geq r$, $d_{r+1} \geq r-2$ and

$$\begin{aligned}&\sum_{i=1}^{p}(d_i - r) + \sum_{i=r-1}^{r-2+p'}(d_i - r + 2) + \sum_{i=r+2}^{q+r+1} d_i \\ &\leq 2(p+p')q + q(q-1) + \sum_{i=p+1}^{r-2} min\{q, d_i - r\} \\ &\quad + \sum_{i=r-1+p'}^{r+1} min\{q, d_i - r + 2\} + \sum_{i=q+r+2}^{n} min\{p+p'+q, d_i\}\end{aligned} \qquad (f)$$

for any p, p' and q, $0 \leq p \leq r-2$, $0 \leq p' \leq 3$ and $0 \leq q \leq n-r-1$.

Theorem 1.8 Let $n \geq r+1$ and $\pi = (d_1, d_2, \ldots, d_n) \in NS_n$, where $\sum_{i=1}^{n} d_i$ is even. Let $d'_{r-2} \geq d'_{r-1} \geq d'_r \geq d'_{r+1}$ be the rearrangement in non-increasing order of $d_{r-2} - r + 1$, $d_{r-1} - r + 1$, $d_r - r + 2$ and $d_{r+1} - r + 2$. Then π is potentially $K_{r+1} - E(P_3)$-graphic if and only if π satisfies one of the following conditions:

(1) $d_{r+1} \geq r$ and (a) holds for any s and t, $0 \leq s \leq r+1$ and $0 \leq t \leq n-r-1$.

(2) $d_{r-1} \geq r$, $d_{r+1} \geq r-1$ and (b) holds for any p, p' and q, $0 \leq p \leq r-1$, $0 \leq p' \leq 2$ and $0 \leq q \leq n-r-1$.

(3) $d_{r-2} \geq r$, $d_r \geq r-1$, $d_{r+1} \geq r-2$ and (c) holds for any p, p' and q, $0 \leq p \leq r-2$, $0 \leq p' \leq 3$ and $0 \leq q \leq n-r-1$.

(4) $d_{r-3} \geq r$, $d_{r+1} \geq r-1$ and (d) holds for any p, p' and q, $0 \leq p \leq r-3$, $0 \leq p' \leq 4$ and $0 \leq q \leq n-r-1$.

(5) $d_{r-3} \geq r$, $d_{r-1} \geq r-1$, $d_{r+1} \geq r-2$ and

$$\begin{aligned}&\sum_{i=1}^{p}(d_i - r) + \sum_{i=r-2}^{r-3+p'} d'_i + \sum_{i=r+2}^{q+r+1} d_i \\ &\leq 2(p+p')q + q(q-1) + \sum_{i=p+1}^{r-3} min\{q, d_i - r\} \\ &\quad + \sum_{i=r-2+p'}^{r+1} min\{q, d'_i\} + \sum_{i=q+r+2}^{n} min\{p+p'+q, d_i\}\end{aligned} \tag{g}$$

for any p, p' and q, $0 \leq p \leq r-3$, $0 \leq p' \leq 4$ and $0 \leq q \leq n-r-1$.

Theorem 1.9 Let $n \geq r+1$ and $\pi = (d_1, d_2, \ldots, d_n) \in NS_n$, where $\sum_{i=1}^{n} d_i$ is even. Let $d'_{r-2} \geq d'_{r-1} \geq d'_r \geq d'_{r+1}$ be the rearrangement in non-increasing order of $d_{r-2} - r + 1, d_{r-1} - r + 1, d_r - r + 1$ and $d_{r+1} - r + 3$. Then π is potentially $K_{r+1} - E(K_{1,3})$-graphic if and only if π satisfies one of the following conditions:

(1) $d_{r+1} \geq r$ and (a) holds for any s and t, $0 \leq s \leq r+1$ and $0 \leq t \leq n-r-1$.

(2) $d_{r-1} \geq r$, $d_{r+1} \geq r-1$ and (b) holds for any p, p' and q, $0 \leq p \leq r-1$, $0 \leq p' \leq 2$ and $0 \leq q \leq n-r-1$.

(3) $d_{r-2} \geq r$, $d_r \geq r-1$, $d_{r+1} \geq r-2$ and (c) holds for any p, p' and q, $0 \leq p \leq r-2$, $0 \leq p' \leq 3$ and $0 \leq q \leq n-r-1$.

(4) $d_{r-3} \geq r, d_r \geq r-1, d_{r+1} \geq r-3$ and

$$\begin{aligned}&\sum_{i=1}^{p}(d_i - r) + \sum_{i=r-2}^{r-3+p'} d'_i + \sum_{i=r+2}^{q+r+1} d_i \\ &\leq q(q-1) + 2(p+p')q + \sum_{i=p+1}^{r-3} min\{q, d_i - r\} \\ &\quad + \sum_{i=r-2+p'}^{r+1} min\{q, d'_i\} + \sum_{i=q+r+2}^{n} min\{p+p'+q, d_i\}\end{aligned} \tag{h}$$

for any p, p' and q, $0 \leq p \leq r-3$, $0 \leq p' \leq 4$ and $0 \leq q \leq n-r-1$.

Theorem 1.10 Let $n \geq r+1$ and $\pi = (d_1, d_2, \ldots, d_n) \in NS_n$, where $\sum_{i=1}^{n} d_i$ is even. Let $d'_{r-3} \geq d'_{r-2} \geq d'_{r-1} \geq d'_r \geq d'_{r+1}$ be the rearrangement in non-increasing order of $d_{r-3}-r+1, d_{r-2}-r+1, d_{r-1}-r+1, d_r-r+1$ and $d_{r+1}-r+2$. Then π is potentially $K_{r+1}-E(K_2\cup P_2)$-graphic if and only if π satisfies one of the following conditions:

(1) $d_{r+1}\geq r$ and (a) holds for any s and t, $0\leq s\leq r+1$ and $0\leq t\leq n-r-1$.

(2) $d_{r-1}\geq r$, $d_{r+1}\geq r-1$ and (b) holds for any p, p' and q, $0\leq p\leq r-1$, $0\leq p'\leq 2$ and $0\leq q\leq n-r-1$.

(3) $d_{r-2}\geq r$, $d_r\geq r-1$, $d_{r+1}\geq r-2$ and (c) holds for any p, p' and q, $0\leq p\leq r-2$, $0\leq p'\leq 3$ and $0\leq q\leq n-r-1$.

(4) $d_{r-3}\geq r$, $d_{r+1}\geq r-1$ and (d) holds for any p, p' and q, $0\leq p\leq r-3$, $0\leq p'\leq 4$ and $0\leq q\leq n-r-1$.

(5) $d_{r-4}\geq r, d_r\geq r-1, d_{r+1}\geq r-2$ and

$$\begin{aligned}&\sum_{i=1}^{p}(d_i-r)+\sum_{i=r-3}^{r-4+p'}d'_i+\sum_{i=r+2}^{q+r+1}d_i\\ &\leq q(q-1)+2(p+p')q+\sum_{i=p+1}^{r-4}min\{q,d_i-r\}\\ &\quad+\sum_{i=r-3+p'}^{r+1}min\{q,d'_i\}+\sum_{i=q+r+2}^{n}min\{p+p'+q,d_i\}\end{aligned}\qquad (i)$$

for any p, p' and q, $0\leq p\leq r-4$, $0\leq p'\leq 5$ and $0\leq q\leq n-r-1$.

2. The Proofs of Theorems

Each of the following known results will be useful as we proceed with the proofs of Theorems. The proof technique of Theorem 1.6–1.11 is using network flows (see also [8]). We shall use a simple version of a general result of Fulkerson et al. [5]. Let H be a simple graph on the vertex set $V(H)=\{v_1,v_2,\dots,v_n\}$. We say that H satisfies the odd-cycle condition, if between any two disjoint odd cycles there is an edge.

Theorem 2.1 [5] Assume that $H=(V(H),E(H))$ satisfies the odd-cycle condition, where $V(H)=\{v_1,v_2,\dots,v_n\}$. There exists a subgraph $G\subseteq H$ such that every vertex v_i has degree d_i, if and only if

(1) $\sum_{i=1}^{n}d_i$ is even,

(2) for every $A,B\subseteq V(H)$ such that $A\cap B=\emptyset$, we have

$$\sum_{v_i\in A}d_i\leq|\{(v_i,v_j):v_iv_j\in E(H),v_i\in A,v_j\in V(H)\setminus B\}|+\sum_{v_i\in B}d_i.$$

Theorem 2.2 [6] If π has a realization G containing H as a subgraph, then there exists a realization G' of π containing H so that the vertices of H have the $|H|$ largest degrees of π.

We also need the following

Lemma 2.1 If $\pi = (d_1, d_2, \ldots, d_n)$ has a realization containing H as an induced subgraph so that the vertices of H have the largest degrees of π, then there exists a realization G of π with the vertex set $V(G) = \{v_1, \cdots, v_n\}$ such that $d_G(v_i) = d_i$ for $1 \le i \le n$, $G[\{v_1, v_2, \ldots, v_{|V(H)|}\}] = H$ and $d_H(v_1) \ge d_H(v_2) \ge \cdots \ge d_H(v_{|V(H)|})$.

Proof. Assume that G is a realization of π with the vertex set $V(G) = \{v_1, \cdots, v_n\}$ such that $d_G(v_i) = d_i$ for $1 \le i \le n$ and $G[\{v_1, v_2, \ldots, v_{|V(H)|}\}] = H$. If $d_H(v_1) \ge d_H(v_2) \ge \cdots \ge d_H(v_{|V(H)|})$, then G is a required realization of π. Otherwise, there exist vertices v_i and v_j such that $d_H(v_j) \ge d_H(v_i)$ and $d_G(v_j) < d_G(v_i)$. Since $d_G(v_j) < d_G(v_i)$, we have $d_G(v_i) - d_H(v_i) > d_G(v_j) - d_H(v_j)$. For convenience, set $A = \{v_{|V(H)|+1}, v_{|V(H)|+2}, \ldots, v_n\}$ and $B = N_A(v_i) \cap N_A(v_j)$. Clearly, $|N_A(v_i) \setminus B| - |N_A(v_j) \setminus B| \ge d_G(v_i) - d_G(v_j)$. Now form a new realization G' of π as follows. Suppose that the edges between v_i and $N_A(v_i) \setminus B$ are $v_iu_1, v_iu_2, \ldots, v_iu_{|N_A(v_i)\setminus B|}$. Then

$$G' = G - \{v_iu_1, v_iu_2, \ldots, v_iu_{d_G(v_i)-d_G(v_j)}\} + \{v_ju_1, v_ju_2, \ldots, v_ju_{d_G(v_i)-d_G(v_j)}\}$$

is a new realization of π. In G', H is still an induced subgraph of $\{v_1, v_2, \ldots, v_{|V(H)|}\}$, $d_{G'}(v_i) = d_G(v_j)$, $d_{G'}(v_j) = d_G(v_i)$ and $d_{G'}(v_i) \le d_{G'}(v_j)$. If there exist such pair v_s and v_t in G' such that $d_H(v_t) \ge d_H(v_s)$ and $d_{G'}(v_t) < d_{G'}(v_s)$, then we repeat this process until no such pair remains. □

In order to prove Theorem 1.6–1.10, we also need more definitions as follows. If $\pi = (d_1, d_2, \ldots, d_n)$ has a realization G with the vertex set $V(G) = \{v_1, v_2, \cdots, v_n\}$ such that $d_G(v_i) = d_i$ for $1 \le i \le n$ and $G[\{v_1, v_2, \ldots, v_{r+1}\}] = K_{r+1} - E(3K_2)$ (denoted by H) so that $d_H(v_i) = r$ for $1 \le i \le r-5$ and $d_H(v_i) = r-1$ for $r-4 \le i \le r+1$, then π is said to be *potentially* $A_{r+1} - E(3K_2)$*-graphic.* If $\pi = (d_1, d_2, \ldots, d_n)$ has a realization G with the vertex set $V(G) = \{v_1, v_2, \cdots, v_n\}$ such that $d_G(v_i) = d_i$ for $1 \le i \le n$ and $G[\{v_1, v_2, \ldots, v_{r+1}\}] = K_{r+1} - E(K_3)$ (denoted by H) so that $d_H(v_i) = r$ for $1 \le i \le r-2$ and $d_H(v_i) = r-2$ for $r-1 \le i \le r+1$, then π is said to be *potentially* $A_{r+1} - E(K_3)$. Further, if $\pi = (d_1, d_2, \ldots, d_n)$ has a realization G with the vertex set $V(G) = \{v_1, v_2, \cdots, v_n\}$ such that $d_G(v_i) = d_i$ for $1 \le i \le n$ and $G[\{v_1, v_2, \ldots, v_{r+1}\}] = K_{r+1} - E(P_3)$ (denoted by H) so that $d_H(v_i) = r$ for $1 \le i \le r-3$, $d_H(v_i) = r-1$ for $r-2 \le i \le r-1$ and $d_H(v_i) = r-2$ for $r \le i \le r+1$, then π is said to be *potentially* $A_{r+1} - E(P_3)$*-graphic.* If $\pi = (d_1, d_2, \ldots, d_n)$ has a realization G with the vertex set $V(G) = \{v_1, v_2, \cdots, v_n\}$ such that $d_G(v_i) = d_i$ for $1 \le i \le n$ and $G[\{v_1, v_2, \ldots, v_{r+1}\}] = K_{r+1} - E(K_{1,3})$ (denoted by H) so that $d_H(v_i) = r$ for $1 \le i \le r-3$, $d_H(v_i) = r-1$ for $r-2 \le i \le r$ and $d_H(v_{r+1}) = r-3$, then π is said to be *potentially* $A_{r+1} - E(K_{1,3})$*-graphic.* In addition, if $\pi = (d_1, d_2, \ldots, d_n)$ has a realization G with the vertex set $V(G) = \{v_1, v_2, \cdots, v_n\}$ such that $d_G(v_i) = d_i$ for $1 \le i \le n$ and $G[\{v_1, v_2, \ldots, v_{r+1}\}] = K_{r+1} - E(K_2 \cup P_2)$ (denoted by H) so that $d_H(v_i) = r$ for $1 \le i \le r-4$, $d_H(v_i) = r-1$ for $r-3 \le i \le r$ and $d_H(v_{r+1}) = r-2$, then π is said to be *potentially $A_{r+1} - E(K_2 \cup P_2)$-graphic.*

The proof of Theorem 1.6. Assume that π is potentially $K_{r+1} - E(3K_2)$-graphic. If π is potentially $K_{r+1} - E(2K_2)$-graphic, then π satisfies one of (1)–(3) of Theorem 1.5. If π is not potentially $K_{r+1} - E(2K_2)$-graphic, then by Theorem 2.2 and Lemma 2.1, π is potentially $A_{r+1} - E(3K_2)$-graphic, we may let G be a realization of π with the vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ such that $d_G(v_i) = d_i$ for $1 \le i \le n$ and $G[\{v_1, v_2, \ldots, v_{r+1}\}] = K_{r+1} - E(3K_2)$ so that the six endvertices of three removed edges from K_{r+1} are exactly those vertices with degrees d_{r-4}, d_{r-3}, d_{r-2}, d_{r-1}, d_r and d_{r+1}. For $0 \le p \le r-5$, $0 \le p' \le 6$ and $0 \le q \le n-r-1$, denote $P = \{v_i | 1 \le i \le p\}$, $P' = \{v_i | r-4 \le i \le r-5+p'\}$, $R = \{v_i | p+1 \le i \le r-5\}$, $R' = \{v_i | r-4+p' \le i \le r+1\}$, $Q = \{v_i | r+2 \le i \le q+r+1\}$ and $S = \{v_i | q+r+2 \le i \le n\}$. The removal of the edges induced by $\{v_1, v_2, \ldots, v_{r+1}\}$ results in a graph G' in which all degrees in $\{v_1, v_2, \ldots, v_{r-3}\}$ are reduced by r and all degrees in $\{v_{r-4}, v_{r-3}, v_{r-2}, v_{r-1}, v_r, v_{r+1}\}$ are reduced by $r-1$. There are at most $(p+p')q$ edges between $P \cup P'$ and Q and the degree sum in the subgraph induced by Q is at most $q(q-1)$. Therefore,

$$m = \sum_{i=1}^{p}(d_i - r) + \sum_{i=r-4}^{r-5+p'}(d_i - r + 1) + \sum_{i=r+2}^{q+r+1} d_i - (2(p+p')q + q(q-1))$$

is the minimum number of edges of G' with exactly one endvertex in $P \cup P' \cup Q$. On the other hand, the maximum number of edges of G' with exactly one endvertex in $R \cup R' \cup S$ is

$$\begin{aligned} M \;=\; & \sum_{i=p+1}^{r-5} min\{q, d_i - r\} + \sum_{i=r-4+p'}^{r+1} min\{q, d_i - r + 1\} \\ & + \sum_{i=q+r+2}^{n} min\{p + p' + q, d_i\}. \end{aligned}$$

Graph G' witnesses that $m \le M$ is true. Thus the necessity is proved.

We now prove the sufficiency. If π satisfies one of (1)–(3) of Theorem 1.6, then π is potentially $K_{r+1} - E(2K_2)$-graphic by Theorem 1.5, which is sufficient to show that π is potentially $K_{r+1} - E(3K_2)$-graphic. Assume that π satisfies (4) of Theorem 1.6. Let $\pi' = (d_1', \ldots, d_{r+1}', d_{r+2}', \ldots, d_n')$, where $d_i' = d_i - r$ for $1 \le i \le r-5$, $d_i' = d_r - r + 1$ for $r-4 \le i \le r+1$ and $d_i' = d_i$ for $r+2 \le i \le n$. Let H be the graph obtained from K_n with the vertex set $V(K_n) = \{v_1, v_2, \ldots, v_n\}$ by deleting all edges between v_i

and v_j for any $i, j \in \{1, 2, \ldots, r+1\}$. It is easy to see that π is potentially $A_{r+1} - E(3K_2)$-graphic if and only if H has a subgraph G with the degree sequence π' such that every vertex v_i has degree d_i'. Observe that between any two disjoint odd cycles of H there is an edge. Therefore, H satisfies the odd-cycle condition and we may apply Theorem 2.1.

Let $K = \{v_1, \ldots, v_{r-5}\}$, $K' = \{v_{r-4}, \ldots, v_{r+1}\}$ and $A, B \subseteq V(H)$ such that $A \cap B = \emptyset$. Let $A_1 = A \cap K, A_1' = A \cap K', A_2 = A \setminus (K \cup K'), B_1 = B \cap K, B_1' = B \cap K', B_2 = B \setminus (K \cup K')$, and set $p = |A_1|, p' = |A_1'|, q = |A_2|, b_1 = |B_1|, b_1' = |B_1'|, b_2 = |B_2|$. For convenience, we denote the left and right hand side of (e) by $L(p, p', q)$ and $R(p, p', q)$, respectively. Let

$$\begin{aligned} L'(A,B) &= \sum_{v_i \in A} d_i' = \sum_{v_i \in A_1} (d_i - r) + \sum_{v_i \in A_1'} (d_i - r + 1) + \sum_{v_i \in A_2} d_i, \\ R'(A,B) &= |\{(v_i, v_j) : v_i v_j \in E(H), v_i \in A, v_j \in V(H) \setminus B\}| + \sum_{v_i \in B} d_i' \\ &= |\{(v_i, v_j) : v_i v_j \in E(H), v_i \in A, v_j \in V(H) \setminus B\}| \\ &\quad + \sum_{v_i \in B_1} (d_i - r) + \sum_{v_i \in B_1'} (d_i - r + 1) + \sum_{v_i \in B_2} d_i. \end{aligned}$$

Clearly, $L'(A,B) \le L(p, p', q)$. $|\{(v_i, v_j) : v_i v_j \in E(H), v_i \in A, v_j \in V(H) \setminus B\}|$ is the number of counting the edges of H between A and $V(H) \setminus (A \cup B)$ and double counting the edges induced by A. Thus we get

$$\begin{aligned} &|\{(v_i, v_j) : v_i v_j \in E(H), v_i \in A, v_j \in V(H) \setminus B\}| \\ &= 2(p+p')q + q(q-1) + q(r-5-p-b_1) + q(6-p'-b_1') \\ &\quad +(p+p'+q)(n-(r+1)-q-b_2) \\ &= 2(p+p')q + q(q-1) + \sum_{i=p+1}^{r-5-b_1} q + \sum_{i=r-4+p'}^{r+1-b_1'} q + \sum_{i=q+r+2}^{n-b_2} (p+p'+q). \text{ So,} \end{aligned}$$

$$\begin{aligned} R'(A,B) &= |\{(v_i, v_j) : v_i v_j \in E(H), v_i \in A, v_j \in V(H) \setminus B\}| \\ &\quad + \sum_{v_i \in B_1} (d_i - r) + \sum_{v_i \in B_1'} (d_i - r + 1) + \sum_{v_i \in B_2} d_i \\ &\ge 2(p+p')q + q(q-1) + \sum_{i=p+1}^{r-5-b_1} q + \sum_{i=r-4+p'}^{r+1-b_1'} q \\ &\quad + \sum_{i=q+r+2}^{n-b_2} (p+p'+q) + \sum_{i=r-4-b_1}^{r-5} (d_i - r) \\ &\quad + \sum_{i=r+2-b_1'}^{r+1} (d_i - r + 1) + \sum_{i=n+1-b_2}^{n} d_i \\ &\ge 2(p+p')q + q(q-1) + \sum_{i=p+1}^{r-5} min\{q, d_i - r\} \\ &\quad + \sum_{i=r-4+p'}^{r+1} min\{q, d_i - r + 1\} \\ &\quad + \sum_{i=q+r+2}^{n} min\{p+p'+q, d_i\} = R(p, p', q). \end{aligned}$$

It follows from $L(p,p',q) \le R(p,p',q)$ that $L'(A,B) \le R'(A,B)$. By Theorem 2.1, H has a subgraph G with the degree sequence π' such that every vertex v_i has degree d'_i. Hence π is potentially $A_{r+1} - E(3K_2)$-graphic. Thus, the sufficiency is proved. □

The proof of Theorem 1.7 Assume that π is potentially $K_{r+1} - E(K_3)$-graphic. If π is potentially $K_{r+1} - E(P_2)$-graphic, then π satisfies one of (1)–(3) of Theorem 1.4. If π is not potentially $K_{r+1} - E(P_2)$-graphic, then π is potentially $A_{r+1} - E(K_3)$-graphic, we let G be a realization of π with the vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ such that $d_G(v_i) = d_i$ for $1 \le i \le n$ and $G[\{v_1, v_2, \ldots, v_{r+1}\}] = K_{r+1} - E(K_3)$ so that the three endvertices of three removed edges from K_{r+1} are exactly those vertices with degrees d_{r-1}, d_r and d_{r+1}. For $0 \le p \le r-2$, $0 \le p' \le 3$ and $0 \le q \le n-r-1$, denote $P = \{v_i | 1 \le i \le p\}$, $P' = \{v_i | r-1 \le i \le r-2+p'\}$, $R = \{v_i | p+1 \le i \le r-2\}$, $R' = \{v_i | r-1+p' \le i \le r+1\}$, $Q = \{v_i | r+2 \le i \le q+r+1\}$ and $S = \{v_i | q+r+2 \le i \le n\}$. The removal of the edges induced by $\{v_1, v_2, \ldots, v_{r+1}\}$ results in a graph G' in which all degrees in $\{v_1, v_2, \ldots, v_{r-3}\}$ are reduced by r and all degrees in $\{v_{r-1}, v_r, v_{r+1}\}$ are reduced by $r-2$.

$$m = \sum_{i=1}^{p}(d_i - r) + \sum_{i=r-1}^{r-2+p'}(d_i - r + 2) + \sum_{i=r+2}^{q+r+1} d_i - (2(p+p')q + q(q-1))$$

is the minimum number of edges of G' with exactly one endvertex in $P \cup P' \cup Q$ and

$$\begin{aligned} M \quad = \quad & \sum_{i=p+1}^{r-2} min\{q, d_i - r\} + \sum_{i=r-1+p'}^{r+1} min\{q, d_i - r + 2\} \\ & + \sum_{i=q+r+2}^{n} min\{p + p' + q, d_i\} \end{aligned}$$

is the maximum number of edges of G' with exactly one endvertex in $R \cup R' \cup S$. Graph G' witnesses that $m \le M$ is true.

We now prove the sufficiency. If π satisfies one of (1)–(3) of Theorem 1.7, then π is potentially $K_{r+1} - E(P_2)$-graphic by Theorem 1.4, which is sufficient to show that π is potentially $K_{r+1} - E(K_3)$-graphic. Assume that π satisfies (4) of Theorem 1.7. Let $\pi' = (d'_1, \ldots, d'_{r+1}, d'_{r+2}, \ldots, d'_n)$, where $d'_i = d_i - r$ for $1 \le i \le r-2$, $d'_i = d_r - r + 2$ for $r-1 \le i \le r+1$ and $d'_i = d_i$ for $r+2 \le i \le n$. Let H be the graph obtained from K_n with the vertex set $V(K_n) = \{v_1, v_2, \ldots, v_n\}$ by deleting all edges between v_i and v_j for any $i, j \in \{1, 2, \ldots, r+1\}$. It is easy to see that π is potentially $A_{r+1} - E(K_3)$-graphic if and only if H has a subgraph G with the degree sequence π' such that every vertex v_i has degree d'_i. Observe that H satisfies the odd-cycle condition.

Let $K = \{v_1, v_2, \ldots, v_{r-2}\}$, $K' = \{v_{r-1}, v_r, v_{r+1}\}$ and $A, B \subseteq V(H)$ such that $A \cap B = \emptyset$. Let $A_1 = A \cap K, A_1' = A \cap K', A_2 = A \setminus (K \cup K'), B_1 = B \cap K, B_1' = B \cap K', B_2 = B \setminus (K \cup K')$, and set $p = |A_1|, p' = |A_1'|, q = |A_2|, b_1 = |B_1|, b_1' = |B_1'|, b_2 = |B_2|$. For convenience, we denote the left and right hand side of (f) by $L(p, p', q)$ and $R(p, p', q)$, respectively. Let

$$\begin{aligned} L'(A,B) &= \sum_{v_i \in A} d_i' = \sum_{v_i \in A_1} (d_i - r) + \sum_{v_i \in A_1'} (d_i - r + 2) + \sum_{v_i \in A_2} d_i, \\ R'(A,B) &= |\{(v_i, v_j) : v_i v_j \in E(H), v_i \in A, v_j \in V(H) \setminus B\}| + \sum_{v_i \in B} d_i' \\ &= |\{(v_i, v_j) : v_i v_j \in E(H), v_i \in A, v_j \in V(H) \setminus B\}| \\ &\quad + \sum_{v_i \in B_1} (d_i - r) + \sum_{v_i \in B_1'} (d_i - r + 2) + \sum_{v_i \in B_2} d_i. \end{aligned}$$

Clearly, $L'(A,B) \le L(p, p', q)$ and

$$\begin{aligned} &|\{(v_i, v_j) : v_i v_j \in E(H), v_i \in A, v_j \in V(H) \setminus B\}| \\ &= 2(p+p')q + q(q-1) + q(r-2-p-b_1) + q(3-p'-b_1') \\ &\quad + (p+p'+q)(n-(r+1)-q-b_2) \\ &= 2(p+p')q + q(q-1) + \sum_{i=p+1}^{r-2-b_1} q + \sum_{i=r-1+p'}^{r+1-b_1'} q + \sum_{i=q+r+2}^{n-b_2} (p+p'+q). \end{aligned}$$

Therefore,

$$\begin{aligned} R'(A,B) &= |\{(v_i, v_j) : v_i v_j \in E(H), v_i \in A, v_j \in V(H) \setminus B\}| \\ &\quad + \sum_{v_i \in B_1} (d_i - r) + \sum_{v_i \in B_1'} (d_i - r + 2) + \sum_{v_i \in B_2} d_i \\ &\ge 2(p+p')q + q(q-1) + \sum_{i=p+1}^{r-2-b_1} q + \sum_{i=r-1+p'}^{r+1-b_1'} q \\ &\quad + \sum_{i=q+r+2}^{n-b_2} (p+p'+q) + \sum_{i=r-1-b_1}^{r-2} (d_i - r) \\ &\quad + \sum_{i=r+2-b_1'}^{r+1} (d_i - r + 2) + \sum_{i=n+1-b_2}^{n} d_i \\ &\ge 2(p+p')q + q(q-1) + \sum_{i=p+1}^{r-2} min\{q, d_i - r\} \\ &\quad + \sum_{i=r-1+p'}^{r+1} min\{q, d_i - r + 2\} + \sum_{i=q+r+2}^{n} min\{p+p'+q, d_i\} \\ &= R(p, p', q). \end{aligned}$$

It follows from $L(p, p', q) \le R(p, p', q)$ that $L'(A,B) \le R'(A,B)$. By Theorem 2.1, H has a subgraph G with the degree sequence π' such that every vertex v_i has degree d_i'. □

The proof of Theorem 1.8 Assume that π is potentially $K_{r+1} - E(P_3)$-graphic. If π is potentially $K_{r+1} - E(2K_2)$-graphic, then π satisfies one of (1)–(3) of Theorem 1.5. If π is potentially $K_{r+1} - E(P_2)$-graphic, then π satisfies one of (1)–(3) of Theorem 1.4. If π is not potentially $K_{r+1} - E(2K_2)$-graphic and π is not potentially $K_{r+1} - E(P_2)$-graphic, then π is potentially $A_{r+1} - E(P_3)$-graphic, we let G be a realization of π with the vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ such that $d_G(v_i) = d_i$ for $1 \le i \le n$ and $G[\{v_1, v_2, \ldots, v_{r+1}\}] = K_{r+1} - E(P_3)$ (denoted by H) so that $d_H(v_{r-2}) = d_H(v_{r-1}) = r - 1$ and $d_H(v_{r+1}) = d_H(v_r) = r - 2$. The removal of the edges induced by $\{v_1, v_2, \ldots, v_{r+1}\}$ results in a graph G' in which all degrees in $\{v_1, v_2, \ldots, v_{r-3}\}$ are reduced by r, both degrees in $\{v_{r-2}, v_{r-1}\}$ are reduced by $r - 1$ and both degrees in $\{v_r, v_{r+1}\}$ are reduced by $r - 2$. Let $d'_{r-2} \ge d'_{r-1} \ge d'_r \ge d'_{r+1}$ be the rearrangement in non-increasing order of $d_{r-2} - r + 1, d_{r-1} - r + 1, d_r - r + 2, d_{r+1} - r + 2$. For $0 \le p \le r-3$, $0 \le p' \le 4$ and $0 \le q \le n - r - 1$, denote $P = \{v_i | 1 \le i \le p\}$, $P' = \{v | v \in \{v_{r-2}, v_{r-1}, v_r, v_{r+1}\}$ and $d_{G'}(v) \in \{d'_{r-2}, \ldots, d'_{r-3+p'}\}\}$, $R = \{v_i | p+1 \le i \le r-3\}$, $R' = \{v_{r-2}, v_{r-1}, v_r, v_{r+1}\} \setminus P'$, $Q = \{v_i | r+2 \le i \le q+r+1\}$ and $S = \{v_i | q+r+2 \le i \le n\}$.

$$m = \sum_{i=1}^{p}(d_i - r) + \sum_{i=r-2}^{r-3+p'} d'_i + \sum_{i=r+2}^{q+r+1} d_i - (2(p+p')q + q(q-1))$$

is the minimum number of edges of G' with exactly one endvertex in $P \cup P' \cup Q$ and

$$M = \sum_{i=p+1}^{r-3} min\{q, d_i - r\} + \sum_{i=r-2+p'}^{r+1} min\{q, d'_i\} + \sum_{i=q+r+2}^{n} min\{p+p'+q, d_i\}$$

is the maximum number of edges of G' with exactly one endvertex in $R \cup R' \cup S$. Graph G' witnesses that $m \le M$ is true.

We now prove the sufficiency. If π satisfies one of (1)–(4) of Theorem 1.8, then π is potentially $K_{r+1} - E(2K_2)$-graphic by Theorem 1.5 and π is potentially $K_{r+1} - E(P_2)$-graphic by Theorem 1.4, which is sufficient to show that π is potentially $K_{r+1} - E(P_3)$-graphic. Assume that π satisfies (5) of Theorem 1.8. Let $\pi' = (d'_1, \ldots, d'_{r+1}, d'_{r+2}, \ldots, d'_n)$, where $d'_i = d_i - r$ for $1 \le i \le r-3$, $d'_{r-2} = d_{r-2} - r + 1$, $d'_{r-1} = d_{r-1} - r + 1$, $d'_r = d_r - r + 2$, $d'_{r+1} = d_{r+1} - r + 2$ and $d'_i = d_i$ for $r + 2 \le i \le n$. Let H be the graph obtained from K_n with the vertex set $V(K_n) = \{v_1, v_2, \ldots, v_n\}$ by deleting all edges between v_i and v_j for any $i, j \in \{1, 2, \ldots, r+1\}$. It is easy to see that π is potentially $A_{r+1} - E(P_3)$-graphic if and only if H has a subgraph G with the degree sequence π' such that every vertex v_i has degree d'_i. Observe that H satisfies the odd-cycle condition.

Let $K = \{v_1, v_2, \ldots, v_{r-3}\}$, $K' = \{v_{r-2}, v_{r-1}, v_r, v_{r+1}\}$ and $A, B \subseteq V(H)$ such that $A \cap B = \emptyset$. Let $A_1 = A \cap K, A'_1 = A \cap K', A_2 = A \setminus (K \cup$

$K'), B_1 = B \cap K, B_1' = B \cap K', B_2 = B \setminus (K \cup K')$, and set $p = |A_1|, p' = |A_1'|, q = |A_2|, b_1 = |B_1|, b_1' = |B_1'|, b_2 = |B_2|$. For convenience, we denote the left and right hand side of (g) by $L(p, p', q)$ and $R(p, p', q)$, respectively. Let

$$\begin{aligned} L'(A,B) &= \sum_{v_i \in A} d_i' = \sum_{v_i \in A_1} (d_i - r) + \sum_{v_i \in A_1'} d_i' + \sum_{v_i \in A_2} d_i, \\ R'(A,B) &= |\{(v_i, v_j) : v_i v_j \in E(H), v_i \in A, v_j \in V(H) \setminus B\}| + \sum_{v_i \in B} d_i' \\ &= |\{(v_i, v_j) : v_i v_j \in E(H), v_i \in A, v_j \in V(H) \setminus B\}| \\ &\quad + \sum_{v_i \in B_1} (d_i - r) + \sum_{v_i \in B_1'} d_i' + \sum_{v_i \in B_2} d_i. \end{aligned}$$

Clearly, $L'(A, B) \le L(p, p', q)$ and

$$\begin{aligned} &|\{(v_i, v_j) : v_i v_j \in E(H), v_i \in A, v_j \in V(H) \setminus B\}| \\ &= 2(p + p')q + q(q - 1) + q(r - 3 - p - b_1) + q(4 - p' - b_1') \\ &\quad + (p + p' + q)(n - (r + 1) - q - b_2) \\ &= 2(p + p')q + q(q - 1) + \sum_{i=p+1}^{r-3-b_1} q + \sum_{i=r-2+p'}^{r+1-b_1'} q + \sum_{i=q+r+2}^{n-b_2} (p + p' + q). \end{aligned}$$

Therefore,

$$\begin{aligned} R'(A,B) &= |\{(v_i, v_j) : v_i v_j \in E(H), v_i \in A, v_j \in V(H) \setminus B\}| \\ &\quad + \sum_{v_i \in B_1} (d_i - r) + \sum_{v_i \in B_1'} d_i' + \sum_{v_i \in B_2} d_i \\ &\ge 2(p + p')q + q(q - 1) + \sum_{i=p+1}^{r-3-b_1} q + \sum_{i=r-2+p'}^{r+1-b_1'} q \\ &\quad + \sum_{i=q+r+2}^{n-b_2} (p + p' + q) + \sum_{i=r-2-b_1}^{r-3} (d_i - r) \\ &\quad + \sum_{i=r+2-b_1'}^{r+1} d_i' + \sum_{i=n+1-b_2}^{n} d_i \\ &\ge 2(p + p')q + q(q - 1) + \sum_{i=p+1}^{r-3} min\{q, d_i - r\} \\ &\quad + \sum_{i=r-2+p'}^{r+1} min\{q, d_i'\} + \sum_{i=q+r+2}^{n} min\{p + p' + q, d_i\} \\ &= R(p, p', q). \end{aligned}$$

It follows from $L(p, p', q) \le R(p, p', q)$ that $L'(A, B) \le R'(A, B)$. By Theorem 2.1, H has a subgraph G with the degree sequence π' such that every vertex v_i has degree d_i'. □

The proof of Theorem 1.9 Assume that π is potentially $K_{r+1} - E(K_{1,3})$-graphic. If π is potentially $K_{r+1} - E(P_2)$-graphic, then π satisfies one of (1)–(3) of Theorem 1.4. If π is not potentially $K_{r+1} - E(P_2)$-graphic, then π is potentially $A_{r+1} - E(K_{1,3})$-graphic, we let G be a realization of π with the vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ such that $d_G(v_i) = d_i$ for $1 \le i \le n$ and $G[\{v_1, v_2, \ldots, v_{r+1}\}] = K_{r+1} - E(K_{1,3})$ (denoted by H) so that $d_H(v_{r-2}) = d_H(v_{r-1}) = d_H(v_r) = r-1$ and $d_H(v_{r+1}) = r-3$. The removal of the edges induced by $\{v_1, v_2, \ldots, v_{r+1}\}$ results in a graph G' in which all degrees in $\{v_1, v_2, \ldots, v_{r-3}\}$ are reduced by r, all degrees in $\{v_{r-2}, v_{r-1}, v_r\}$ are reduced by $r-1$ and the degree of v_{r+1} is reduced by $r-3$. Let $d'_{r-2} \ge d'_{r-1} \ge d'_r \ge d'_{r+1}$ be the rearrangement in non-increasing order of $d_{r-2}-r+1, d_{r-1}-r+1, d_r-r+1$ and $d_{r+1}-r+3$. For $0 \le p \le r-3$, $0 \le p' \le 4$ and $0 \le q \le n-r-1$, denote $P = \{v_i | 1 \le i \le p\}$, $P' = \{v | v \in \{v_{r-2}, v_{r-1}, v_r, v_{r+1}\}$ and $d_{G'}(v) \in \{d'_{r-2}, \ldots, d'_{r-3+p'}\}\}$, $R = \{v_i | p+1 \le i \le r-3\}$, $R' = \{v_{r-2}, v_{r-1}, v_r, v_{r+1}\} \setminus P'$, $Q = \{v_i | r+2 \le i \le q+r+1\}$ and $S = \{v_i | q+r+2 \le i \le n\}$.

$$m = \sum_{i=1}^{p}(d_i - r) + \sum_{i=r-2}^{r-3+p'} d'_i + \sum_{i=r+2}^{q+r+1} d_i - (2(p+p')q + q(q-1))$$

is the minimum number of edges of G' with exactly one endvertex in $P \cup P' \cup Q$ and

$$M = \sum_{i=p+1}^{r-3} min\{q, d_i - r\} + \sum_{i=r-2+p'}^{r+1} min\{q, d'_i\} + \sum_{i=q+r+2}^{n} min\{p+p'+q, d_i\}$$

is the maximum number of edges of G' with exactly one endvertex in $R \cup R' \cup S$. Graph G' witnesses that $m \le M$ is true.

We now prove the sufficiency. If π satisfies one of (1)–(3) of Theorem 1.9, then π is potentially $K_{r+1} - E(P_2)$-graphic by Theorem 1.4, which is sufficient to show that π is potentially $K_{r+1} - E(K_{1,3})$-graphic. Assume that π satisfies (4) of Theorem 1.9. Let $\pi' = (d'_1, \ldots, d'_{r+1}, d'_{r+2}, \ldots, d'_n)$, where $d'_i = d_i - r$ for $1 \le i \le r-3$, $d'_i = d_i - r + 1$ for $r-2 \le i \le r$, $d'_{r+1} = d_{r+1} - r + 3$ and $d'_i = d_i$ for $r+2 \le i \le n$. Let H be the graph obtained from K_n with the vertex set $V(K_n) = \{v_1, v_2, \ldots, v_n\}$ by deleting all edges between v_i and v_j for any $i, j \in \{1, 2, \ldots, r+1\}$. It is easy to see that π is potentially $A_{r+1} - E(K_{1,3})$-graphic if and only if H has a subgraph G with the degree sequence π' such that every vertex v_i has degree d'_i. Observe that H satisfies the odd-cycle condition.

Let $K = \{v_1, v_2, \ldots, v_{r-3}\}$, $K' = \{v_{r-2}, v_{r-1}, v_r, v_{r+1}\}$ and $A, B \subseteq V(H)$ such that $A \cap B = \emptyset$. Let $A_1 = A \cap K, A'_1 = A \cap K', A_2 = A \setminus (K \cup K'), B_1 = B \cap K, B'_1 = B \cap K', B_2 = B \setminus (K \cup K')$, and set $p = |A_1|, p' = |A'_1|, q = |A_2|, b_1 = |B_1|, b'_1 = |B'_1|, b_2 = |B_2|$. For convenience, we denote the left and right hand side of (h) by $L(p, p', q)$ and $R(p, p', q)$, respectively.

Let

$$\begin{aligned} L'(A,B) &= \sum_{v_i\in A} d'_i = \sum_{v_i\in A_1}(d_i-r) + \sum_{v_i\in A'_1} d'_i + \sum_{v_i\in A_2} d_i, \\ R'(A,B) &= |\{(v_i,v_j): v_iv_j\in E(H), v_i\in A, v_j\in V(H)\setminus B\}| + \sum_{v_i\in B} d'_i \\ &= |\{(v_i,v_j): v_iv_j\in E(H), v_i\in A, v_j\in V(H)\setminus B\}| \\ &\quad + \sum_{v_i\in B_1}(d_i-r) + \sum_{v_i\in B'_1} d'_i + \sum_{v_i\in B_2} d_i. \end{aligned}$$

Clearly, $L'(A,B)\le L(p,p',q)$ and

$$\begin{aligned} &|\{(v_i,v_j): v_iv_j\in E(H), v_i\in A, v_j\in V(H)\setminus B\}| \\ &= 2(p+p')q + q(q-1) + q(r-3-p-b_1) + q(4-p'-b'_1) \\ &\quad +(p'+q+p)(n-(r+1)-b_2-q) \\ &= 2(p+p')q + q(q-1) + \sum_{i=p+1}^{r-3-b_1} q + \sum_{i=r-2+p'}^{r+1-b'_1} q + \sum_{i=q+r+2}^{n-b_2}(p+p'+q). \end{aligned}$$

Therefore,

$$\begin{aligned} R'(A,B) &= |\{(v_i,v_j): v_iv_j\in E(H), v_i\in A, v_j\in V(H)\setminus B\}| \\ &\quad + \sum_{v_i\in B_1}(d_i-r) + \sum_{v_i\in B'_1} d'_i + \sum_{v_i\in B_2} d_i \\ &\ge 2(p+p')q + q(q-1) + \sum_{i=p+1}^{r-3-b_1} q + \sum_{i=r-2+p'}^{r+1-b'_1} q \\ &\quad + \sum_{i=q+r+2}^{n-b_2}(p+p'+q) + \sum_{i=r-2-b_1}^{r-3}(d_i-r) \\ &\quad + \sum_{i=r+2-b'_1}^{r+1} d'_i + \sum_{i=n+1-b_2}^{n} d_i \\ &\ge 2(p+p')q + q(q-1) + \sum_{i=p+1}^{r-3} min\{q, d_i-r\} \\ &\quad + \sum_{i=r-2+p'}^{r+1} min\{q, d'_i\} + \sum_{i=r+2+q}^{n} min\{p+p'+q, d_i\} \\ &= R(p,p',q). \end{aligned}$$

It follows from $L(p,p',q)\le R(p,p',q)$ that $L'(A,B)\le R'(A,B)$. By Theorem 2.1, H has a subgraph G with the degree sequence π' such that every vertex v_i has degree d'_i. Hence π is potentially $A_{r+1}-E(K_{1,3})$-graphic. □

The proof of Theorem 1.10 Assume that π is potentially $K_{r+1}-E(K_2\cup P_2)$-graphic. If π is potentially $K_{r+1}-E(2K_2)$-graphic, then π satisfies one of (1)–(3) of Theorem 1.5. If π is potentially $K_{r+1}-E(P_2)$-graphic, then π satisfies one of (1)–(3) of Theorem 1.4. If π is not potentially $K_{r+1}-E(2K_2)$-graphic and π is not potentially $K_{r+1}-E(P_2)$-graphic, then π is potentially $A_{r+1}-E(K_2\cup P_2)$-graphic, we let G be a realization of π

with the vertex set $V(G)=\{v_1,v_2,\dots,v_n\}$ such that $d_G(v_i)=d_i$ for $1\le i\le n$ and $G[\{v_1,v_2,\dots,v_{r+1}\}]=K_{r+1}-E(K_2\cup P_2)$ (denoted by H) so that $d_H(v_{r-3})=d_H(v_{r-2})=d_H(v_{r-1})=d_H(v_r)=r-1$ and $d_H(v_{r+1})=r-2$. The removal of the edges induced by $\{v_1,v_2,\dots,v_{r+1}\}$ results in a graph G' in which all degrees in $\{v_1,v_2,\dots,v_{r-4}\}$ are reduced by r, all degrees in $\{v_{r-3},v_{r-2},v_{r-1},v_r\}$ are reduced by $r-1$ and the degree of v_{r+1} is reduced by $r-2$. Let $d'_{r-3}\ge d'_{r-2}\ge d'_{r-1}\ge d'_r\ge d'_{r+1}$ be the rearrangement in non-increasing order of $d_{r-3}-r+1, d_{r-2}-r+1, d_{r-1}-r+1, d_r-r+1$ and $d_{r+1}-r+2$. For $0\le p\le r-4$, $0\le p'\le 5$ and $0\le q\le n-r-1$, denote $P=\{v_i|1\le i\le p\}$, $P'=\{v|v\in\{v_{r-3},v_{r-2},v_{r-1},v_r,v_{r+1}\}$ and $d_{G'}(v)\in\{d'_{r-3},d'_{r-2},\dots,d'_{r-4+p'}\}\}$, $R=\{v_i|p+1\le i\le r-4\}$, $R'=\{v_{r-3},v_{r-2},v_{r-1},v_r,v_{r+1}\}\setminus P'$, $Q=\{v_i|r+2\le i\le q+r+1\}$ and $S=\{v_i|q+r+2\le i\le n\}$.

$$m=\sum_{i=1}^{p}(d_i-r)+\sum_{i=r-3}^{r-4+p'}d'_i+\sum_{i=r+2}^{q+r+1}d_i-(2(p+p')q+q(q-1))$$

is the minimum number of edges of G' with exactly one endvertex in $P\cup P'\cup Q$ and

$$M=\sum_{i=p+1}^{r-4}min\{q,d_i-r\}+\sum_{i=r-3+p'}^{r+1}min\{q,d'_i\}+\sum_{i=q+r+2}^{n}min\{p+p'+q,d_i\}$$

is the maximum number of edges of G' with exactly one endvertex in $R\cup R'\cup S$. Graph G' witnesses that $m\le M$ is true.

We now prove the sufficiency. If π satisfies one of (1)–(4) of Theorem 1.10, then π is potentially $K_{r+1}-E(2K_2)$-graphic by Theorem 1.5 and π is potentially $K_{r+1}-E(P_2)$-graphic by Theorem 1.4, which is sufficient to show that π is potentially $K_{r+1}-E(K_2\cup P_2)$-graphic. Assume that π satisfies (5) of Theorem 1.10. Let $\pi'=(d'_1,\dots,d'_{r+1},d'_{r+2},\dots,d'_n)$, where $d'_i=d_i-r$ for $1\le i\le r-4$, $d'_i=d_i-r+1$ for $r-3\le i\le r$, $d'_{r+1}=d_{r+1}-r+2$ and $d'_i=d_i$ for $r+2\le i\le n$. Let H be the graph obtained from K_n with the vertex set $V(K_n)=\{v_1,v_2,\dots,v_n\}$ by deleting all edges between v_i and v_j for any $i,j\in\{1,2,\dots,r+1\}$. It is easy to see that π is potentially $A_{r+1}-E(K_2\cup P_2)$-graphic if and only if H has a subgraph G with the degree sequence π' such that every vertex v_i has degree d'_i. Observe that H satisfies the odd-cycle condition.

Let $K=\{v_1,v_2,\dots,v_{r-4}\}$, $K'=\{v_{r-3},v_{r-2},v_{r-1},v_r,v_{r+1}\}$ and $A,B\subseteq V(H)$ such that $A\cap B=\emptyset$. Let $A_1=A\cap K, A'_1=A\cap K', A_2=A\setminus(K\cup K'), B_1=B\cap K, B'_1=B\cap K', B_2=B\setminus(K\cup K')$, and set $p=|A_1|, p'=|A'_1|, q=|A_2|, b_1=|B_1|, b'_1=|B'_1|, b_2=|B_2|$. For convenience, we denote the left and right hand side of (i) by $L(p,p',q)$ and

$R(p,p',q)$, respectively. Let

$$\begin{aligned} L'(A,B) &= \sum_{v_i\in A} d'_i = \sum_{v_i\in A_1}(d_i-r)+\sum_{v_i\in A'_1} d'_i+\sum_{v_i\in A_2} d_i, \\ R'(A,B) &= |\{(v_i,v_j): v_iv_j\in E(H), v_i\in A, v_j\in V(H)\setminus B\}|+\sum_{v_i\in B} d'_i \\ &= |\{(v_i,v_j): v_iv_j\in E(H), v_i\in A, v_j\in V(H)\setminus B\}| \\ &\quad +\sum_{v_i\in B_1}(d_i-r)+\sum_{v_i\in B'_1} d'_i+\sum_{v_i\in B_2} d_i. \end{aligned}$$

Clearly, $L'(A,B)\le L(p,p',q)$ and

$$\begin{aligned} &|\{(v_i,v_j): v_iv_j\in E(H), v_i\in A, v_j\in V(H)\setminus B\}| \\ &= 2(p+p')q+q(q-1)+q(r-4-p-b_1)+q(5-p'-b'_1) \\ &\quad +(p'+q+p)(n-(r+1)-b_2-q) \\ &= 2(p+p')q+q(q-1)+\sum_{i=p+1}^{r-4-b_1} q+\sum_{i=r-3+p'}^{r+1-b'_1} q+\sum_{i=q+r+2}^{n-b_2}(p+p'+q). \end{aligned}$$

Therefore,

$$\begin{aligned} R'(A,B) &= |\{(v_i,v_j): v_iv_j\in E(H), v_i\in A, v_j\in V(H)\setminus B\}| \\ &\quad +\sum_{v_i\in B_1}(d_i-r)+\sum_{v_i\in B'_1} d'_i+\sum_{v_i\in B_2} d_i \\ &\ge 2(p+p')q+q(q-1)+\sum_{i=p+1}^{r-4-b_1} q+\sum_{i=r-3+p'}^{r+1-b'_1} q \\ &\quad +\sum_{i=q+r+2}^{n-b_2}(p+p'+q)+\sum_{i=r-3-b_1}^{r-4}(d_i-r) \\ &\quad +\sum_{i=r+2-b'_1}^{r+1} d'_i+\sum_{i=n+1-b_2}^{n} d_i \\ &\ge 2(p+p')q+q(q-1)+\sum_{i=p+1}^{r-4} min\{q,d_i-r\} \\ &\quad +\sum_{i=r-3+p'}^{r+1} min\{q,d'_i\}+\sum_{i=r+2+q}^{n} min\{p+p'+q,d_i\} \\ &= R(p,p',q). \end{aligned}$$

It follows from $L(p,p',q)\le R(p,p',q)$ that $L'(A,B)\le R'(A,B)$. By Theorem 2.1, H has a subgraph G with the degree sequence π' such that every vertex v_i has degree d'_i. □

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