Some results on transverse Steiner quadruple systems of type $g^t u^1$

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Abstract

A transverse Steiner quadruple system or *TSQS* is a triple $(X, \mathcal{H}, \mathcal{B})$ where X is a v-element set of *points*, $\mathcal{H} = \{H_1, H_2, \ldots, H_r\}$ is a partition of X into *holes* and $\mathcal{B}$ is a collection of transverse 4-element subsets with respect to $\mathcal{H}$ called *blocks* such that every transverse 3-element subset is in exactly one block. In this article, transverse Steiner quadruple systems with t holes of size g and 1 hole of size u are studied. Constructions based on the use of s-fans are given, including a construction for quadrupling the number of holes of size g. New results on systems with 6 and 11 holes are obtained, and constructions for $TSQS(x^n(2x)^1)$ and $TSQS(4^n2^1)$ are provided.

1 Introduction

Given a partition $\mathcal{H} = \{H_1, H_2, \ldots, H_r\}$ of a set X, a subset $T \subseteq X$ is *transverse* with respect to $\mathcal{H}$ if $|T \cap H_i| = 0$ or 1 for each $i = 1, 2, \ldots, r$. A *transverse t-design* with parameters t-(v, k, λ) is a triple $(X, \mathcal{H}, \mathcal{B})$ where X is a v-element set of *points*, $\mathcal{H} = \{H_1, H_2, \ldots, H_r\}$ is a partition of X into *holes* and $\mathcal{B}$ is a collection of transverse k-element subsets with respect to $\mathcal{H}$ called *blocks* such that every transverse t-element subset is in exactly λ blocks. The transverse t-(v, k, λ) designs of type 1^v are the (ordinary) t-designs. A transverse t-(v, k, λ) design with $\lambda = 1$, is a *transverse Steiner system.* A *transverse Steiner triple system* or transverse STS(v) is a transverse 2-$(v, 3, 1)$ design. The focus in this article is on transverse 3-$(v, 4, 1)$ designs, which are refered to as *transverse Steiner quadruple systems* or $TSQS(v)$. Let $h_i = |H_i|$ be the size of the hole $H_i \in \mathcal{H}$. The *type* of a transverse t-design is the multi-set $\{h_1, h_2, \ldots, h_r\}$ of hole sizes. It is custom to write $s_1^{u_1} s_2^{u_2} \ldots s_m^{u_m} = h_1 h_2 \cdots h_r$ for the type of a transverse t-design with u_i holes of size s_i, $i = 1, 2, \ldots, m$. If all the holes have the same size h, then the transverse t-design is said to be *uniform.* Such a design would have type h^u for some u. A 3-$(12, 4, 1)$ design transverse to the holes $\mathcal{H} = \{\{0, 1\}, \{2, 3\}, \{4, 5\}, \{6, 7\}, \{a, b, c, d\}\}$ is displayed in Figure 1. This is an example of a TSQS of type $2^4 4^1$ with holes $\{0,1\}$, $\{2,3\}$, $\{4,5\}$, $\{6,7\}$, $\{a,b,c,d\}$.

$\{3,5,7,a\}$ $\{3,5,6,b\}$ $\{3,4,7,d\}$ $\{3,4,6,c\}$ $\{2,5,7,c\}$ $\{2,5,6,d\}$
$\{2,4,7,b\}$ $\{2,4,6,a\}$ $\{1,5,7,b\}$ $\{1,5,6,c\}$ $\{1,4,7,a\}$ $\{1,4,6,d\}$
$\{1,3,7,c\}$ $\{1,3,6,a\}$ $\{1,3,5,d\}$ $\{1,3,4,b\}$ $\{1,2,7,d\}$ $\{1,2,6,b\}$
$\{1,2,5,a\}$ $\{1,2,4,c\}$ $\{0,5,7,d\}$ $\{0,5,6,a\}$ $\{0,4,7,c\}$ $\{0,4,6,b\}$
$\{0,3,7,b\}$ $\{0,3,6,d\}$ $\{0,3,5,c\}$ $\{0,3,4,a\}$ $\{0,2,7,a\}$ $\{0,2,6,c\}$
$\{0,2,5,b\}$ $\{0,2,4,d\}$

Figure 1: A TSQS of type $2^4 4^1$

If $(X,\mathcal{H},\mathcal{B})$ is a transverse t-(v,k,λ) design, and $x \in H \in \mathcal{H}$, then $(X',\mathcal{H}',\mathcal{B}')$ is a transverse $(t-1)$-$(v-|H|,k-1,\lambda)$ design, where:

$$\begin{aligned} X' &= X \setminus \{H\}, \\ \mathcal{H}' &= \mathcal{H} \setminus \{H\} \text{ and} \\ \mathcal{B}' &= \{B \setminus \{x\} : x \in B \in \mathcal{B}\}. \end{aligned}$$

The design $(X',\mathcal{H}',\mathcal{B}')$ is called the *derived design* of $(X,\mathcal{H},\mathcal{B})$ with respect to x.

A transverse 2-$(v,k,1)$ design is also called a *group divisible design* (GDD). Let K and G be sets of positive integers and let λ be a positive integer. A *group divisible design of index* λ *and order* v ((K,λ)-GDD) is a triple $(V,\mathcal{G},\mathcal{B})$, where V is a finite set of cardinality v, $\mathcal{G}$ is a partition of V into parts (*groups*) whose sizes lie in G, and $\mathcal{B}$ is a family of subsets (*blocks*) of V which satisfy the properties:

1. If $B \in \mathcal{B}$, then $|B| \in \mathcal{K}$.
2. Every pair of distinct elements of V occurs in exactly λ blocks or one group, but not both.
3. $|\mathcal{G}| > 1$.

If $K = \{k\}$, then the (K,λ)-GDD is a (k,λ)-GDD. If $\lambda = 1$, the GDD is denoted by K-GDD. Furthermore, a $(\{k\},1)$-GDD is a k-GDD. Necessary and sufficient conditions for the existence of a 3-GDD of type t^u were proved by Hanani in 1975.

Theorem 1.1 (Hanani, 1975 [9]) *Let u and t be positive integers. There exists a 3-GDD of type t^u if and only if $u \geq 3$ and the conditions in the following table are satisfied.*

t	u
1, 5 (mod 6)	1, 3 (mod 6)
2, 4 (mod 6)	0, 1 (mod 3)
3 (mod 6)	1 (mod 2)
0 (mod 6)	no constraint

The uniform case for $TSQS$ has been studied extensively. It is easy to construct a $TSQS$ of type w^4. Simply take as the holes $H_i = \{(x,i) : x \in Z_w\}$ for $i = 1,2,3,4$ and blocks $\mathcal{B} = \{(x,1),(x,2),(x,3),(x,4) : x_1 + x_2 + x_3 + x_4 = 0\}$. Verifying that this indeed is a transverse SQS of type w^4 is straightforward. The following theorem is a generalization of this and was proved by Mills in 1990.

Theorem 1.2 (Mills, 1990 [20]) *For $u \geq 4$, $u \neq 5$ a* TSQS *of type h^u exists if and only if hu is even and $h(u-1)(u-2) \equiv 0 \pmod 3$.*

With regards to the case $u = 5$, in 2003 Lauinger, et. al. [18] showed there exists a $TSQS$ of type h^5 for all $h \equiv 0, 4, 6$, or $8 \pmod{12}$. This result has been recently improved by L. Ji [12], who showed in [13] that there exists a TSQS of type h^5 if h is even and $h \not\equiv 10$ or $26 \pmod{48}$.

The necessary and sufficient conditions for the existence of a 3-GDD of type $g^t u^1$ were established by Colbourn, Hoffman and Rees in 1992.

Theorem 1.3 (Colbourn, Hoffman, Rees, 1992 [7]) *Let g, t, and u be nonnegative integers. There exists a 3-*GDD *of the type $g^t u^1$ if and only if the following conditions are satisfied:*

1. *if $g > 0$, then $t \geq 3$, or $t = 2$ and $u = g$, or $t = 1$ and $u = 0$, or $t = 0$;*
2. *$u \leq g(t-1)$ or $gt = 0$;*
3. *$g(t-1) + u \equiv 0 \pmod 2$ or $gt = 0$;*
4. *$gt \equiv 0 \pmod 2$ or $u = 0$;*
5. *$\frac{1}{2}g^2 t(t-1) + gtu \equiv 0 \pmod 3$.*

In this article the focus is on $TSQS(g^t u^1)$. The necessary conditions for the existence of a $TSQS$ of type $g^t u^1$ were given in [16].

Theorem 1.4 (Keranen, Kreher, 2007 [16]) *If a $TSQS(g^t u^1)$ exists where $t \geq 4$, then $0 \leq u \leq g(t-2)$, $\binom{t}{3}g^3 + \binom{t}{2}g^2 u \equiv 0 \pmod 4$, and*

1. *if $g \equiv 0 \pmod 6$, then u is even;*
2. *if $g \equiv 1$ or $5 \pmod 6$, then $t \equiv 1$ or $3 \pmod 6$ and*
   - *a.) if $t \equiv 1 \pmod 6$, then u is odd;*
   - *b.) if $g \equiv 1 \pmod 6$ and $t \equiv 3 \pmod 6$, then $u \equiv 1 \pmod 6$;*
   - *c.) if $g \equiv 5 \pmod 6$ and $t \equiv 3 \pmod 6$, then $u \equiv 5 \pmod 6$;*
3. *if $g \equiv 2$ or $4 \pmod 6$, then $t \equiv 0$ or $1 \pmod 3$ and*
   - *a.) if $t \equiv 0 \pmod 3$, then $u \equiv g \pmod 6$;*
   - *b.) if $t \equiv 1 \pmod 3$, then u is even;*
4. *if $g \equiv 3 \pmod 6$, then $t \equiv 1 \pmod 2$ and u is odd.*

One way to obtain $TSQS$ with t holes of size g and 1 hole of size u is to begin with a large set of transverse Steiner triple systems of type g^t and then adjoin new point to each of the triples in each system. Lauinger, Kreher, Rees, and Stinson were able to obtain the following result using this idea.

Theorem 1.5 (Lauinger, et. al., 2005 [18] *There exists a* $TSQS(m^s((s-2)m)^1)$ *if and only if* $s(s-1)m^2 \equiv 0 \pmod 6$, $(s-1)m \equiv 0 \pmod 2$, *and* $(m,s) \neq (1,7)$.

Following are some more useful results on small $TSQS$ which can be found in [18].

Theorem 1.6 (Lauinger, et. al.,2003 [18]) *If there exists a* $TSQS$ *of type* $\{h_1, h_2, \ldots, h_k\}$, *then there exists a* $TSQS$ *of type* $\{wh_1, wh_2, \ldots, wh_k\}$ *for all* $w \geq 1$.

Theorem 1.7 (Lauinger, et. al.,2003 [18]) *If* mn *is even and there exists a* $TSQS$ *of type* $(mn)^r(s+t)^1$ *and a* $TSQS$ *of type* $m^n s^1 t^1$ *then there exists a* $TSQS$ *of type* $m^{rn} s^1 t^1$.

Theorem 1.8 *There exists TSQSs of types* $2^4 4^1$, $2^7 4^1$, $2^1 4^4$, *and* $3^5 9^1$.

2 New Results on $TSQS(g^t u^1)$

2.1 Six Holes

Keranen and Kreher established a number of results on $TSQS$ with 5 holes. In this section, similar methods are used to provide a result on $TSQS$ with 6 holes.

Theorem 2.1 *(Keranen, Kreher, 2007 [16], [17]) If* $g \equiv 0 \pmod 4$, *then there exists a* $TSQS(g^4 u^1)$ *if and only if* u *is even and* $0 \leq u \leq 2g$, *except possibly when* $g^4 u^1$ *is one of the following exceptions:*

1. $8^4 u^1$, $u = 2, 6, 10, 14$,
2. $12^4 u^1$, $u = 2, 4, 8, 10, 14, 16, 20, 22$,
3. $24^4 u^1$, $u \equiv 2, 10 \pmod{12}$, $2 \leq u \leq 48$,
4. $40^4 u^1$, $u \equiv 2 \pmod 4$, $2 \leq u \leq 80$,
5. $(4n)^4 u^1$, $n \equiv 2, 10 \pmod{12}$, $n > 10$ *and* $u \equiv 2 \pmod 4$, $10 \leq u \leq 8n$,
6. $(4n)^4 u^1$, $n \equiv 6 \pmod{12}$, $n > 6$ *and* $u \equiv 2, 10 \pmod{12}$, $10 \leq u \leq 8n$.

Theorem 2.2 (Keranen, Kreher, 2007 [16]) *If* $g \equiv 2 \pmod 4$, $g \neq 6$, u *is doubly even, and* $0 \leq u \leq 2g$, *there exists a* $\text{TSQS}(g^4 u^1)$.

An *orthogonal array* of *size* N, with k *constraints* (or of *degree* k), s *levels* (or of *order* s), and *strength* t, denoted $\text{OA}_\lambda(N, k, s, t)$, is a $k \times N$ array with entries from a set of $s \geq 2$ symbols, having the property that in every $t \times N$ sub-matrix, every $t \times 1$ column vector appears the same number $\lambda = \frac{N}{s^t}$ times. The parameter λ is the *index* of the orthogonal array. An $\text{OA}_\lambda(N, k, s, t)$ is also denoted by $\text{OA}_\lambda(t, k, s)$; in this notation, if t is omitted it is understood to be 2, and if λ is omitted it is understood to be 1. The following theorem, related to orthogonal arrays was proved by Bush in 1952 [5].

Theorem 2.3 (Bush, 1952 [5]) *If s is a prime power and $t < s$, then an* $\mathrm{OA}_1(t, s+1, s)$ *exists. Moreover, if $s \geq 4$ is a power of 2, an* $\mathrm{OA}_1(3, s+2, s)$ *exists.*

A uniform transverse t-(kh, k, λ) design of type h^k is equivalent to an orthogonal array of order h, strength t, index λ and degree k. The following theorem gives the main construction that Keranen and Kreher used to obtain a number of results on $TSQS$ with five holes (see [16]).

Theorem 2.4 (Keranen, Kreher, 2007 [16]) *If there exists an* $\mathrm{OA}(3, k+1, n)$ *and a* $\mathrm{TSQS}(w^k u_i^1)$ *for all* $i = 1, 2, \cdots, n$, *then there exists a* $\mathrm{TSQS}((wn)^k (\sum_{i=1}^{n} u_i)^1)$.

Ji and Yin have recently found new orthogonal arrays of strength 3 and degree 6.

Theorem 2.5 *(Ji, Yin, [14]) Let v be a positive integer which satisfies $gcd(v, 4) \neq 2$ and $gcd(v, 18) \neq 3$. Then there is an $OA(3, 6, v)$.*

These OAs can be used as ingredients for constructing $TSQS$ with 6 holes. It is known that there exist $TSQS(3^6)$ by Theorem 1.2 and $TSQS(3^5 9^1)$ by Theorem 1.8. Therefore, applying Theorem 2.4 with $k = 5$, $w = 3$, and $u_i \in \{3, 9\}$ yields the following result.

Theorem 2.6 *There exists a $TSQS((3v)^5 (\sum_{i=1}^{n} u_i)^1)$ for $u_i \in \{3, 9\}$ for all v such that $gcd(v, 4) \neq 2$ and $gcd(v, 18) \neq 3$.*

2.2 *g-FSQS*

In a Steiner quadruple system $(X, \mathcal{B})$, if the block set $\mathcal{B}$ can be partitioned into disjoint subsets $\mathcal{B}_1, \cdots, \mathcal{B}_s$ and $\mathcal{A}$ such that each $(X, \mathcal{B}_i)$ is a 2-$(v, 4, 1)$ for $1 \leq i \leq s$, then the SQS is called an *s-fan* denoted by s-FSQS(v). In a 1-FSQS(v), the 2-$(v, 4, 1)$ is also called a *spanning block design*. A TSQS of type r^m is called a *transverse s*-FSQS *of type* r^m if its block set $\mathcal{B}$ can be partitioned into disjoint subsets $\mathcal{B}_1, \cdots, \mathcal{B}_s$ and $\mathcal{A}$ such that each $\mathcal{B}_i$ is the block set of a transverse 2-$(v, 4, 1)$ of type r^m for $1 \leq i \leq s$. Ji and Zhu established the following in [15].

Lemma 2.7 (Ji, Zhu, 2003 [15]) *For $v \geq 4$ and $v \neq 6, 10$, there exists a 1-FSQS of type v^4.*

Lemma 2.8 (Ji, Zhu, 2003 [15]) *There exists a transverse $1 - FSQS$ of type* 4^{10}.

These additional results on s-fans can be found in [13], [1], and [23].

Theorem 2.9 (Ji, [13]) *A $1 - FSQS(v)$ exists if and only if $v = 12k + 4$ is a positive integer.*

Theorem 2.10 (Baker, 1976 [1] and Teirlinck, 1994 [23]) *There is a $(\frac{v-2}{2}) - FSQS(v)$ when $v = 4^n$, or $v = 2(q^n + 1)$ for all $n \geq 1$ and $q \in \{7, 31\}$.*

The following useful lemmas are based on the use of g-$FSQS$.

Lemma 2.11 *If there exists a $g-FSQS(n)$ and a $TSQS(x^4u_i^1)$, for all $i = 1,2,\cdots,g$, then there exists a $TSQS(x^n(\sum_{i=1}^g u_i)^1)$.*

Proof. A $g-FSQS(v)$ is equivalent to a SQS(V,B) that contains g disjoint $2-(v,4,1)$ designs, $D_1, D_2, \cdots, D_g$. Write $u = \sum_{i=1}^g u_i$. Let $X = \{1,2,3,\cdots,x\}$ and $U = \{1,2,\cdots,u\}$. Partition U into subsets $U_1, U_2, \cdots, U_g$ where $|U_i| = u_i$. For each block $B = \{a,b,c,d\} \in D_i$, construct a $TSQS(x^4u_i^1)$ on $(B \times X) \cup U_i$ with holes $\{a\} \times X$, $\{b\} \times X$, $\{c\} \times X$, $\{d\} \times X$, U_i. A $TSQS(x^4)$ exists for all x (see Section 1), so for each block $B = \{a,b,c,d\} \notin D_i$, construct a $TSQS(x^4)$ on $(B \times X)$ with holes $\{a\} \times X$, $\{b\} \times X$, $\{c\} \times X$, $\{d\} \times X$. This makes a $TSQS(x^nu^1)$ on $(V \times X) \cup U$. □

Lemma 2.12 *If there exists a transverse $1-FSQS$ of type v^n and there exists a $TSQS(x^4u^1)$, then there exists a $TSQS((vx)^nu^1)$.*

Proof. A transverse $1-FSQS$ of type $(v)^n$ is equivalent to a $TSQS(v^n)$ that contains a transverse $2-(vn,4,1)$ design of type v^n. Let $W = \{1,2,\cdots,x\}$ and $U = \{1,2,\cdots,u\}$. For each block $B = \{a,b,c,d\}$ in the 2-design, construct a $TSQS(x^4u^1)$ on $(B \times W) \cup U$ with holes $\{a\} \times W$, $\{b\} \times W$, $\{c\} \times W$, $\{d\} \times W$, U. A $TSQS(x^4)$ exists for all x (see Section 1), so for each block $B = \{a,b,c,d\}$ not in the 2-design, construct a $TSQS(x^4)$ on $B \times W$ with holes $\{a\} \times W$, $\{b\} \times W$, $\{c\} \times W$, $\{d\} \times W$. □

Because there exists a transverse $1-FSQS$ of type 4^{10} by Theorem 2.8, and there exists a $TSQS(x^4u^1)$ whenever the conditions in Theorems 2.1 and 2.2 are satisfied, Lemma 2.12 can be applied with $v = 4$ and $n = 10$ to obtain the following.

Theorem 2.13 *There exists a $TSQS((4x)^{10}u^1)$ whenever $x \equiv 0 \pmod 4$, u is even and $0 \le u \le 2x$, except possibly when (x,u) is one of the following:*

1. $x = 8$ *and* $u \in \{2,6,10,14\}$
2. $x = 12$ *and* $u \in \{2,4,8,10,14,16,20,22\}$
3. $x = 24$ *and* $u \equiv 2,10 \pmod{12}$, $2 \le u \le 48$
4. $x = 40$ *and* $u \equiv 2 \pmod 4$, $2 \le u \le 80$
5. $x = 4n$, $n \equiv 2,10 \pmod{12}$, $n > 10$ *and* $u \equiv 2 \pmod 4$, $10 \le u \le 8n$
6. $x = 4n$, $n \equiv 6 \pmod{12}$, $n > 6$ *and* $u \equiv 2,10 \pmod{12}$, $10 \le u \le 8n$.

Furthermore, there exists a $TSQS((4x)^{10}u^1)$ whenever $x \equiv 2 \pmod 4$, $x \ne 6$, u is doubly even, and $0 \le u \le 2x$.

Consider any pair of holes H_1 and H_2 that have the same even cardinality m. Let $\mathcal{F}_i = \{F_{i_1}, F_{i_2}, \cdots, F_{i_{m-1}}\}$ be a one-factorization of the complete graph G_i on H_i, $i = 1,2$. Pair the one-factors of $\mathcal{F}_1$ and $\mathcal{F}_2$ to construct blocks. That is for $i \ne j$ take as blocks $\{a,b,c,d\}$ where $\{a,b\} \in \mathcal{F}_{i_k}$ and $\{c,d\} \in \mathcal{F}_{j_k}$, $k = 1,2,\ldots,m-1$. These blocks will cover triples consisting of two points from H_1 and one point from H_2 or one point from H_1 and two points from H_2. This construction is referred to as the *doubling one-factorization* or *DOF* construction. Following is a construction for quadrupling the number of holes of size g which is based on a variation of this.

Theorem 2.14 *If there exists TSQSs of type $g^t u^1$, g^4, and $(tg)^4 u^1$ where t is even, then there exists a $TSQS(g^{4t}u^1)$.*

Proof. Let H_1, H_2, H_3, H_4 and X be disjoint sets with $|H_i| = tg$ for $i = 1,2,3,4$, and $|X| = u$. Construct a $TSQS((tg)^4u^1)$ where the H_is and X are the holes. For each $i = 1,2,3,4$, partition H_i into t subsets $H_{i_1}, H_{i_2}, \ldots, H_{i_t}$ where $|H_{i_j}| = g$ for $j = 1,2,\ldots,t$. Construct a $TSQS(g^tu^1)$ on each $H_i \cup X$ with holes $H_{i_1}, H_{i_2}, \cdots, H_{i_t}, X$. Now apply the DOF construction to each pair of holes, H_i and H_j as follows. Construct the complete graph G_i and G_j on H_i and H_j with the points $H_{i_1}, H_{i_2}, \ldots, H_{i_t}$ and $H_{j_1}, H_{j_2}, \ldots, H_{i_t}$ respectively. Let $\mathcal{F}_l = \{F_{l_1}, F_{l_2}, \ldots, F_{l_{t-1}}\}$ be a one-factorization of G_l for $l = i$ and $l = j$. Pair the one-factors of $\mathcal{F}_i$ and $\mathcal{F}_j$ to obtain blocks

$$\{H_{i_1}, H_{i_2}, H_{j_1}, H_{j_2} : \{H_{i_1}, H_{i_2}\} \in \mathcal{F}_i, \text{ and } \{H_{j_1}, H_{j_2}\} \in \mathcal{F}_j\}.$$

On each such block, construct a $TSQS(g^4)$. □

There exists a $TSQS((4m)^4)$ by Theorem 1.2. Also, by Theorem 2.1, there exists a $TSQS((4m)^4 2^1)$ for $m \geq 4$, and there exists a $TSQS((4 \cdot 4m)^4 2^1)$ for $m \geq 1$. So apply Theorem 2.14 with $g = 4m$, $t = 4$ and $u = 2$ to obtain a $TSQS((4m)^{16} 2^1)$. This theorem can be applied again with $t = 16$ and $u = 2$ to get a $TSQS((4m)^{64} 2^1)$. Thus recursively applying this theorem and noting the existence of a $TSQS(4^4 2^1)$ from Theorem 1.8 yields the following result.

Corollary 2.15 *There exists a $TSQS((4m)^{4^k} 2^1)$ for all $m, k \geq 1$, $m \neq 2$ or 3.*

2.3 $TSQS(x^n(2x)^1)$

In this section, necessary and sufficient conditions for the existence of a $TSQS(2^n 4^1)$ are established. As a consequence, results on $TSQS$ of type $(2n)^n u^1$ are obtained. The technique is also extended to obtain some $TSQS$s of type $x^n(2x)^1$. This first result uses a $g - FSQS$ to obtain $TSQS$ with v holes of size 2 and one long hole.

Theorem 2.16 *There exists a $TSQS(2^v u^1)$ for every $0 \leq u \leq 2(v-2)$ such that $u \equiv 0 \pmod 4$ when $v = 4^n$, or $v = 2(q^n+1)$ for all $n \geq 1$ and $q \in \{7, 31\}$.*

Proof. For these v, there exists a $(\frac{v-2}{2}) - FSQS(v)$ by Theorem 2.10. So apply Lemma 2.11 with $g = (v-2)/2$, $x = 2$, and $u_i = \{0, 4\}$. □

Theorem 2.17 *There exists a $TSQS(2^n 4^1)$ if and only if $n \equiv 1 \pmod 3$.*

Proof. Suppose there exists a $TSQS(2^n 4^1)$. Derive with respect to a point, x, in the hole of size 4 to get a transverse $2 - (2n, 3, 1)$ design of type 2^n, (i.e. a $\{3\} -$ GDD of type 2^n). By Theorem 1.1, this implies that $n \equiv 0$ or $1 \pmod 3$. Now derive with respect to a point, y, in a hole of size 2 to get a transverse $2 - (2n+2, 3, 1)$ design of type $2^{n-1}4^1$, (i.e. a $\{3\} -$ GDD of type $2^{n-1}4^1$). By Theorem 1.3, this implies that $\frac{1}{2}(2)^2(n-1)(n-2) + 2(n-1)4 \equiv 0 \pmod 3$. This simplifies to $2(n-1)(n+2) \equiv 0 \pmod 3$, so $n \equiv 1 \pmod 3$.

If $r = 1$ or $r = 2$ then there exists a $TSQS(2^4 4^1)$ and a $TSQS(2^7 4^1)$ by Theorem 1.8. There exists a $TSQS(6^{r+1})$, for all $r \geq 0$, $r \neq 2$, by Theorem 1.2. As previously stated, a $TSQS(2^4 4^1)$ exists. So apply Theorem 1.7 with $m = 2$, $n = 3$, $s = 2$, and $t = 4$.

□

The following result is obtained as a consequence of Theorem 2.17.

Corollary 2.18 *There exists a* $TSQS((2n)^n u^1)$ *for all even* u *such that* $0 \leq u \leq 4n$ *whenever* $n \equiv 1 \pmod 3$, $n \neq 4$, *is a prime power.*

Proof. If $n \equiv 1 \pmod 3$ is a prime power, $n \neq 4$, then there exists an $OA(3, n+1, n)$ by Theorem 2.3. By Theorem 2.17, there exists a $TSQS(2^n 4^1)$. It is known by Theorem 1.2 that a $TSQS(2^n)$ exists and a $TSQS(2^{n+1})$ exists. Therefore, apply Theorem 2.4 with $w = 2$ and $u_i \in \{0, 2, 4\}$, where $u = \sum_{i=1}^n u_i$. □

The necessary conditions for the existence of a $TSQS$ of type $x^n(2x)^1$ are as follows.

Theorem 2.19 *If a* $TSQS(x^n(2x)^1)$ *exists where* $n \geq 4$ *and* $0 \leq (2x) \leq x(n-2)$, *then* x *is even, and if* $x \equiv 2$ *or* $4 \pmod 6$ *then* $n \equiv 1 \pmod 3$.

Proof. If a $TSQS(x^n(2x)^1)$ exists, then by Theorem 1.4, because $2x$ is even x must also be even. If $x \equiv 0 \pmod 6$, there are no other conditions. However, if $x \equiv 2$ or $4 \pmod 6$, then $n \equiv 0$ or $1 \pmod 3$. If $n \equiv 0 \pmod 3$, then $(2x) \equiv x \pmod 6$. But $x \equiv 2$ or $4 \pmod 6$, so this case will never happen. Therefore, $n \equiv 1 \pmod 3$. □

Theorem 2.20 *For all even* x *and* $n \geq 0$, *there exists a* $TSQS(x^{3n+1}(2x)^1)$.

Proof. There exists $TSQS$s of types $2^4 4^1$ and $2^7 4^1$ by Theorem 1.8. So applying Theorem 1.6 with $w = \frac{x}{2}$ to each gives $TSQS$s of type $x^4(2x)^1$ and $x^7(2x)^1$.

It is given in Theorem 1.2 that a $TSQS((3x)^{n+1})$ exists for all $n \geq 3$, $n \neq 4$. But $3x \equiv 0$ or $6 \pmod{12}$ for all $x \equiv 2$ or $4 \pmod 6$, so a $(3x)^5$ also exists. (See the discussion following Theorem 1.2.) So for any $n \geq 3$, apply Theorem 1.7 with $m = x$, $n = 3$, $t = 2x$, $s = x$, and $r = n$. □

The above theorem establishes sufficient conditions for the existence of a $TSQS(x^n(2x)^1)$ when $x \equiv 2$ or $4 \pmod 6$. When $x \equiv 0 \pmod 6$, open cases are when $n \equiv 0, 2 \pmod 3$.

2.4 $TSQS(4^n 2^1)$

Theorem 2.21 *If there exists a* $TSQS(4^n 2^1)$ *then* $n \equiv 1 \pmod 3$.

Proof. Suppose there exists a $TSQS(4^n 2^1)$. Derive with respect to a point, x, in the hole of size 2 to obtain a transverse $2-(4n, 3, 1)$ design of type 4^n, (i.e. a $\{3\}-$GDD of type 4^n). By Theorem 1.1, this implies that $n \equiv 0$ or $1 \pmod 3$. Now derive with respect to a point, y, in a hole of size 4 to get a transverse $2-(4n-2, 3, 1)$ design of type $4^{n-1}2^1$, (i.e. a $\{3\}-$GDD of type $4^{n-1}2^1$). By Theorem 1.3, this implies that $\frac{1}{2}(4)^2(n-1)(n-2) + 4(n-1)2 \equiv 0 \pmod 3$. This simplifies to $8(n-1)(n-1) \equiv 0 \pmod 3$, so $n \equiv 1 \pmod 3$. □

The next result gives a method for finding the more general $TSQS(4^v u^1)$, thus it can be applied to $u = 2$.

Theorem 2.22 *There exists a $TSQS(4^v u^1)$ for every even $0 \le u \le v-2$ when $v = 4^n$, or $v = 2(q^n + 1)$ for all $n \ge 1$ and $q \in \{7, 31\}$.*

Proof. For these v, there exists a $(\frac{v-2}{2}) - FSQS(v)$ by Theorem 2.10. So apply Lemma 2.11 with $v = (v-2)/2$, $x = 4$, and $u_i \in \{0, 2\}$. □

Theorem 2.23 *There exists a $TSQS(4^n 2^1)$ when $n = 12k + 4$ is a positive integer.*

Proof. Theorem 2.9 says that when $n = 12k+4$ is a positive integer there exists a $1 - FSQS(n)$. Therefore, apply Lemma 2.11 with $g = 1$, $x = 4$, and $u_i = 2$. □

The following design was constructed by using the backtracking algorithm found in [18].

Theorem 2.24 *There exists a $TSQS(4^7 2^1)$.*

Proof. Develop the following 26 baseblocks:

$\{0,2,8,24\}$,	$\{0,2,9,20\}$,	$\{0,2,10,23\}$,	$\{1,2,6,28\}$,
$\{1,2,7,21\}$,	$\{1,2,9,22\}$,	$\{1,2,11,23\}$,	$\{2,6,10,18\}$,
$\{2,6,11,16\}$,	$\{2,6,12,17\}$,	$\{2,6,13,20\}$,	$\{2,6,15,23\}$,
$\{2,6,19,22\}$,	$\{2,6,21,24\}$,	$\{2,6,25,27\}$,	$\{2,7,10,15\}$,
$\{2,7,13,19\}$,	$\{2,7,14,20\}$,	$\{2,7,16,18\}$,	$\{2,7,17,23\}$,
$\{2,8,10,19\}$,	$\{2,8,16,22\}$,	$\{2,8,21,27\}$,	$\{2,9,12,25\}$,
$\{2,9,14,23\}$			.

with the automorphisms in the group generated by
$(2,6,10,14,18,22,26)(3,7,11,15,19,23,27)(4,8,12,16,20,24,28)$
$(5,9,13,17,21,25,29)$
and
$(2,3,4,5)(6,7,8,9)(10,11,12,13)(14,15,16,17)(18,19,20,21)$
$(22,23,24,25)(26,27,28,29)$.

The holes are: $\{0,1\}$, $\{2,3,4,5\}$, $\{6,7,8,9\}$, $\{10,11,12,13\}$, $\{14,15,16,17\}$, $\{18,19,20,21\}$, $\{22,23,24,25\}$, $\{26,27,28,29\}$. □

Theorem 2.25 *If there exists a $TSQS(4^n 2^1)$, then there exists a $TSQS(4^{4n-3} 2^1)$, for all $n \ge 5$, $n \ne 7, 11$.*

Proof. There exists a transverse $1 - FSQS$ of type $(n-1)^4$ for all $n \ge 5$, $n \ne 7, 11$, by Lemma 2.7. This is equivalent to a $TSQS((n-1)^4)$ that contains a transverse $2 - (v,4,1)$ design of type $(n-1)^4$. Let $W = \{1,2,3,4\}$ and $U = \{1,2,3,4,5,6\}$. For each block $B = \{a,b,c,d\}$ in the 2-design, construct a $TSQS(4^4 6^1)$ on $(B \times W) \cup U$ with holes $\{a\} \times W$, $\{b\} \times W$, $\{c\} \times W$, $\{d\} \times W$, U. This makes a $TSQS((4(n-1))^4 6^1)$. Now apply Theorem 1.7 with $m = 4$, $n = n-1$, $r = s = 4$, and $t = 2$. □

Remark: The next Theorem is similar to Theorem 2.25, however it gives some new designs. For example, we can make a $TSQS(4^{22} 2^1)$ with this theorem

($k = 3$ and $n = 7$), but we could not have used Theorem 2.25 to make it. (Set $4n - 3 = 22$ and try to solve for n.)

Theorem 2.26 *If there exists TSQSs of types* $4^{k+1}2^1$ *and* $(4k)^n6^1$*, then there exists a* $TSQS(4^{nk+1}2^1)$.

Proof. Apply Theorem 1.7 with $m = 4$, $n = k$, $r = n$, $s = 4$, and $t = 2$. □

The open cases for $TSQS(4^n2^1)$s are whenever $n \not\equiv 4 \pmod{12}$ and Theorems 2.22, 2.25, or 2.26 cannot be applied. A list of some small n regarding the open cases for $TSQS(4^n2^1)$s is

$$n = 10, 19, 31, 34, 37, 43, 46, 55, 58, 70, 73.$$

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