$c\lambda$−optimally connected mixed Cayley graph*

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Abstract: The cyclic edge-connectivity of a cyclically separable graph G, denoted by $c\lambda(G)$, is the minimum cardinality of all edge subsets, where edge subset F such that $G - F$ is disconnected and at least two of its components contain cycles. Since $c\lambda(G) \leq \zeta(G)$, where $\zeta(G)$=min$\{\omega(A)|A$ induces a shortest cycle in $G\}$, for any cyclically separable graph G, a cyclically separable graph G is said to be cyclically optimal if $c\lambda(G) = \zeta(G)$. The mixed Cayley graph is a kind of semi-regular graphs. The cyclic edge-connectivity is a widely studied parameter, which can be used to measure the reliability of network. Because the previous work studied the cyclically optimal mixed Cayley graphs with girth $g \geq 5$, this paper focuses on the mixed Cayley graphs with girth $g < 5$, and gives some sufficient and necessary conditions for these graphs to be cyclically optimal.

Keywords: Mixed Cayley graph; Cyclic edge-cut; Cyclically optimal; Cyclic edge-atom.

1 Introduction

Let $G = (V, E)$ be a finite, undirected and simple graph. Call an edge set F a *cyclic edge-cut* if $G - F$ is disconnected and at least two of its components contain cycles, and denoted by g the *girth* of G is the length of a shortest cycle of G. Hence, a graph has a cyclic edge-cut if and only if it has two disjoint cycles. Lovász [8] characterized all multigraphs without two disjoint cycles. We call those graphs, which do have cyclic edge-cut, *cyclically separable*. Following [9], Plummer defined the *cyclic edge-connectivity* of G, denoted by $c\lambda(G)$, as follows: If G is not connected, then $c\lambda(G) = 0$;

*Foundation item: The research is supported by NSFC (No.11171283, No.11401510), XJEDU (No.2013S03).

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If G is connected but does not have two disjoint cycles, then $c\lambda(G) = \infty$; Otherwise, $c\lambda(G)$ is the minimum cardinality over all cyclic edge-cuts of G.

For disjoint vertex sets A, B of G, $[A, B]$ is the set of edges with one end in A and the other end in B. $G[A]$ is the subgraph of G induced by vertex set A, and $\overline{A}$ is the complement of A in G, and $\omega(A) = |[A, \overline{A}]|$ is the number of edges between A and $\overline{A}$ in G. If $[A, \overline{A}]$ is a minimum cyclic edge-cut, then both $G[A]$ and $G[\overline{A}]$ are connected. Define $\zeta(G)$=min$\{\omega(A)|A$ induces a shortest cycle in $G\}$. In [1], Wang and Zhang showed that any cyclically separable graph satisfies $c\lambda(G) \leq \zeta(G)$. Hence, a cyclically separable graph G is called *cyclically optimal* if $c\lambda(G) = \zeta(G)$. That is, for any cyclically separable graph G, if G is not cyclically optimal, then $c\lambda(G) < \zeta(G)$.

In [12], Nedela and Skoviera studied the existence of the cyclic edge-cut in cubic multigraphs, and showed that a connected cubic graph G has no cyclic edge cut if and only if it is isomorphic to one of K_4, $K_{3,3}$ or θ_2 (the multigraph with two vertices and three edges between them). Xu and Liu in [6] showed that a k-regular simple graph G with $k \geq 3$ which is not K_4, K_5 and $K_{3,3}$ is cyclically separable, and $c\lambda(G) \leq \zeta(G)$. Furthermore, they proved that a connected k-regular vertex transitive graph G with $k \geq 4$ ($k \neq 5$) and girth $g(G) \geq 5$ is cyclically optimal. Lou and Wang [2] gave a polynomial algorithm deciding whether $c\lambda(G) < \infty$ for multigraph G. Wang and Zhang in [1] showed that any vertex transitive graph with regularity degree $k \geq 4$ and girth $g \geq 5$ is cyclically optimal. Chen et al. in [7] studied super edge-connectivity of mixed Cayley graph. Tian and Meng in [16] showed that any half vertex transitive regular graph with $g(G) \geq 6$ is cyclically optimal. Lin et al. [4] studied the cyclic edge-connectivity of graphs with double orbits of same size. Tian and Meng in [17] studied λ'-optimally of mixed Cayley graph. Yang et al. reported the edge-connectivity of graphs with two orbits of the same size in [14]. Yang et al. in [15] studied super 2-restricted and 3-restricted edge-connected vertex transitive graphs. Lin et al. in [3] studied super restricted edge connectivity of regular graphs with two orbits. Yang et al. in [13] studied the cyclically optimality of the regular double orbits graphs, and they gave that if girth $g \geq 5$ and the Cayley graphs $Cay(G, S_0)$ and $Cay(G, S_1)$ are connected, then the mixed Cayley graph $MC(G, S_0, S_1, S_2)$ is cyclically optimal, where S_0, S_1 and S_2 are subsets of G, and $1_G \notin S_i$, $S_i^{-1} = S_i$ for $i \in \{0, 1\}$. In this paper, we study the cyclically optimal connected mixed Cayley graph with girth $g \leq 4$.

2 Preliminaries

A vertex set A is a *cyclic edge-fragment* if $[A, \overline{A}]$ is a minimum cyclic edge-cut. A cyclic edge-fragment with the minimum cardinality is called a *cyclic edge-atom*. Without confusion, the fragment will stand for the cyclic edge-fragment and the atom will stand for the cyclic edge-atom. It is easy to know, if A is a fragment, then $\overline{A}$ is also a fragment, and both $G[A]$ and $G[\overline{A}]$ are connected. Clearly, if A is an atom, and A' is a proper subset of A such that $[A', \overline{A'}]$ is a cyclic edge-cut, then $\omega(A') > \omega(A)$.

Mader [11] and Watkins [10] first proposed the concepts of fragment and atom, and the fragment and atom play an important role in studying various kinds of connectivity. An atom is said to be trivial if it induces a cycle of G, otherwise it is non-trivial. For a vertex u and a vertex set A. $N_A(u)=\{v \in A | v$ is adjacent with $u\}$ is the set of neighbors of u in A, denoted by $d_A(u) = |N_A(u)|$ the degree of u in A.

Lemma 2.1. *([1]) Let G be a connected graph with $\delta(G) \geq 3$ and girth $g \geq 5$, or $\delta(G) \geq 4$ and order $n \geq 6$, then G is cyclically separable.*

Lemma 2.2. *([1]) Let G be a connected graph with $\delta(G) \geq 3$ and A be a fragment, then*

(1) $\delta(G[A]) \geq 2$;

(2) If $\delta(G[A]) \geq 3$, then $d_A(v) \geq d_{\overline{A}}(v)$ holds for any $v \in A$;

(3) If $G[A]$ is not a cycle and v is a vertex in A with $d_A(v) = 2$, then $d_A(v) \geq d_{\overline{A}}(v)$;

(4) If $\delta(G) \geq 4$, and A is a non-trivial atom of G, then $\delta(G[A]) \geq 3$. Furthermore, $d_A(v) > d_{\overline{A}}(v)$ holds for any $v \in G$.

Definition 2.3. *([7]) Let G be a finite group, and S_0, S_1 and S_2 be subsets of G, where $1_G \notin S_i$ and $S_i^{-1} = S_i$ for $i \in \{0,1\}$. The mixed Cayley graph $X = MC(G, S_0, S_1, S_2)$ has vertex set $V(X) = G \times \{0,1\}$ and edge set $E(X) = E_0 \cup E_1 \cup E_2$, where $E_i = \{\{(g,i),(s_i g,i)\} : g \in G, s_i \in S_i\}$ for $i \in \{0,1\}$, and $E_2 = \{\{(g,0),(s_2 g,1)\} : g \in G, s_2 \in S_2\}$.*

In the following, we assume that $X = MC(G, S_0, S_1, S_2)$ is a connected mixed Cayley graph. $V(X) = V_0 \cup V_1$, where $V_i = G \times \{i\}, X_i = (V_i, E_i)$ for $i \in \{0,1\}$, and $k_i = |S_i|$ for $i \in \{0,1,2\}$. Clearly, the minimum degree of X is $\delta(X)=\min\{k_0 + k_2, k_1 + k_2\}$ and the maximum degree of X is $\Delta(X)=\max\{k_0+k_2, k_1+k_2\}$. Denote $MR(G) = \{MR(g)|MR(g) : (h,i) \to (hg,i)$ for $g,h \in G$ and $i \in \{0,1\}\}$. Clearly, $MR(G)$ acts transitively both on V_0 and on V_1.

Lemma 2.4. *Let G be a cyclically separable connected graph with $\delta \geq 3$ and girth g. Assume $\zeta(G) = (\delta - 2) \cdot g$. For any vertex set A of G with $|A| \geq g$, if $G[A]$ is a forest, then $\omega(A) \geq \zeta(G) + 2$.*

Proof. Since $G[A]$ is a forest, we have $|E(G[A])| \leq |A| - 1$, and

$$\begin{aligned}\omega(A) &= \Sigma_{v\in A} d(v) - 2|E(G[A])| \geq \delta \cdot |A| - 2(|A| - 1) \\ &= (\delta - 2) \cdot |A| + 2 \geq (\delta - 2) \cdot g + 2 = \zeta(G) + 2. \square\end{aligned}$$

Lemma 2.5. *Let G be a cyclically separable connected graph with $\delta \geq 3$ and girth g. Assume $\zeta(G) = (\delta - 2) \cdot g$. If G is not cyclically optimal, then for any two distinct atoms A and B with $A \cap B \neq \phi$, $|A \cap B| \leq g - 1$, and $|A| = |B| \leq 2(g-1)$.*

Proof. Assume $|A \cap B| \geq g$. By the minimality of atom, we have $|A| = |B| \leq \frac{|V(G)|}{2}$. Thus $|\overline{A \cup B}| = |V(G)| - |A| - |B| + |A \cap B| \geq g$. In the following, we show $\omega(A \cap B) > c\lambda(G)$. If $G[A \cap B]$ contains cycles, since $G[\overline{A \cap B}] \supseteq G[\overline{A}]$ and $G[\overline{A}]$ contains cycles, $[A \cap B, \overline{A \cap B}]$ is a cyclic edge-cut, and $\omega(A \cap B) > \omega(A) = c\lambda(G)$. If $G[A \cap B]$ is a forest, then $\omega(A \cap B) \geq \zeta(G) + 2$ by lemma 2.4, and $\omega(A \cap B) > c\lambda(G)$.

By a similar argument as above, we have $\omega(\overline{A \cup B}) \geq c\lambda(G)$. By $\omega(A \cup B) = \omega(\overline{A} \cup \overline{B})$ and the submodular inequality, we have

$$2c\lambda(G) < \omega(A \cap B) + \omega(A \cup B) \leq \omega(A) + \omega(B) = 2c\lambda(G),$$

a contradiction.

Assume $|A| = |B| > 2(g-1)$, then $|A \cap \overline{B}| = |\overline{A} \cap B| \geq g$, by a similar argument as above, we also have a contradiction.□

Lemma 2.6. *Let G be a cyclically separable connected graph with $\delta \geq 3$ and girth g. Assume $\zeta(G) = (\delta - 2) \cdot g$. If C is the shortest cycle with $\omega(C) = \zeta(G)$, then $[C, \overline{C}]$ is a cyclic edge-cut.*

Proof. Suppose $[C, \overline{C}]$ is not a cyclic edge-cut, then $\overline{C}$ is a forest. Since G is cyclically separable, we have $|V(G)| \geq 2g$. Thus $|\overline{C}| \geq g$, and $\omega(\overline{C}) \geq \zeta(G) + 2$ by lemma 2.4. Thus $\omega(C) = \omega(\overline{C}) > \zeta(G)$, a contradiction.□

Lemma 2.7. *Let G be a cyclically separable connected graph with $\delta \geq 3$ and girth g. Assume $\zeta(G) = (\delta - 2) \cdot g$. Then G is cyclically optimal if and only if every atom of G is trivial.*

Proof. Let A be an atom of G. Suppose A is trivial, then $|E(G[A])| = |A| = g$, and we have

$$c\lambda(G) = \omega(A) = \Sigma_{v \in A} d(v) - 2|E(G[A])| \geq \delta \cdot |A| - 2|A| = (\delta - 2) \cdot g = \zeta(G).$$

Since $c\lambda(G) \leq \zeta(G)$, we have $c\lambda(G) = \zeta(G)$. Hence G is cyclically optimal.

Conversely, G is cyclically optimal. Let C be a shortest cycle with $\omega(C) = \zeta(G)$, then $\omega(C) = c\lambda(G)$. $[C, \overline{C}]$ is a cyclic edge-cut by lemma 2.6. So $V(C)$ is a fragment. Since any atom of G contains at least g vertices, we find that C is an atom, and so every atom of G has g vertices. Let A be an arbitrary atom, then $G[A]$ contains a cycle, and $G[A]$ is a shortest cycle, and so A is a trivial atom.□

Lemma 2.8. *Let G be a cyclically separable connected graph with $\delta \geq 3$ and girth g. Assume $\zeta(G) = (\delta - 2) \cdot g$. If G is not cyclically optimal with $\delta \geq 6$ and $g = 3$, or $\delta \geq 4$ and $g \geq 4$, then for any two distinct atoms A and B of G, $A \cap B = \phi$.*

Proof. Since G is not cyclically optimal, A and B are two non-trivial atoms by lemma 2.7. Suppose $A \cap B \neq \phi$, then $|A| = |B| \leq 2(g-1)$ and $|A \cap B| \leq g - 1$ by lemma 2.5, $\delta(G[A]) \geq 3$ and $\delta(G[B]) \geq 3$ by lemma 2.2 (4).

Case 1. $g = 3$. Then $|A| = |B| \leq 4$ and $|A \cap B| \leq 2$.

Since $\delta(G[A]) \geq 3$ and $\delta(G[B]) \geq 3$, $A \cong B \cong K_4$, and $|E(G[A])| = 6$. We have

$$c\lambda(G) = \omega(A) = \Sigma_{v \in A} d(v) - 2|E(G[A])| \geq \delta \cdot |A| - 2 \times 6$$

$$= 4\delta - 12 = 3(\delta - 2) + \delta - 6 \geq \zeta(G) > c\lambda(G)$$

for $\delta \geq 6$, a contradiction.

Case 2. $g \geq 4$.

Subcase 2.1. $g = 4$. Then $|A| = |B| \leq 6$ and $|A \cap B| \leq 3$.

If $|A| = |B| = 4$, then A is a cycle, which contradicts to that A is non-trivial. If $|A| = |B| = 5$, then $|E(G[A])| \geq \frac{15}{2}$ by $\delta(G[A]) \geq 3$. By Turán theorem [5], we have $|E(G[A])| \leq |E(T_{2,5})| = 6$ for $g = 4$, a contradiction. Thus $|A| = |B| = 6$. Since $\delta(G[A]) \geq 3$ and $\delta(G[B]) \geq 3$, we have $G[A] \cong G[B] \cong K_{3,3}$, and $|E(G[A])| = 9$.

If $|A \cap B| = 1$, suppose $A \cap B = \{u\}$, then $d_{A \cap B}(u) = 0$, and $d(u) \geq d_A(u) + d_B(u) - d_{A \cap B}(u) = 6$. We have

$$c\lambda(G) = \omega(A) = \Sigma_{v \in A} d(v) - 2|E(G[A])| \geq \delta \cdot (|A| - 1) + d(u) - 2 \times 9$$

$$\geq 4(\delta - 2) + \delta + d(u) - 10 \geq 4(\delta - 2) = \zeta(G) > c\lambda(G)$$

for $\delta \geq 4$, a contradiction.

If $|A \cap B| = 2$, suppose $A \cap B = \{v_1, v_2\}$, then $d_{A \cap B}(v_i) \leq 1$, and $d(v_i) \geq d_A(v_i) + d_B(v_i) - d_{A \cap B}(v_i) \geq 5$ for $i \in \{1, 2\}$. So $d(v_1) + d(v_2) \geq 10$, and we have

$$c\lambda(G) = \omega(A) = \Sigma_{v \in A} d(v) - 2|E(G[A])| \geq \delta \cdot (|A| - 2) + d(v_1) + d(v_2) - 2 \times 9$$

$$= 4(\delta - 2) + d(v_1) + d(v_2) - 10 \geq 4(\delta - 2) = \zeta(G) > c\lambda(G),$$

a contradiction.

If $|A \cap B| = 3$, suppose $A \cap B = \{v_1, v_2, v_3\}$, then at least two elements of $A \cap B$ satisfy $d_{A \cap B}(v_i) \leq 1$ by $g = 4$. Without loss of generality, assume $d_{A \cap B}(v_i) \leq 1$, then $d(v_i) \geq d_A(v_i) + d_B(v_i) - d_{A \cap B}(v_i) \geq 5$ for $i \in \{1, 2\}$. So $d(v_1) + d(v_2) \geq 10$, and we have

$$c\lambda(G) = \omega(A) = \Sigma_{v \in A} d(v) - 2|E(G[A])| \geq \delta \cdot (|A| - 2) + d(v_1) + d(v_2) - 2 \times 9$$

$$= 4(\delta - 2) + d(v_1) + d(v_2) - 10 \geq 4(\delta - 2) = \zeta(G) > c\lambda(G),$$

a contradiction.

Subcase 2.2. $g \geq 5$.

By lemma 2.1, $G[A]$ and $G[B]$ contain two disjoint cycles,respectively, which implies $|A| = |B| \geq 2g > 2(g-1)$, a contradiction.□

Assume Φ is a group of permutations of a set T, and A is a proper, non-trivial subset of T, if $\varphi \in \Phi$ such that either $\varphi(A) = A$ or $\varphi(A) \cap A = \phi$, then A is an *imprimitive block* for Φ.

Lemma 2.9. *Let $X = MC(G, S_0, S_1, S_2)$ be a connected graph, $k_i = |S_i|$ for $i \in \{0, 1, 2\}$. Assume A is an atom of X, and $Y = X[A]$ is the subgraph of X induced by A, and $\zeta(X) = (\delta - 2) \cdot g$. If X is not cyclically optimal with $g = 3$ and $\delta \geq 6$, or $g \geq 4$ and $\delta \geq 4$, then*

(1) When $A \subseteq V_i$ and $A = H \times \{i\}$, V_i is a disjoint union of distinct atoms, and $Y = X[A]$ is a r-regular vertex transitive graph for $i = 0$ or 1, where $r \geq 3$. If $1_G \in H$, then $H \leq G$;

(2) When $A_i = A \cap V_i \neq \phi$, and $A_i = H_i \times \{i\}$, $V(X)$ is a disjoint union of distinct atoms, and $Y_i = X[A_i]$ is a r_i-regular vertex transitive graph for $i \in \{0, 1\}$. and $|A_0| = |A_1|$. If $1_G \in H_i$. then $H_i \leq G$ for $i = 0$ or 1.

Proof. Obviously, A is an imprimitive block of X by lemma 2.8.

(1) Without loss of generality, we assume $A \subseteq V_0$. Since X_0 is a vertex transitive graph, each vertex of X_0 lies in some atom, and so V_0 is the disjoint union of distinct atoms by lemma 2.8. Since for any $g \in G, MR(g)$ is an automorphism of X, for any $(g,0), (h,0) \in A, MR(g^{-1}h)(A)$ is also an atom of X and $MR(g^{-1}h)(A) \cap A \neq \phi$, and so $MR(g^{-1}h)(A) = A$. Obviously, the restriction of $MR(g^{-1}h)$ to A induces an automorphism of Y, which maps $(g,0)$ to $(h,0)$, so $Y = X[A]$ is r-regular vertex transitive graph. and $r \geq 3$ by lemma 2.2 (4).

If $1_G \in H$. then $(1_G, 0) \in A$. For any $(g, 0) \in A$, we have $(1_G, 0) \in MR(g^{-1})(A) = Ag^{-1}$, so $Ag^{-1} = A$. Since $(g,0), (h,0) \in A$, we have $(hg^{-1}, 0) \in A$, and for any $g, h \in H$, we have $hg^{-1} \in H$. Thus $H \leq G$.

(2) By a similar argument to (1), we have $V(X)$ is a disjoint union of distinct atoms. For $i \in \{0, 1\}$, let $V(X) = \cup_{j=1}^{k} MR(g_j)(A)(g_1 = 1_G)$, then $V_i = \cup_{j=1}^{k} MR(g_j)(A_i)$. Since $|V_0| = |V_1|, |A_0| = |A_1|$. For any $(g,i), (h,i) \in A_i$. $MR(g^{-1}h)(A)$ is also an atom of X and $MR(g^{-1}h)(A) \cap A \neq \phi$, we have $MR(g^{-1}h)(A) = A$ and $MR(g^{-1}h)(A_i) = A_i$. Thus the restriction of $MR(g^{-1}h)$ to A_i induces an automorphism of Y_i, which maps (g,i) to (h,i). It implies that $Y_i = X[A_i]$ is a r_i-regular vertex transitive graph for $i \in \{0, 1\}$.

If $1_G \in H_i$, then $(1_G, i) \in A_i$ for $i = 0$ or 1. Assume $i = 0$. For any $(g, 0) \in A_0$, we have $(1_G, 0) \in MR(g^{-1})(A) = Ag^{-1}$, $Ag^{-1} = A$, and so

$A_0g^{-1} = A_0$. Thus for any $(g,0),(h,0) \in A_0$, $(hg^{-1},0) \in A_0$, that is, we have $hg^{-1} \in H_0$ for any $g,h \in H_0$,. Thus $H_0 \leq G$.□

3 $c\lambda-$optimally connected mixed Cayley graph with $g = 3$

If $k_0 = k_1$ in $X = MC(G, S_0, S_1, S_2)$, where $k_i = |S_i|$ for $i \in \{0,1,2\}$, then $X = MC(G, S_0, S_1, S_2)$ is a k-regular graph, where $k = k_0 + k_2 = k_1 + k_2$.

Lemma 3.1. *Let $X = MC(G, S_0, S_1, S_2)$ be a $k-$regular connected graph with $k \geq 6$ and girth $g = 3$. Assume A is an atom of X. If X is not cyclically optimal, then $|A| > k - 2$.*

Proof. Assume $a = |A|$. Since A is an atom of X, and X is not cyclically optimal, we have $\omega(A) = c\lambda(X) < \zeta(X) = (k-2)\cdot g$.

Considering the sum of degrees of all vertices of A, we have

$$k \cdot a = \Sigma_{v\in A} d(v) \leq a(a-1) + \omega(A) < a^2 - a + 3(k-2),$$

that is $(a-3)(k-a-2) < 0$, which implies $a > k-2$ for $a = |A| > g = 3$.□

Lemma 3.2. *Let $X = MC(G, S_0, S_1, S_2)$ be a connected graph with $\delta \geq 6$ and girth $g = 3$. Assume A is an atom of X, $k_1 > k_0$, and X_0 contains a cycle of length g. If X is not cyclically optimal, then:*

(1) When $A \subseteq V_0$, $|A| > \delta - 2$;

(2) When $A \subseteq V_1$, $|A| > \Delta - 2$;

(3) When $A \cap V_i \neq \phi$ for $i \in \{0,1\}$, $|A| > \frac{\delta+\Delta-4}{2}$.

Proof. Assume $a = |A|$. Since A is an atom of X, and X is not cyclically optimal, $\omega(A) = c\lambda(X) < \zeta(X)$. Since X_0 contains a cycle of length g, $\zeta(X) = (\delta - 2)\cdot g$.

(1) When $A \subseteq V_0$,we have

$$\delta \cdot a = \Sigma_{v\in A} d(v) \leq a(a-1) + \omega(A) < a^2 - a + 3(\delta - 2),$$

that is $(a-3)(\delta - a - 2) < 0$, hence $a > \delta - 2$ for $a = |A| > g = 3$.

(2) When $A \subseteq V_1$, we have

$$\Delta \cdot a = \Sigma_{v\in A} d(v) \leq a(a-1) + \omega(A) < a^2 - a + 3(\delta - 2),$$

that is $(a-3)(\Delta - a - 2) < 0$, hence $a > \Delta - 2$.

(3) When $A_i = A \cap V_i \neq \phi$ for $i \in \{0,1\}$, $|A_0| = |A_1|$ by lemma 2.9, we have

$$\delta \cdot \frac{a}{2} + \Delta \cdot \frac{a}{2} = \Sigma_{v\in A} d(v) \leq a(a-1) + \omega(A) < a^2 - a + 3(\delta - 2),$$

that is $(a-3)(\delta + \Delta - 2a - 4) < 0$, hence $a > \frac{\delta+\Delta-4}{2}$.□

Theorem 3.3. *Let $X = MC(G, S_0, S_1, S_2)$ be a k-regular connected graph with $k \geq 6$ and girth $g = 3$. Assume $G_i = \langle S_i \rangle$ for $i \in \{0, 1\}$ and $S_2 = \{s_2\}$. Then X is not cyclically optimal if and only if X satisfies one of the following conditions:*

(1) There exists a subgroup H of G satisfied $|H| < (k-2) \cdot g$ and $G_i \leq H$ for $i \in \{0, 1\}$;

(2) There exists a subgroup H of G and an element $s_i \in S_i$, satisfied $|H| < \frac{g}{2} \cdot (k-2)$ and $\langle S_i \setminus \{s_i\} \rangle \leq H$ for $i \in \{0, 1\}$;

(3) There exists a subgroup H of G, a proper, inverse-closed subset $S_0' = \{s_{01}, ..., s_{0m}\}$ of S_0, and a proper, inverse-closed subset $S_1' = \{s_{11}, ..., s_{1n}\}$ of $S_1 (1 \leq m + n \leq 4)$, satisfied $|H| < \frac{g}{m+n} \cdot (k-2), \langle S_0 \setminus S_0' \rangle \leq H$ and $\langle S_1 \setminus S_1' \rangle \leq s_2 H s_2^{-1}$.

Proof. We first prove the if part.

(1) Assume $i = 0$. Let $A = H \times \{0\}$. Since $G_0 \leq H$, $\delta(X[A]) = k_0 = |S_0| > 2$, and so $X[A]$ contains cycles. $\delta(X_1) = k_1 = |S_1| > 2$, we have X_1 contains cycles, and so $X[\overline{A}]$ contains cycles. Thus $[A, \overline{A}]$ is a cyclic edge-cut, and we have

$$c\lambda(X) \leq \omega(A) = (k - k_0) \cdot |A| = |H| < (k-2) \cdot g = \zeta(X).$$

Hence X is not cyclically optimal.

(2) Assume $i = 0$. Let $A = H \times \{0\}$, since $\langle S_0 \setminus \{s_0\} \rangle \leq H$, $\delta(X[A]) = k_0 - 1 > 2$, and $X[A]$ contains cycles. $\delta(X_1) = k_1 > 2$, we have X_1 contains cycles, and so $X[\overline{A}]$ contains cycles. Thus $[A, \overline{A}]$ is a cyclic edge-cut, we have

$$c\lambda(X) \leq \omega(A) = (k - k_0 + 1) \cdot |A| = 2|H| < (k-2) \cdot g = \zeta(X).$$

Hence X is not cyclically optimal.

(3) Let $A = H \times \{0\} \cup s_2 H \times \{1\}, A_i = A \cap X_i$ for $i \in \{0, 1\}$. Since S_0' and S_1' are proper subsets of S_0 and S_1, respectively, $\langle S_0 \setminus S_0' \rangle \leq H$, and $\langle S_1 \setminus S_1' \rangle \leq s_2 H s_2^{-1}$, we have $\delta(X[A]) \geq 2$, and so $X[A]$ contains cycles. Since $1 \leq m + n \leq 4$, $\delta(X[\overline{A}]) \geq k - (m+n) \geq 2$, and so $X[\overline{A}]$ contains cycles. Thus $[A, \overline{A}]$ is a cyclic edge-cut, and we have

$$c\lambda(X) \leq \omega(A) = m \cdot |A_0| + n \cdot |A_1| = (m+n) \cdot |H| < (k-2) \cdot g = \zeta(X).$$

Hence X is not cyclically optimal.

Now we prove the only if part. Let A be an atom. Since X is not cyclically optimal and girth $g = 3$, we see that $|A| > k - 2$ by lemma 3.1.

Case 1. $A \subseteq V_0$ or $A \subseteq V_1$.

If $A \subseteq V_0$, let $A = H \times \{0\}$, without loss of generality, assume $1_G \in H$, then $H \leq G$ and $Y = X[A]$ is a r_0-regular graph by lemma 2.9. We have

$$c\lambda(X) = \omega(A) = (k - r_0) \cdot |A| < \zeta(X) = 3(k-2).$$

Since $|A| > k-2$, $r_0 > k-3$. Since $k = k_0 + k_2$ and $k_2 = |S_2| = 1$, $r_0 \geq k_0 - 1$. Obviously, $r_0 \leq k_0$, then $r_0 = k_0$ or $r_0 = k_0 - 1$.

If $r_0 = k_0$, then $G_0 \leq H$ and $|H| < (k-2) \cdot g$.

If $r_0 = k_0 - 1$, then there exists an element $s_0 \in S_0$ satisfied $\langle S_0 \backslash \{s_0\} \rangle \leq H$ and $|H| < \frac{g}{2} \cdot (k-2)$.

In a similar manner, if $A \subseteq V_1$, then X is not cyclically optimal only if there exists a subgroup H of G satisfied $G_1 \leq H$ and $|H| < (k-2) \cdot g$, or there exists a subgroup H of G and an element $s_1 \in S_1$, satisfied $\langle S_1 \backslash \{s_1\} \rangle \leq H$ and $|H| < \frac{g}{2} \cdot (k-2)$.

Case 2. $A_i = A \cap V_i \neq \phi$ for $i \in \{0, 1\}$.

Let $A_i = H_i \times \{i\}$. Without loss of generality, assume $1_G \in H_0$, then $H_0 \leq G$ by lemma 2.9. For any $h \in H_0$, we have $MR(h)(A) \cap A \neq \phi$, then $MR(h)(A) = A$ by lemma 2.8. Since $H_1 h = H_1$, $H_1 H_0 = H_1$. Since $X[A]$ is connected, we have $H_1 = s_2 H_0$.

Since $H_0 \leq G$, we have $G = \cup_{j=1}^{k} H_0 g_j (g_1 = 1_G)$. Hence $V(X) = \cup_{j=1}^{k} MR(g_j)(A)$, $V_0 = \cup_{j=1}^{k} MR(g_j)(A_0) = \cup_{j=1}^{k} H_0 g_j \times \{0\}$, and $V_1 = \cup_{j=1}^{k} MR(g_j)(A_1) = \cup_{j=1}^{k} s_2 H_0 g_j \times \{1\}$.

The subgraph $Y_i = X[A_i]$ is regular by lemma 2.9 for $i \in \{0, 1\}$, and $Y' = X[A] \setminus (E(Y_0) \cup E(Y_1))$ is also a regular graph. Assume that Y_i is a r_i-regular graph. Obviously, Y' is a 1-regular graph. We have

$$c\lambda(X) = \omega(A) = (k_0 - r_0) \cdot |A_0| + (k_1 - r_1) \cdot |A_1| < \zeta(X) = 3(k-2).$$

Since $|A| > k-2$, we have $|A_0| = |A_1| \geq \frac{k-1}{2}$. Let $k_0 - r_0 = m$ and $k_1 - r_1 = n$. then $1 \leq m + n \leq 4$ and $|H_0| < \frac{3}{m+n} \cdot (k-2)$.

Since $k_0 - r_0 = m \leq 4$, there exists a proper, inverse-closed subset $S_0' = \{s_{01}, \ldots, s_{0m}\}$ of S_0 satisfied $\langle S_0 \setminus S_0' \rangle \leq H_0$. Suppose $1_G \in s_2 H_0 g_s$, then $1_G = s_2 h_0 g_s$, and $g_s = h_0^{-1} s_2^{-1}$ for some $h_0 \in H_0$. Since $MR(g_s)(A)$ is also an atom, and $k_1 - r_1 = n \leq 4$, there exists a proper, inverse-closed subset $S_1' = \{s_{11}, \ldots, s_{1n}\}$ of S_1 satisfied $\langle S_1 \setminus S_1' \rangle \leq s_2 H_0 g_s = s_2 H_0 s_2^{-1}$.□

Theorem 3.4. *Let $X = MC(G, S_0, S_1, S_2)$ be a k-regular connected graph with $k \geq 6$ and girth $g = 3$. Assume $G_i = \langle S_i \rangle$ for $i \in \{0, 1\}$, $G_2 = \langle S_2^{-1} S_2 \rangle$, $k_0 = k_1 \geq 2$ and $k_2 \geq 2$, then X is not cyclically optimal if and only if X satisfies one of the following conditions:*

(1) There exists a subgroup H of G satisfied $|H| < \frac{g}{k_2} \cdot (k-2) (k_2 = 2)$ and $G_i \leq H$ for $i \in \{0, 1\}$;

(2) There exists a subgroup H of G, an inverse-closed subset $S_0' = \{s_{01}, \ldots, s_{0m}\}$ of S_0, an inverse-closed subset $S_1' = \{s_{11}, \ldots, s_{1n}\}$ of $S_1 (1 \leq m + n \leq 4)$, and an element $s_2 \in S_2$, satisfied $|H| < \frac{g}{m+n} \cdot (k-2)$, $\langle S_0 \setminus S_0' \rangle \leq H$, $\langle S_1 \setminus S_1' \rangle \leq s_2 H s_2^{-1}$, and $G_2 \leq H$;

(3) There exists a subgroup H of G, a proper, inverse-closed subset $S_0' = \{s_{01}, \ldots, s_{0m}\}$ of S_0, a proper, inverse-closed subset $S_1' = \{s_{11}, \ldots, s_{1n}\}$

of $S_1 (0 \leq m+n \leq 2)$, *and two elements* $s_2, s_2' \in S_2$, *satisfied* $|H| < \frac{g}{m+n+2} \cdot (k-2)$, $\langle S_0 \setminus S_0' \rangle \leq H$, $\langle S_1 \setminus S_1' \rangle \leq s_2 H s_2^{-1}$, *and* $\langle (S_2 \setminus \{s_2'\})^{-1} (S_2 \setminus \{s_2'\}) \rangle \leq H$;

(4) There exists a subgroup H *of* G, *and three elements* $s_2, s_{21}, s_{22} \in S_2$, *satisfied* $|H| < \frac{g}{4} \cdot (k-2)$, $G_0 \leq H$, $G_1 \leq s_2 H s_2^{-1}$, *and* $\langle (S_2 \setminus \{s_{21}, s_{22}\})^{-1} (S_2 \setminus \{s_{21}, s_{22}\}) \rangle \leq H$.

Proof. We first prove the if part.

(1) Assume $i = 0$. Let $A = H \times \{0\}$. Since $G_0 \leq H$, $\delta(X[A]) = k_0 \geq 2$, and $X[A]$ contains cycles. $\delta(X_1) = k_1 \geq 2$, we have X_1 contains cycles, and so $X[\overline{A}]$ contains cycles. Thus $[A, \overline{A}]$ is a cyclic edge-cut, and we have

$$c\lambda(X) \leq \omega(A) = (k - k_0) \cdot |A| = k_2 \cdot |H| < (k-2) \cdot g = \zeta(X).$$

Hence X is not cyclically optimal.

In the following, let $A = H \times \{0\} \cup s_2 H \times \{1\}$, where $s_2 \in S_2$. Assume $A_i = A \cap V_i$ for $i \in \{0, 1\}$.

(2) Since $\langle S_2^{-1} S_2 \rangle = G_2 \leq H$, for any $s_i, s_j \in S_2$, $s_i^{-1} s_j \in H$, and so $s_j \in s_2 H$. Thus $\delta(X[A]) \geq k_2 \geq 2$, and so $X[A]$ contains cycles. Since $1 \leq m + n \leq 4$, $\delta(X[\overline{A}]) \geq k - (m+n) \geq 2$, and so $X[\overline{A}]$ contains cycles. Then $[A, \overline{A}]$ is a cyclic edge-cut, and we have

$$c\lambda(X) \leq \omega(A) = m \cdot |A_0| + n \cdot |A_1| = (m+n) \cdot |H| < (k-2) \cdot g = \zeta(X).$$

Hence X is not cyclically optimal.

(3) Since $\langle (S_2 \setminus \{s_2'\})^{-1} (S_2 \setminus \{s_2'\}) \rangle \leq H$, for any $s_i, s_j \in S_2 \setminus \{s_2'\}$, $s_i^{-1} s_j \in H$, and so $s_j \in s_2 H$. Since S_0' and S_1' are proper subset of S_0 and S_1, respectively, $\delta(X[A]) \geq 1 + k_2 - 1 = k_2 \geq 2$, and so $X[A]$ contains cycles. Since $0 \leq m + n \leq 2$, we have $\delta(X[\overline{A}]) \geq k - (m+n) > 2$, and so $X[\overline{A}]$ contains cycles. Thus $[A, \overline{A}]$ is a cyclic edge-cut, and we have

$$\begin{aligned} c\lambda(X) &\leq \omega(A) = (m+1) \cdot |A_0| + (n+1) \cdot |A_1| \\ &= (m+n+2) \cdot |H| < (k-2) \cdot g = \zeta(X). \end{aligned}$$

Hence X is not cyclically optimal.

(4) Since $G_0 \leq H, G_1 \leq s_2 H s_2^{-1}$, and X is a $k-$regular graph, $\delta(X[A]) \geq k_0 + 1 > 2$, and so $X[A]$ contains cycles. Since $\langle (S_2 \setminus \{s_{21}, s_{22}\})^{-1} (S_2 \setminus \{s_{21}, s_{22}\}) \rangle \leq H$, $\delta(X[\overline{A}]) \geq k - 2 > 2$, and so $X[\overline{A}]$ contains cycles. Then $[A, \overline{A}]$ is a cyclic edge-cut, and we have

$$c\lambda(X) \leq \omega(A) = 2|A_0| + 2|A_1| = 4|H| < (k-2) \cdot g = \zeta(X).$$

Hence X is not cyclically optimal.

Now we prove the only if part. Let A be an atom. Since X is not cyclically optimal, and girth $g = 3$, $|A| > k - 2$ by lemma 3.1.

Case 1. $A \subseteq V_0$ or $A \subseteq V_1$.

If $A \subseteq V_0$, let $A = H \times \{0\}$, without loss of generality, assume $1_G \in H$, then $H \leq G$ and $Y = X[A]$ is a r_0-regular graph by lemma 2.9. We have

$$c\lambda(X) = \omega(A) = (k - r_0) \cdot |A| < \zeta(X) = 3(k-2).$$

Since $|A| > k-2$, $r_0 > k-3$. Since $k = k_0 + k_2$ and $k_2 \geq 2$, $r_0 \geq k_0$. Obviously, $r_0 \leq k_0$. Then $r_0 = k_0$, $G_0 \leq H$, and $|H| < \frac{g}{k_2} \cdot (k-2)$. $|H| = |A| > k-2$, we have $k_2 = 2$.

In a similar manner, if $A \subseteq V_1$, then X is not cyclically optimal only if there exists a subgroup H of G satisfied $G_1 \leq H$, $|H| < \frac{g}{k_2} \cdot (k-2)$, and $k_2 = 2$.

Case 2. $A_i = A \cap V_i \neq \phi$ for $i \in \{0, 1\}$.

Let $A_i = H_i \times \{i\}$. Without loss of generality, assume $1_G \in H_0$, then $H_0 \leq G$ by lemma 2.9. For any $h \in H_0$, $MR(h)(A) = A$, we have $H_1 h = H_1$, and $H_1 H_0 = H_1$. Since $X[A]$ is connected and $H_0 \leq G$, we have $H_1 = s_2 H_0$, and $G = \cup_{j=1}^{k} H_0 g_j (g_1 = 1_G)$. Hence $V(X) = \cup_{j=1}^{k} MR(g_j)(A)$, $V_0 = \cup_{j=1}^{k} MR(g_j)(A_0) = \cup_{j=1}^{k} H_0 g_j \times \{0\}$, and $V_1 = \cup_{j=1}^{k} MR(g_j)(A_1) = \cup_{j=1}^{k} s_2 H_0 g_j \times \{1\}$.

The subgraph $Y_i = X[A_i]$ is r_i-regular by lemma 2.9, and $Y' = X[A] \setminus (E(Y_0) \cup E(Y_1))$ is a d-regular graph, we have

$$\begin{aligned} c\lambda(X) &= \omega(A) = (k_0 - r_0 + k_2 - d) \cdot |A_0| + (k_1 - r_1 + k_2 - d) \cdot |A_1| \\ &< \zeta(X) = 3(k-2). \end{aligned}$$

Let $k_0 - r_0 = m$, $k_1 - r_1 = n$ and $k_2 - d = t$. Since $|A_1| = |A_0| = |H_0|$, $|H_0| < \frac{g}{m+n+2t} \cdot (k-2)$. Since $|A| > k-2$ and $3(k-2) > (m+t) \cdot |A_0| + (n+t) \cdot |A_1| \geq t \cdot |A_0| + t \cdot |A_1| = t \cdot |A|$, $t < 3$. We consider three subcases in the following:

Subcase 2.1. $k_2 - d = 0$.

Since $k_2 - d = 0$, $1 \leq m+n \leq 4$. Obviously, $r_0 \geq 0$ and $r_1 \geq 0$, so $m \leq k_0$ and $n \leq k_1$, and there exists an inverse-closed subset $S_0' = \{s_{01}, ..., s_{0m}\}$ of S_0 satisfied $\langle S_0 \setminus S_0' \rangle \leq H_0$, and an inverse-closed subset $S_1' = \{s_{11}, ..., s_{1n}\}$ of S_1 satisfied $\langle S_1 \setminus S_1' \rangle \leq s_2 H_0 s_2^{-1}$. Since $k_2 - d = 0$, $sH_0 = s_2 H_0$ for any element $s \in S_2$. Hence $G_2 \leq H_0$.

Subcase 2.2. $k_2 - d = 1$.

Since $k_2 - d = 1$, $0 \leq m+n \leq 2$. Obviously, $r_0 \geq 1$, which implies $m < k_0$. Thus there exists a proper, inverse-closed subset $S_0' = \{s_{01}, ..., s_{0m}\}$ of S_0 satisfied $\langle S_0 \setminus S_0' \rangle \leq H_0$. Suppose $1_G \in s_2 H_0 g_s$, thus $1_G = s_2 h_0 g_s$ and $g_s = h_0^{-1} s_2^{-1}$ for some $h_0 \in H_0$. Since $MR(g_s)(A)$ is also an atom and $k_1 - r_1 = n < k_1$. there exists a proper, inverse-closed subset $S_1' = \{s_{11}, ..., s_{1n}\}$ of S_1 satisfied $\langle S_1 \setminus S_1' \rangle \leq s_2 H_0 g_s = s_2 H_0 s_2^{-1}$. Since $k_2 - d = 1$, there exists an element $s_2' \in S_2$, and $s_{2j} H_0 = s_2 H_0$ for any element $s_{2j} \in S_2 \setminus \{s_2'\}$. Hence $\langle (S_2 \setminus \{s_2'\})^{-1} (S_2 \setminus \{s_2'\}) \rangle \leq H_0$.

Subcase 2.3. $k_2 - d = 2$.

$m + n = 0$ for $k_2 - d = 2$. Since $k_0 - r_0 = m = 0$, $G_0 \le H_0$. Suppose $1_G \in s_2 H_0 g_s$, thus $1_G = s_2 h_0 g_s$ and $g_s = h_0^{-1} s_2^{-1}$ for some $h_0 \in H_0$. Since $MR(g_s)(A)$ is also an atom and $k_1 - r_1 = n = 0$, we have $G_1 \le s_2 H_0 g_s = s_2 H_0 s_2^{-1}$. Since $k_2 - d = 2$, there exists two elements $s_{21}, s_{22} \in S_2$, and $s_{2j} H_0 = s_2 H_0$ for any element $s_{2j} \in S_2 \setminus \{s_{21}, s_{22}\}$. Hence $\langle (S_2 \setminus \{s_{21}, s_{22}\})^{-1}(S_2 \setminus \{s_{21}, s_{22}\})\rangle \le H_0$.□

With the similar manner to that for theorem 3.3 and 3.4, we have the following two theorems.

Theorem 3.5. *Let $X = MC(G, S_0, S_1, S_2)$ be a connected graph with $\delta \ge 6$ and girth $g = 3$. Assume $G_i = \langle S_i \rangle$ for $i \in \{0,1\}$, and $S_2 = \{s_2\}$. If $k_1 > k_0$, and X_0 contains a cycle of length g, then X is not cyclically optimal if and only if X satisfies one of the following conditions:*

(1) There exists a subgroup H of G satisfied $|H| < (\delta - 2)\cdot g$ and $G_i \le H$ for $i \in \{0,1\}$;

(2) There exists a subgroup H of G and an element $s_i \in S_i$, satisfied $|H| < \frac{g}{2}\cdot(\delta - 2)$ and $\langle S_i \setminus \{s_i\}\rangle \le H$ for $i \in \{0,1\}$;

(3) There exists a subgroup H of G, a proper, inverse-closed subset $S_0' = \{s_{01}, ..., s_{0m}\}$ of S_0, and a proper, inverse-closed subset $S_1' = \{s_{11}, ..., s_{1n}\}$ of $S_1 (1 \le m + n \le 4)$, satisfied $|H| < \frac{g}{m+n}\cdot(\delta - 2)$, $\langle S_0 \setminus S_0' \rangle \le H$, and $\langle S_1 \setminus S_1' \rangle \le s_2 H s_2^{-1}$.

Theorem 3.6. *Let $X = MC(G, S_0, S_1, S_2)$ be a connected graph with $\delta \ge 6$ and girth $g = 3$. Assume $G_i = \langle S_i \rangle$ for $i \in \{0,1\}$, and $G_2 = \langle S_2^{-1} S_2 \rangle$. If $k_1 > k_0 \ge 2, k_2 \ge 2$, and X_0 contains a cycle of length g, then X is not cyclically optimal if and only if X satisfies one of the following conditions:*

(1) There exists a subgroup H of G satisfied $|H| < \frac{g}{k_2}\cdot(\delta - 2)(k_2 = 2)$ and $G_i \le H$ for $i \in \{0,1\}$;

(2) There exists a subgroup H of G, an inverse-closed subset $S_0' = \{s_{01}, ..., s_{0m}\}$ of S_0, an inverse-closed subset $S_1' = \{s_{11}, ..., s_{1n}\}$ of $S_1 (1 \le m + n \le 4)$, and an element $s_2 \in S_2$, satisfied $|H| < \frac{g}{m+n}\cdot(\delta - 2)$, $\langle S_0 \setminus S_0' \rangle \le H$, $\langle S_1 \setminus S_1' \rangle \le s_2 H s_2^{-1}$, and $G_2 \le H$;

(3) There exists a subgroup H of G, a proper, inverse-closed subset $S_0' = \{s_{01}, ..., s_{0m}\}$ of S_0, a proper, inverse-closed subset $S_1' = \{s_{11}, ..., s_{1n}\}$ of $S_1 (0 \le m+n \le 2)$, and two elements $s_2, s_2' \in S_2$, satisfied $|H| < \frac{g}{m+n+2}\cdot(\delta - 2)$, $\langle S_0 \setminus S_0' \rangle \le H$, $\langle S_1 \setminus S_1' \rangle \le s_2 H s_2^{-1}$, and $\langle (S_2 \setminus \{s_2'\})^{-1}(S_2 \setminus \{s_2'\})\rangle \le H$;

(4) There exists a subgroup H of G, and three elements $s_2, s_{21}, s_{22} \in S_2$, satisfied $|H| < \frac{g}{4}\cdot(\delta - 2)$, $G_0 \le H$, $G_1 \le s_2 H s_2^{-1}$, and $\langle (S_2 \setminus \{s_{21}, s_{22}\})^{-1}(S_2 \setminus \{s_{21}, s_{22}\})\rangle \le H$.

4 $c\lambda-$optimally connected mixed Cayley graph with $g = 4$

Lemma 4.1. *Let $X = MC(G, S_0, S_1, S_2)$ be a $k-$regular connected graph with $k \geq 4$ and girth $g = 4$. Assume A is an atom of X. If X is not cyclically optimal, then $|A| > 2k - 4$.*

Proof. Assume $a = |A|$. Since A is an atom of X, and X is not cyclically optimal, we have $\omega(A) = c\lambda(X) < \zeta(X) = (k-2) \cdot g$.

By Turán theorem, we have $|E(A)| \leq |E(T_{2,n})|$. Considering the sum of degrees of all vertices of A, we have

$$k \cdot a = \Sigma_{v \in A} d(v) \leq 2 \cdot \frac{a}{2} \cdot \frac{a}{2} + \omega(A) < \frac{a^2}{2} + 4(k-2),$$

that is $(a-4)(2k-a-4) < 0$. Hence $|A| = a > 2k-4$ for $a = |A| > g = 4$.□

Lemma 4.2. *Let $X = MC(G, S_0, S_1, S_2)$ be a connected graph with $\delta \geq 4$ and girth $g = 4$. Assume A is an atom of X, $k_1 > k_0$, and X_0 contains a cycle of length g. If X is not cyclically optimal, then*

(1) When $A \subseteq V_0$, $|A| > 2\delta - 4$;

(2) When $A \subseteq V_1$, $|A| > 2\Delta - 4$;

(3) When $A \cap V_i \neq \phi (i = 0, 1)$, $|A| > \delta + \Delta - 4$.

Proof. Assume $a = |A|$. Since A is an atom of X, and X is not cyclically optimal, we have $\omega(A) = c\lambda(X) < \zeta(X) = (\delta - 2) \cdot g$.

(1) When $A \subseteq V_0$, we have

$$\delta \cdot a = \Sigma_{v \in A} d(v) \leq 2 \cdot \frac{a}{2} \cdot \frac{a}{2} + \omega(A) < \frac{a^2}{2} + 4(\delta - 2),$$

that is $(a-4)(2\delta - a - 4) < 0$. Hence $|A| = a > 2\delta - 4$ for $a = |A| > g = 4$.

(2) When $A \subseteq V_1$, we have

$$\Delta \cdot a = \Sigma_{v \in A} d(v) \leq 2 \cdot \frac{a}{2} \cdot \frac{a}{2} + \omega(A) < \frac{a^2}{2} + 4(\delta - 2),$$

that is $(a-4)(2\Delta - a - 4) < 0$. Hence $|A| = a > 2\Delta - 4$.

(3) When $A_i = A \cap V_i \neq \phi$ for $i \in \{0, 1\}$, $|A_0| = |A_1|$ by lemma 2.9, and we have

$$\delta \cdot \frac{a}{2} + \Delta \cdot \frac{a}{2} = \Sigma_{v \in A} d(v) \leq 2 \cdot \frac{a}{2} \cdot \frac{a}{2} + \omega(A) < \frac{a^2}{2} + 4(\delta - 2),$$

that is $(a-4)(\delta + \Delta - a - 4) < 0$. Hence $|A| = a > \delta + \Delta - 4$.□

With the similar manner to that for theorem 3.3 and 3.4, we obtain the following four theorems.

Theorem 4.3. *Let $X = MC(G, S_0, S_1, S_2)$ be a k-regular connected graph with $k \geq 4$ and girth $g = 4$. Assume $G_i = \langle S_i \rangle$ for $i \in \{0,1\}$, and $S_2 = \{s_2\}$. Then X is not cyclically optimal if and only if X satisfies one of the following conditions:*

(1) There exists a subgroup H of G satisfied $|H| < (k-2)\cdot g$ and $G_i \leq H$ for $i \in \{0,1\}$;

(2) There exists a subgroup H of G, a proper, inverse-closed subset $S_0' = \{s_{01}, ..., s_{0m}\}$ of S_0, and a proper, inverse-closed subset $S_1' = \{s_{11}, ..., s_{1n}\}$ of $S_1 (1 \leq m + n \leq 3)$, satisfied $|H| < \frac{g}{m+n} \cdot (k-2)$, $\langle S_0 \setminus S_0' \rangle \leq H$, and $\langle S_1 \setminus S_1' \rangle \leq s_2 H s_2^{-1}$.

Theorem 4.4. *Let $X = MC(G, S_0, S_1, S_2)$ be a k-regular connected graph with $k \geq 4$ and girth $g = 4$. Assume $G_i = \langle S_i \rangle$ for $i \in \{0,1\}$, and $G_2 = \langle S_2^{-1} S_2 \rangle$. If $k_0 = k_1 \geq 2$ and $k_2 \geq 2$. Then X is not cyclically optimal if and only if X satisfies one of the following conditions:*

(1) There exists a subgroup H of G, an inverse-closed subset $S_0' = \{s_{01}, ..., s_{0m}\}$ of S_0, an inverse-closed subset $S_1' = \{s_{11}, ..., s_{1n}\}$ of $S_1 (1 \leq m + n \leq 3)$, and an element $s_2 \in S_2$, satisfied $|H| < \frac{g}{m+n} \cdot (k-2)$, $\langle S_0 \setminus S_0' \rangle \leq H$. $\langle S_1 \setminus S_1' \rangle \leq s_2 H s_2^{-1}$, and $G_2 \leq H$;

(2) There exists a subgroup H of G, a proper, inverse-closed subset $S_0' = \{s_{01}, ..., s_{0m}\}$ of S_0, a proper, inverse-closed subset $S_1' = \{s_{11}, ..., s_{1n}\}$ of $S_1 (0 \leq m+n \leq 1)$, and two elements $s_2, s_2' \in S_2$, satisfied $|H| < \frac{g}{m+n+2} \cdot (k-2)$, $\langle S_0 \setminus S_0' \rangle \leq H$, $\langle S_1 \setminus S_1' \rangle \leq s_2 H s_2^{-1}$, and $\langle (S_2 \setminus \{s_2'\})^{-1} (S_2 \setminus \{s_2'\}) \rangle \leq H$.

Theorem 4.5. *Let $X = MC(G, S_0, S_1, S_2)$ be a connected graph with $\delta \geq 4$ and girth $g = 4$. Assume $G_i = \langle S_i \rangle$ for $i = 0, 1$ and $S_2 = \{s_2\}$. If $k_1 > k_0$, and X_0 contains a cycle of length g, then X is not cyclically optimal if and only if X satisfies one of the following conditions:*

(1) There exists a subgroup H of G satisfied $|H| < (\delta-2)\cdot g$ and $G_i \leq H$ for $i \in \{0,1\}$;

(2) There exists a subgroup H of G, a proper, inverse-closed subset $S_0' = \{s_{01}, ..., s_{0m}\}$ of S_0, a proper, inverse-closed subset $S_1' = \{s_{11}, ..., s_{1n}\}$ of $S_1 (1 \leq m + n \leq 3)$, satisfied $|H| < \frac{g}{m+n} \cdot (\delta - 2)$, $\langle S_0 \setminus S_0' \rangle \leq H$, and $\langle S_1 \setminus S_1' \rangle \leq s_2 H s_2^{-1}$.

Theorem 4.6. *Let $X = MC(G, S_0, S_1, S_2)$ be a connected graph with $\delta \geq 4$ and girth $g = 4$. Assume $G_i = \langle S_i \rangle$ for $i = 0, 1$ and $G_2 = \langle S_2^{-1} S_2 \rangle$. If $k_1 > k_0 \geq 2$ and $k_2 \geq 2$, and X_0 contains a cycle of length g, then X is not cyclically optimal if and only if X satisfies one of the following conditions:*

(1) There exists a subgroup H of G, an inverse-closed subset $S_0' = \{s_{01}, ..., s_{0m}\}$ of S_0. an inverse-closed subset $S_1' = \{s_{11}, ..., s_{1n}\}$ of $S_1 (1 \leq$

$m+n \leq 3$), *and an element* $s_2 \in S_2$, *satisfied* $|H| < \frac{g}{m+n} \cdot (\delta - 2)$, $\langle S_0 \setminus S_0' \rangle \leq H$, $\langle S_1 \setminus S_1' \rangle \leq s_2 H s_2^{-1}$, *and* $G_2 \leq H$;

(2) There exists a subgroup H *of* G, *a proper, inverse-closed subset* $S_0' = \{s_{01}, ..., s_{0m}\}$ *of* S_0, *a proper, inverse-closed subset* $S_1' = \{s_{11}, ..., s_{1n}\}$ *of* $S_1 (0 \leq m+n \leq 1)$, *and two elements* $s_2, s_2' \in S_2$, *satisfied* $|H| < \frac{g}{m+n+2} \cdot (\delta - 2)$, $\langle S_0 \setminus S_0' \rangle \leq H$, $\langle S_1 \setminus S_1' \rangle \leq s_2 H s_2^{-1}$, *and* $\langle (S_2 \setminus \{s_2'\})^{-1} (S_2 \setminus \{s_2'\}) \rangle \leq H$.

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