A note on the graphs with given small matching number*

Weihua Yang[a]† Hao Li[b]

[a]Department of Mathematics, Taiyuan University of Technology, 030024 Taiyuan, Shanxi, China

[b]Laboratoire de Recherche en Informatique, UMR 8623, C.N.R.S.-Université de Paris-sud, 91405-Orsay cedex, France

Abstract: In this note we characterize the graphs with a given small matching number. We characterize the graphs with minimum degree at least 2 and matching number at most 3. The characterization when the matching number is at most 2 strengthens the result of Lai and Yan's that characterized the non-supereulerian 2-edge connected graphs with matching at most 2; Furthermore, the characterization of the graphs with matching number at most 3 addresses a conjecture of Lai and Yan in [SuperEulerian graphs and matchings, Applied Mathematics Letters 24 (2011) 1867–1869].

Keywords: Supereulerian graphs, Matching number

1 Introduction

Motivated by the Chinese Postman Problem, Boesch et al. [3] proposed the supereulerian graph problem: determine when a graph has a spanning eulerian subgraph. They indicated that this might be a difficult problem. Pulleyblank [13] showed that such a decision problem, even when restricted to planar graphs, is NP-complete. We refer the readers to [6, 9] for the supereulerian graph problem.

We use [2] for terminology and notation not defined here, and consider simple finite graphs only. In particular, *matching number* of a graph G is the size of the maximum matching in G, denoted by $\alpha'(G)$. We denote by $\delta(G)$ the minimum degree of G. Let m, n be two positive integers. Let

*The research is supported by NSFC (No.11301371), NSF of Shanxi Province (No. 2014021010-2) and The Project-sponsored by SRF for ROCS, SEM.

†Corresponding author.E-mail: ywh222@163.com (W. Yang).

$H_1 \cong K_{2,m}$ and $H_2 \cong K_{2,n}$ be two complete bipartite graphs. Let u_1, v_1 be two nonadjacent vertices of degree m in H_1, and u_2, v_2 be two nonadjacent vertices of degree n in H_2. Let $S_{n,m}$ denote the graph obtained from H_1 and H_2 by identifying v_1 and v_2, and by connecting u_1 and u_2 with a new edge u_1u_2. Note that $S_{1,1}$ is the same as C_5, the 5-cycle. Define $K_{1,3}(1,1,1)$ to be the graph obtained from a 6-cycle $C = u_1u_2u_3u_4u_5u_6u_1$ by adding one vertex u and three edges uu_1, uu_3 and uu_5.

Recently, Lai and Yan in [12] considered supereulerian graph problem with the restriction of matching number of a graph and posed two conjectures as follows.

Conjecture 1. *If G is a 2-edge-connected simple graph with matching number at most 3, then G is supereulerian if and only if G is not one of $\{K_{2,t}, S_{n,m}, K_{1,3}(1,1,1)\}$ where n, m are natural numbers and t is an odd number*

Conjecture 2. *If G is a 3-edge-connected simple graph with matching number at most 5, then G is supereulerian if and only if G is not contractible to the Petersen graph.*

In [1], An and Xiong pointed that the second conjecture is a corollary of a result in Chen [8] and they also pointed that the first conjecture is not true by giving some counterexamples. They revised the first conjecture as a new conjecture in [1], but the revision is not complete yet. We do not list it here, see [1].

In this note we first obtain a characterization for graphs with minimum degree 2 and matching number at most 2 that strengthens the result of Lai and Yan in [12] (the main result in [12] is that every 2-edge-connected graph with $\alpha'(G) \leq 2$ is supereulerian if and only if G is not $K_{2,t}$ for some odd number t); Similarly, a characterization of the graphs with minimum degree at least 2 and matching number at most 3 is obtained which addresses the problem raised by Conjecture 1.

2 The characterization of graphs with minimum degree at least 2 and matching number at most 2

In this section we characterize the graphs with minimum degree 2 and matching number at most 2. For a graph G, a cycle of G is called *dominating cycle* if the cycle contains at least one endvertex of any edge of G.

Theorem 3. *Let G be a graph with minimum degree at least 2 and maximum matching number at most 2. Then $G \in \mathcal{F}_1 = \{G : \delta(G) \geq 2$ and $|V(G)| \leq 5\} \cup \{K_{2,t}\} \cup \{K'_{2,t}\}$, where $K'_{2,t}$ is obtained from $K_{2,t}$ by adding an edge between the two vertices of degree t.*

Proof. Suppose C is the longest cycle of G with length l. If $l \leq 3$, then by $\delta(G) \geq 2$ and $\alpha'(G) \leq 2$ we clearly have either $G \cong K_3$ or G isomorphic to the hourglass, where the hourglass is a graph obtained from K_5 by removing the edges of a C_4. Thus, $G \in \mathcal{F}_1$. Note that $\alpha'(G) \leq 2$, then $l \leq 5$.

Case 1. $l = 5$.

We claim that C is a dominating cycle of G, since other cases will induce $\alpha'(G) \geq 3$. Note that G is connected. If $V(G - C) \neq \emptyset$, the case will induce a matching of size 3. Thus, $|V(G)| = 5$ and then $G \in \{G : \delta(G) \geq 2$ and $|V(G)| \leq 5\} \subseteq \mathcal{F}_1$.

Case 2. $l = 4$.

Suppose $C = x_1x_2x_3x_4x_1$. Clearly, C is a dominating cycle, since otherwise it will induce $\alpha'(G) \geq 3$. If $V(G) = V(C)$, then $G \in \mathcal{F}_1$. Otherwise, let $v \in V(G - C)$. Since $\delta(G) \geq 2$, v has exactly two neighbors in $V(C)$ (otherwise will induces a cycle of length 5). Since $\alpha'(G) \leq 2$, all the vertices in $V(G - C)$ have the same neighbors in $V(C)$. So we clearly have either $G \cong K_{2,t}$ or $G \cong K'_{2,t}$. □

Observation 4. *A graph G with $\delta(G) \geq 2$ and $|V(G)| \leq 5$ is either supereulerian or $G = K_{2,3}$. Moreover, $K_{2,t}$ is supereulerian if t is an even number, and non-supereulerian otherwise.*

Proof. It is easy to obtain the first part by considering its longest cycle. The second part is obvious. □

Corollary 5 (Theorem 2 [12]). *If G is a 2-edge-connected simple graph with matching number at most 2, then G is supereulerian if and only if G is not $K_{2,t}$ for some odd number t.*

Proof. By the Observation 4, the corollary holds since $K'_{2,t}$ is supereulerian. □

3 The graphs with matching number at most 3

We first introduce some special graphs as follows.

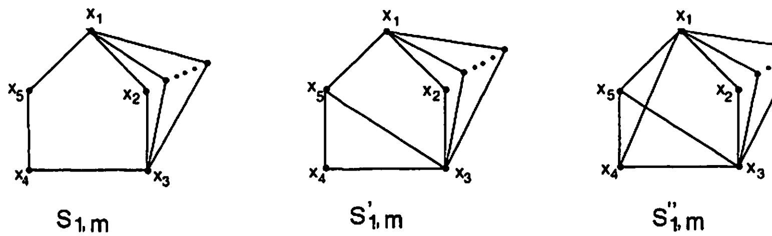

Figure.1 The case $n = 1$.

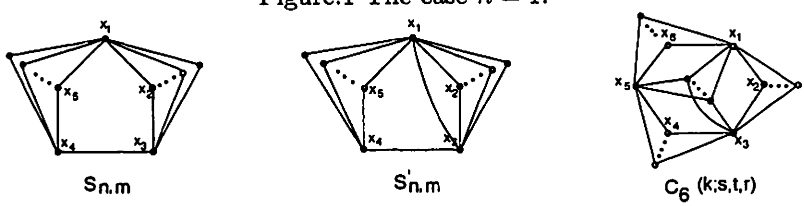

Figure.2 The case $m \geq n \geq 2$.

The following four graphs below are the simple variants of $S_{n.m}$: $S^*_{1,m} = S_{1,m} + x_1x_3, S'^*_{1,m} = S'_{1,m} + x_1x_3, S''^*_{1,m} = S'_{1,m} + x_1x_3, S''_{n,m} = S'_{n,m} + x_1x_4$. Similarly, $C_6(k; s, t)$ contains the following three variants that obtained by adding edges $\{x_1x_3\}, \{x_1x_3, x_3x_5\}$ and $\{x_1x_3, x_3x_5, x_5x_1\}$, respectively. We use $\mathcal{C}$ to denote the set of $C_6(k; s, t, r)$ and its three variants mentioned above. Let $\mathcal{S} = \{S_{1,m}, S'_{1,m}, S''_{1,m}, S^*_{1,m}, S'^*_{1,m}, S''^*_{1,m}, S_{n,m}, S'_{n,m}, S''_{n,m}, K_{2,t}, K'_{2,t}\}$

Theorem 6. *Let G be a graph with minimum degree 2 and matching number at most 3. Then G is in $\mathcal{F}_2 = \{G : |V(G)| \leq 7\} \cup \mathcal{S} \cup \mathcal{C}$.*

Proof. Suppose C is the longest cycle of G with length l. Similarly as in Theorem 4, we have $|V(G)| = 7$ if $l = 7$ and thus $G \in \mathcal{F}_2$.

If $l = 3$, then $G \in \{K_3, H, H', H''\}$, where H denotes the hourglass, H', H'' denote the two different graphs obtained from a hourglass and a triangle by identifying a vertex of a hourglass and a vertex of a triangle. Each of the cases implies $|V(G)| \leq 7$ and then $G \in \mathcal{F}_2$. Thus we may assume $l \geq 4$.

Case 1. $l = 6$

Let $C = x_1x_2x_3x_4x_5x_6x_1$ be a longest cycle of G. Clearly, C is a dominating cycle of G. Suppose $V(G - C) \neq \emptyset$. Since $\alpha'(G) \leq 3$ and $l = 6$, at most one of the two endvertices of an edge $ab \in E(C)$ has neighbors in $V(G - C)$. Thus at most three pairwise nonadjacent vertices of $V(C)$ have neighbors in $V(G - C)$ and assume that they are x_1, x_3, x_5. Suppose there are k vertices of $V(G - C)$ such that each of them is only adjacent to vertices of x_1, x_3, x_5. In this case, it is easy to see that $G \in \mathcal{C}$.

Case 2. $l = 5$

Let $C = x_1x_2x_3x_4x_5x_1$ be a longest cycle of G. Note that $l = 5$ and $\delta(G) \geq 2$. Then if C is not a dominating cycle of G, then G is the graph obtained from a cycle C_5 (a cycle of length 5) and a triangle by identifying a vertex of the C_5 and a vertex of triangle and adding some edges on the C_5. In this case, we have $|V(G)| \leq 7$. Suppose C is a dominating cycle of G. It is easy to see that G contains a $S_{n,m}$ as a subgraph for some m, n. If $m \geq n \geq 2$, then $G \in \{S_{n,m}, S'_{n,m}, S''_{n,m}\}$. If $n = 1$, then $G \in \{S_{1,m}, S'_{1,m}, S''_{1,m}\}$, see Figure 2.

Case 3. $l = 4$

Let $C = x_1x_2x_3x_4x_1$ be a longest cycle of G. Similarly, if C is not a dominating cycle, then $|V(G)| \leq 7$. If C is a dominating cycle, we have $G \cong K_{2,t}$ or $G \cong K'_{2,t}$ for some integer t. □

Note that $S^*_{1,m}, S'^*_{1,m}, S''^*_{1,m}$, and $S''_{n,m}$ are supereulerian. If $G \cong C_6(k; s, t, r)$, $k = 1, s, t, r \geq 1$ and the parities of s, t, r are the same, then G is not supereulerian. In fact, G has 4 vertices of odd degree and only one edge can be removed. So G is not supereulerian. If $k \geq 2$, the $k - 1$ vertices of degree 3 can be used to adjust the parities of s, t, r such that one of them is different from others, then the resulting is supereulerian clearly. So we have the following theorem.

Theorem 7. *Let G be a graph with minimum degree 2 and $\alpha'(G) \leq 3$. Then G is not supereulerian if and only if one of the following holds:*

(1) *If $G \cong S_{n,m}, m \geq n \geq 1$, then one of n, m is an even number;*

(2) *If $G \in \{S'_{1,m}, S''_{1,m}\}$, then m is even number;*

(3) *If $G \cong S'_{n,m}, m \geq n \geq 2$, then x_4 is a vertex of odd degree.*

(4) *If $G \cong C_6(k; s, t, r)$, then $k = 0$ or 1. Moreover, if $k = 1$, then the parities of s, t, r are the same; If $k = 0$, then s, t, r are different.*

Proof. We only prove the necessity. We first claim that a graph $G \notin \mathcal{S} \cup \mathcal{C}$ with at most 7 vertices and $\delta(G) \geq 2$ is supereulerian. In fact, the claim is easily obtained by considering the longest cycle of G. Thus we assume $|V(G)| \geq 8$. By Theorem 6 and the discussion above, (1), (2), (3), (4) are easy to obtain by consider the parities of n, m, s, t, r. □

Remark 8. *Theorem 7 implies Conjecture 1 is not complete and one can revise it easily by using the theorem above.*

4 Acknowledgements

We would like to thank the referees for their helpful comments and useful suggestions.

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