On the path-connectivity of lexicographic product graphs*

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Abstract

The k-path-connectivity $\pi_k(G)$ of a graph G was introduced by Hager in 1986. Recently, Mao investigated the 3-path-connectivity of lexicographic product graphs. Denote by $G \circ H$ the lexicographic product of two graphs G and H. In this paper, we prove that $\pi_4(G \circ H) \geq \pi_4(G)\lfloor \frac{|V(H)|-2}{3} \rfloor + 1$ for any two connected graphs G and H. Moreover, the bound is sharp. We also derive an upper bound of $\pi_4(G \circ H)$, that is, $\pi_4(G \circ H) \leq 2\pi_4(G)|V(H)|$.

Keywords: Connectivity; internally disjoint S-paths; path-connectivity; lexicographic product.

AMS subject classification 2010: 05C05, 05C40, 05C70, 05C76.

1 Introduction

All graphs considered in this paper are undirected, finite and simple. We refer to [2] for graph theoretical notation and terminology not described here. For a graph G, let $V(G)$, $E(G)$ and $\delta(G)$ denote the set of vertices, the set of edges and the minimum degree of G, respectively. The connectivity of a graph is one of the most basic concepts in graph theory. For more details on the connectivity and the edge-connectivity of a graph, we can refer to the survey paper [24].

Steiner tree is popularly used in the physical design of VLSI circuits (see [7, 8, 25]). Steiner tree is also used in computer communication networks (see [6]) and optical wireless communication networks (see [4]). In [9], Hager introduced the concept of the generalized connectivity of a graph. Let G be a nontrivial connected graph of order n and k be an integer with $2 \leq k \leq n$. For a set S with k vertices of $V(G)$, let $\kappa(S)$ denote the maximum number of edge-disjoint Steiner trees $T_1, T_2, \cdots, T_\ell$ in G such that $V(T_i) \cap V(T_j) = S$ for every pair i, j of distinct integers with $1 \leq i, j \leq \ell$. The generalized k-connectivity of G, denoted by $\kappa_k(G)$, is defined as $\kappa_k(G) = \min \kappa(S)$, where the minimum is taken over all k-subsets S of $V(G)$. Thus $\kappa_2(G) = \kappa(G)$, where $\kappa(G)$ is the

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connectivity of G. For more details about the generalized k-connectivity, we can refer to [3, 12, 13, 14, 15, 16, 18, 17, 19, 20, 23].

In [5], Dirac proved that in a $(k-1)$-connected graph there is a path through each k vertices. Later, Hager [10] revised this statement to the question how many internally disjoint paths P_i with the exception of a given set S of k vertices exist such that $S \subseteq V(P_i)$. For a graph $G = (V, E)$ and a set $S \subseteq V(G)$ of at least two vertices, *a path connecting* S (or simply, *a* S*-path*) is a subgraph $P = (V', E')$ of G that is a path with $S \subseteq V'$. Let G be a nontrivial connected graph of order n and k be an integer with $2 \leq k \leq n$. For a set S with k vertices of $V(G)$, let $\pi(S)$ denote the maximum number of edge-disjoint S-paths $P_1, P_2, \cdots, P_\ell$ in G such that $V(P_i) \cap V(P_j) = S$ for every pair i, j of distinct integers with $1 \leq i, j \leq \ell$. The *k-path-connectivity* of G is defined as $\pi_k(G) = \min\{\pi_G(S) \mid S \subseteq V(G), |S| = k\}$. Clearly, $\pi_1(G) = \delta(G)$ and $\pi_2(G) = \kappa(G)$. For $k \geq 3$, $\pi_k(G) \leq \kappa_k(G)$ holds because each path is also a tree.

Recently, Mao [21] investigated the 3-path-connectivity of lexicographic product graphs. In this paper, we will study the sharp lower bound of $\pi_4(G \circ H)$. Recall that the lexicographic product of two graphs G and H, written as $G \circ H$, is defined as follows: $V(G \circ H) = V(G) \times V(H)$, and two distinct vertices (u, v) and (u', v') of $G \circ H$ are adjacent if and only if either $(u, u') \in E(G)$ or $u = u'$ and $(v, v') \in E(H)$. Notice that unlike the Cartesian product, the lexicographic product is a non-commutative product since $G \circ H$ is usually not isomorphic to $H \circ G$.

Observation 1.1 (1) *Let G be a connected graph. Then $\pi_4(G) \leq \delta(G)$.*

(2) *Let G be a connected graph with the minimum degree δ. If G has two adjacent vertices of degree δ, then $\pi_k(G) \leq \delta - 1$.*

In [10], Hager got a sharp lower bound of $\pi_4(G)$.

Lemma 1.1 [10] *For any connected graph G, $\pi_4(G) \geq \frac{1}{3}\kappa(G)$. Moreover, the lower bound is sharp.*

Li et al. [16] obtained the following result.

Lemma 1.2 [16] *For any connected graph G, $\kappa_4(G) \leq \kappa(G)$. Moreover, the upper bound is sharp.*

Yang and Xu [28] investigated the classical connectivity of the lexicographic product of two graphs.

Lemma 1.3 [28] *Let G and H be two graphs. If G is non-trivial, non-complete and connected, then $\kappa(G \circ H) = \kappa(G)|V(H)|$.*

Now we obtain an upper bound of $\pi_4(G \circ H)$ by the above three lemmas.

Theorem 1.1 *Let G and H be two connected graphs. Then*

$$\pi_4(G \circ H) \leq 3\pi_4(G)|V(H)|.$$

Proof. From Lemma 1.1, we have $\pi_4(G \circ H) \geq \frac{1}{3}\kappa(G \circ H)$, hence $\kappa(G \circ H) \leq 3\pi_4(G \circ H)$. By Lemma 1.2, $\pi_4(G \circ H) \leq \kappa_4(G \circ H) \leq \kappa(G \circ H)$. Furthermore, by Lemma 1.3, we have $\pi_4(G \circ H) \leq \kappa(G \circ H) = \kappa(G)|V(H)| \leq 3\pi_4(G)|V(H)|$. The proof is complete. ∎

In Section 2, we will prove the following lower bound of $\pi_4(G \circ H)$.

Theorem 1.2 *Let G and H be two connected graphs. Then*

$$\pi_4(G \circ H) \geq \pi_4(G)\left\lfloor \frac{|V(H)| - 2}{3} \right\rfloor + 1.$$

Moreover, the bound is sharp.

To show the sharpness of the above lower bound, we let $G = P_n$ and $H = P_3$. Clearly, $\pi_4(G) = 1$ and $|V(H)| = 3$. Thus, $\pi_4(P_n \circ P_3) \geq 1$. One can check that $\pi_4(P_n \circ P_3) \leq 1$. Therefore, $\pi_4(P_n \circ P_3) = 1 = \lfloor \frac{|V(H)|-2}{3} \rfloor + 1$.

2 Proof of Theorem 1.2

In this section, let G and H be two connected graphs with $V(G) = \{u_1, u_2, \ldots, u_n\}$ and $V(H) = \{v_1, v_2, \ldots, v_m\}$, respectively. Then $V(G \circ H) = \{(u_i, v_j) \,|\, 1 \leq i \leq n,\ 1 \leq j \leq m\}$. For $v \in V(H)$, we use $G(v)$ to denote the subgraph of $G \circ H$ induced by the vertex set $\{(u_i, v) \,|\, 1 \leq i \leq n\}$. Similarly, for $u \in V(G)$, we use $H(u)$ to denote the subgraph of $G \circ H$ induced by the vertex set $\{(u, v_j) \,|\, 1 \leq j \leq m\}$. In the sequel, let K_n and P_n denote the complete graph and the path with order n, respectively. If G is a connected graph and $x, y \in V(G)$, then the *distance* $d_G(x, y)$ between x and y is the length of a shortest path connecting x and y in G. The degree of a vertex v in G is denoted by $d_G(v)$.

Given a vertex x and a set U of vertices, a *(x, U)-fan* is the set of paths from x to U such that any two of them share only the vertex x. The size of a (x, U)-fan is the number of internally disjoint paths from x to U.

We now introduce the general idea of the proof of Theorem 1.2. In Section 2.1, we first address the 4-path-connectivity of the lexicographic product of a path P and a connected graph H and prove $\pi_4(P \circ H) \geq \lfloor \frac{|V(H)|-2}{3} \rfloor + 1$. After this preparation, we consider the graph $G \circ H$ and prove $\pi_4(G \circ H) \geq \pi_4(G)\lfloor \frac{|V(H)|-2}{3} \rfloor + 1$ in Subsection 2.2.

For the sake of our results, we need to introduce the following two well-known lemmas.

Lemma 2.1 *(Fan Lemma, [27], p-170) A graph is k-connected if and only if it has at least $k + 1$ vertices and, for every choice of x, U with $|U| \geq k$, it has a (x, U)-fan of size k.*

Lemma 2.2 *(Expansion Lemma, [27], p-162) If G is a k-connected graph and G' is obtained from G by adding a new vertex y with at least k neighbors in G, then G' is k-connected.*

Let G be a k-connected graph. Choose $U \subseteq V(G)$ with $|U| = k$. Then the graph G' is obtained from G by adding a new vertex y and joining each vertex of U and the vertex y. We call this operation *an expansion operation at y and U*. Denote the resulting graph G' by $G' = G \vee \{y, U\}$.

2.1 The Lexicographic product of a path and a connected graph

To start with, we introduce the following proposition, which is the preparation of the next subsection.

Proposition 2.1 *Let H be a connected graph and P_n be a path with n vertices. Then $\pi_4(P_n \circ H) \geq \lfloor \frac{|V(H)|-2}{3} \rfloor + 1$. Moreover, the bound is sharp.*

Set $V(H) = \{v_1, v_2, \ldots, v_m\}$ and $V(P_n) = \{u_1, u_2, \ldots, u_n\}$. Without loss of generality, let u_i and u_j be adjacent if and only if $|i - j| = 1$, where $1 \leq i \neq j \leq n$. It suffices to show that $\pi_{P_n \circ H}(S) \geq \lfloor \frac{m-2}{3} \rfloor + 1$ for any $S = \{x, y, z, t\} \subseteq V(P_n \circ H)$, that is, there exist $\lfloor \frac{m-2}{3} \rfloor + 1$ internally disjoint paths connecting S in $P_n \circ H$. We proceed our proof by the following four lemmas.

Lemma 2.3 *If x, y, z, t belongs to the same $V(H(u_i))$ $(1 \leq i \leq n)$, then there exist $\lfloor \frac{m-2}{3} \rfloor + 1$ internally disjoint S-paths.*

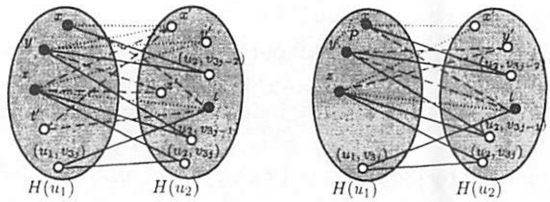
Proof. Without loss of generality, we assume $x, y, z, t \in V(H(u_1))$. Since H is connected, there exists a path connecting x and y in $V(H(u_1))$ and we say P. Then the paths L_j induced by the edges in $\{x(u_2, v_{3j-2}), y(u_2, v_{3j-2}), y(u_2, v_{3j-1}), z(u_2, v_{3j-1}), z(u_2, v_{3j}), t(u_2, v_{3j})\}$ $(1 \leq j \leq \lfloor \frac{m-2}{3} \rfloor)$ together with the path induced by the edges in $\{y(u_2, v_{m-1}), z(u_2, v_{m-1}), z(u_2, v_m), t(u_2, v_m)\} \cup E(P)$ are $\lfloor \frac{m-2}{3} \rfloor + 1$ internally disjoint S-paths, as desired. ∎

Lemma 2.4 *If only three of $\{x, y, z, t\}$ belong to some copy $H(u_i)$ $(1 \leq i \leq n)$, then there exist $\lfloor \frac{m-2}{3} \rfloor + 1$ internally disjoint S-paths.*

Proof. Without loss of generality, we may assume that $x, y, z \in V(H(u_1))$ and $t \in V(H(u_i))$ $(2 \leq i \leq n)$.

At first, we consider the case $i = 2$. We may assume that $x, y, z \in V(H(u_1))$ and $t \in V(H(u_2))$. Let x', y', z' be the vertices corresponding to x, y, z in $H(u_2)$, and let t' be the vertex corresponding to t in $H(u_1)$. Clearly, $H(u_1)$ is connected and so there exists a path P connecting x and y in $H(u_1)$.

If $t' \notin \{x, y, z\}$, without loss of generality, let $\{x, y, z, t'\} = \{(u_1, v_j) \mid m - 3 \leq j \leq m\}$ and $\{x', y', z', t\} = \{(u_2, v_j) \mid m - 3 \leq j \leq m\}$, then the paths L_j induced by the edges in $\{x(u_2, v_{3j-2}), y(u_2, v_{3j-2}), y(u_2, v_{3j-1}), z(u_2, v_{3j-1}), z(u_2, v_{3j}), (u_1, v_{3j})(u_2, v_{3j}), t(u_1, v_{3j})\}$ $(1 \leq j \leq \lfloor \frac{m-4}{3} \rfloor)$ and the path L_1' induced by the edges in $\{xx', x'y, yy', y'z, zt\}$ and the path L_2' induced by the

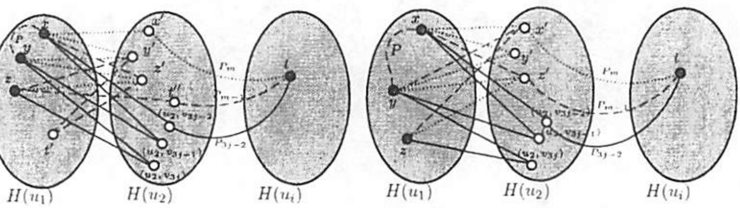


(a) (b)
Figure 1: Graphs for Lemma 2.4.

edges in $\{xt, tt', t'x', x'z, zz', z'y\}$ are $\lfloor \frac{m-4}{3} \rfloor + 2$ internally disjoint S-paths; see Figure 1 (a).

If $t' \in \{x, y, z\}$, without loss of generality, let $t' = z$, $\{x, y, z\} = \{(u_1, v_{m-2}), (u_1, v_{m-1}), (u_1, v_m)\}$ and $\{x', y', t\} = \{(u_2, v_{m-2}), (u_2, v_{m-1}), (u_2, v_m)\}$, then the paths L_j induced by the edges in $\{x(u_2, v_{3j-2}), y(u_2, v_{3j-2}), y(u_2, v_{3j-1}), z(u_2, v_{3j-1}), z(u_2, v_{3j}), (u_1, v_{3j})(u_2, v_{3j}), t(u_1, v_{3j})\}$ $(1 \leq j \leq \lfloor \frac{m-3}{3} \rfloor)$ and the path L_1' induced by the edges in $\{xx', x'z, zt\} \cup E(P)$ and the path L_2' induced by the edges in $\{xt, yt, yy', y'z\}$ are $\lfloor \frac{m-3}{3} \rfloor + 2$ internally disjoint S-paths; see Figure 1 (b).

Next, we consider the case $i \geq 3$. Let $P' = u_2 u_3 \cdots u_i$. Clearly, $\kappa(P' \circ H) \geq m$. From Lemma 2.1, there is a t, U-fan in $P' \circ H$, where $U = V(H(u_2)) = \{(u_2, v_j) \mid 1 \leq j \leq m\}$. There exist m internally disjoint paths $P_1, P_2, \cdots, P_m$ such that P_j $(1 \leq j \leq m)$ is a path connecting t and (u_2, v_j). Since $H(u_1)$ is connected, there exists a path P connecting x and y in $H(u_1)$. Let x', y', z', t'' be the vertices corresponding to x, y, z, t in $H(u_2)$, and let t' be the vertex corresponding to t in $H(u_1)$.



(a) (b)
Figure 2: Graphs for Lemma 2.4.

Suppose that $t' \notin \{x, y, z\}$. Without loss of generality, let $\{x, y, z, t'\} = \{(u_1, v_j) \mid m-3 \leq j \leq m\}$ and $\{x', y', z', t''\} = \{(u_2, v_j) \mid m-3 \leq j \leq m\}$. Then the paths L_j induced by the edges in $\{x(u_2, v_{3j-2}), x(u_2, v_{3j-1}), y(u_2, v_{3j-1}), y(u_2, v_{3j}), z(u_2, v_{3j})\} \cup E(P_{3j-2})$ $(1 \leq j \leq \lfloor \frac{m-4}{3} \rfloor)$ and the path L_1' induced by the edges in $\{xx', xy', yy', yz', z'z\} \cup E(P_m)$ and the path L_2' induced by the edges in $\{xz', z't', t'y', y'z, zt''\} \cup E(P_{m-1}) \cup E(P)$ are $\lfloor \frac{m-4}{3} \rfloor + 2$

internally disjoint S-paths; see Figure 2 (a).

Suppose that $t' \in \{x, y, z\}$. Without loss of generality, let $t' = z$, $x = (u_1, v_m), y = (u_1, v_{m-1}), z = (u_1, v_{m-2})$ and $x' = (u_2, v_m)$, $y' = (u_2, v_{m-1})$, $t'' = (u_2, v_{m-2})$. Then the paths L_j induced by the edges in $\{x(u_2, v_{3j-2}), x(u_2, v_{3j-1}), y(u_2, v_{3j-1}), y(u_2, v_{3j}), z(u_2, v_{3j})\} \cup (P_{3j-2})$ $(1 \leq j \leq \lfloor \frac{m-3}{3} \rfloor)$ and the path L'_1 induced by the edges in $\{xx', xy', yy', yz', z'z\} \cup E(P_m)$ and the path L'_2 induced by the edges in $\{zx', x'y, xz'\} \cup E(P_{m-1}) \cup E(P)$ are $\lfloor \frac{m-3}{3} \rfloor + 2$ internally disjoint S-paths; see Figure 2 (b). ∎

Lemma 2.5 *If two of* $\{x, y, z, t\}$ *belong to some copy* $H(u_i)$ $(1 \leq i \leq n)$, *then there exist* $\lfloor \frac{m-2}{3} \rfloor + 1$ *internally disjoint S-paths.*

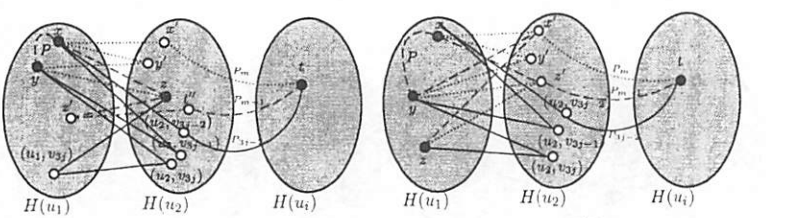
Proof. We have the following cases to be considered.

Case 1. $x, y \in V(H(u_i))$, $z \in V(H(u_j))$ and $t \in V(H(u_k))$, where $i < j < k$, $1 \leq i \leq n-2, 2 \leq j \leq n-1, 3 \leq k \leq n$.

Without loss of generality, we may assume that $x, y \in V(H(u_1))$. Clearly, $H(u_1)$ is connected and so there exists a path P connecting x and y in $H(u_1)$.

Subcase 1.1 $z \in V(H(u_2))$ and $t \in V(H(u_k))$, where $3 \leq k \leq n$.

Consider the case $k \geq 4$. Let $P' = u_2 u_3 \cdots u_k$. Clearly, $\kappa(P' \circ H) \geq m$. From Lemma 2.1, there is a t, U-fan in $P' \circ H$, where $U = V(H(u_2)) = \{(u_2, v_j) \mid 1 \leq j \leq m\}$. Thus, there exist m internally disjoint paths $P_1, P_2, \cdots, P_m$ such that each of P_j $(1 \leq j \leq m)$ is a path connecting t and (u_2, v_j). Let x', y', t'' be the vertices corresponding to x, y, t in $H(u_2)$, and let z', t' be the vertices corresponding to z, t in $H(u_1)$.



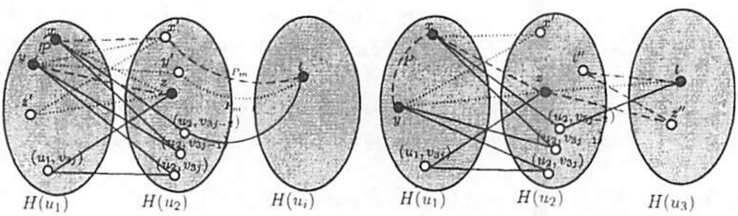
(a) (b)

Figure 3: Graphs for Lemma 2.5.

If x, y, z', t' are distinct vertices in $H(u_1)$. Without loss of generality, let $\{x, y, z', t'\} = \{(u_1, v_j) \mid m-3 \leq j \leq m\}$ and $\{x', y', z, t''\} = \{(u_2, v_j) \mid m - 3 \leq j \leq m\}$. Then the paths L_j induced by the edges in $\{x(u_2, v_{3j-2}), x(u_2, v_{3j-1}), y(u_2, v_{3j-1}), y(u_2, v_{3j}), (u_2, v_{3j})(u_1, v_{3j}), z(u_1, v_{3j})\} \cup E(P_{3j-2})$ $(1 \leq j \leq \lfloor \frac{m-4}{3} \rfloor)$ and the path L'_1 induced by the edges in $\{xx', xy', yy', yz\} \cup E(P_m)$ and the path L'_2 induced by the edges in $\{xz, zz', z't''\} \cup E(P_{m-1}) \cup E(P)$ are $\lfloor \frac{m-4}{3} \rfloor + 2$ internally disjoint S-paths; see Figure 3 (a).

Suppose that three of x, y, z', t' are the same vertices in $H(u_1)$. Without loss of generality, let $y = z' = t'$, and let $\{x, y\} = \{(u_1, v_j) \mid m-1 \le j \le m\}$ and $\{x', z\} = \{(u_2, v_j) \mid m-1 \le j \le m\}$. Then the paths L_j induced by the edges in $\{x(u_2, v_{3j-2}), x(u_2, v_{3j-1}), y(u_2, v_{3j-1}), y(u_2, v_{3j}), (u_2, v_{3j})(u_1, v_{3j}), z(u_1, v_{3j})\} \cup E(P_{3j-2})$ $(1 \le j \le \lfloor \frac{m-2}{3} \rfloor)$ and the path L'_1 induced by the edges in $\{yz, zx, xx'\} \cup E(P_m)$ are $\lfloor \frac{m-2}{3} \rfloor + 1$ internally disjoint S-paths; see Figure 3 (b).

Suppose that two of x, y, z', t' are the same vertices in $H(u_1)$. Without loss of generality, let $z' = t'$ and $\{x, y, z'\} = \{(u_1, v_{m-2}), (u_1, v_{m-1}), (u_1, v_m)\}$ and $\{x', y', z\} = \{(u_2, v_{m-2}), (u_2, v_{m-1}), (u_2, v_m)\}$.Then the paths L_j induced by the edges in $\{x(u_2, v_{3j-2}), x(u_2, v_{3j-1}), y(u_2, v_{3j-1}), y(u_2, v_{3j}), (u_2, v_{3j})(u_1, v_{3j}), z(u_1, v_{3j})\} \cup E(P_{3j-2})$ $(1 \le j \le \lfloor \frac{m-3}{3} \rfloor)$ and the path L'_1 induced by the edges in $\{yz, xx'\} \cup E(P_m) \cup E(P)$ and the path L'_2 induced by the edges in $\{xz, zz', z'x', x'y, yy'\} \cup E(P_{m-1})$ are $\lfloor \frac{m-3}{3} \rfloor + 2$ internally disjoint S-paths; see Figure 4 (a).



(a) (b)

Figure 4: Graphs for Lemma 2.5.

Consider the case $k = 3$. We may assume that $t \in V(H(u_3))$. Let x', y', t'' be the vertices corresponding to x, y, t in $H(u_2)$, z', t' be the vertices corresponding to z, t in $H(u_1)$, x'', y'', z'' be the vertices corresponding to x, y, z in $H(u_3)$. Clearly, $H(u_1)$ is connected and so there exists a path P connecting x and y in $H(u_1)$.

If x, y, z', t' are distinct vertices in $H(u_1)$, without loss of generality, let $\{x, y, z', t'\} = \{(u_1, v_j) \mid m-3 \le j \le m\}$ and $\{x', y', z, t''\} = \{(u_2, v_j) \mid m-3 \le j \le 4\}$ and $\{x'', y'', z'', t\} = \{(u_3, v_j) \mid m-3 \le j \le m\}$. Then the paths L_j induced by the edges in $\{x(u_2, v_{3j-2}), x(u_2, v_{3j-1}), y(u_2, v_{3j-1}), y(u_2, v_{3j}), (u_2, v_{3j})(u_1, v_{3j}), z(u_1, v_{3j}), t(u_2, v_{3j-2})\}$ $(1 \le j \le \lfloor \frac{m-4}{3} \rfloor)$ and the path L'_1 induced by the edges in $\{xx', yx', yz, zt\}$ and the path L'_2 induced by the edges in $\{xz, zz'', z''t'', t''t\} \cup E(P)$ are $\lfloor \frac{m-4}{3} \rfloor + 2$ internally disjoint S-paths; see Figure 4 (b).

Suppose that three of x, y, z', t' are the same vertices in $H(u_1)$. Without loss of generality, let $y = z' = t'$, and $\{x, y\} = \{(u_1, v_j) \mid m-1 \le j \le m\}$ and $\{x', z\} = \{(u_2, v_j) \mid m-1 \le j \le m\}$,and $\{x'', t\} = \{(u_3, v_j) \mid m-1 \le j \le m\}$. Then the paths L_j induced by the edges in $\{x(u_2, v_{3j-2}), x(u_2, v_{3j-1}),$

$y(u_2, v_{3j-1}), y(u_2, v_{3j}), (u_2, v_{3j})(u_1, v_{3j}), z(u_1, v_{3j}), t(u_2, v_{3j-2})\}$ $(1 \leq j \leq \lfloor \frac{m-2}{3} \rfloor)$ and the path L_1' induced by the edges in $\{xx', yx', yz, zt\}$ are $\lfloor \frac{m-2}{3} \rfloor + 1$ internally disjoint S-paths; see Figure 5 (a).

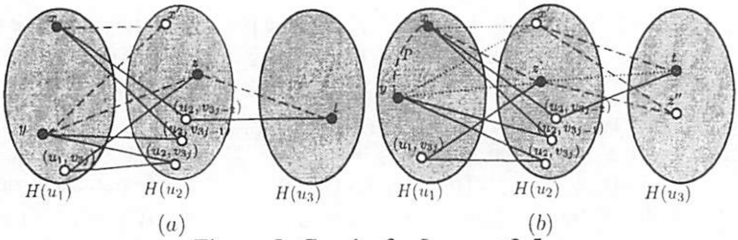


Figure 5: Graphs for Lemma 2.5.

Suppose that two of x, y, z', t' are the same vertices in $H(u_1)$. Without loss of generality, let $z' = t'$ and $\{x, y, z'\} = \{(u_1, v_{m-2}), (u_1, v_{m-1}), (u_1, v_m)\}$ and $\{x', y', z\} = \{(u_2, v_{m-2}), (u_2, v_{m-1}), (u_2, v_m)\}$.Then the paths L_j induced by the edges in $\{x(u_2, v_{3j-2}),\ x(u_2, v_{3j-1}), y(u_2, v_{3j-1}), y(u_2, v_{3j}), (u_2, v_{3j})$ $(u_1, v_{3j}), z(u_1, v_{3j}), t(u_2, v_{3j-2})\}$ $(1 \leq j \leq \lfloor \frac{m-3}{3} \rfloor)$ and the path L_1' induced by the edges in $\{xx', yx', yz, zt\}$ and the path L_2' induced by the edges in $\{xz, zz'', z''x', x't\} \cup E(P)$ are $\lfloor \frac{m-3}{3} \rfloor + 2$ internally disjoint S-paths; see Figure 5 (b).

Subcase 1.2 $z \in V(H(u_j))$ and $t \in V(H(u_k))$, where $3 \leq j \leq n-2$, $4 \leq k \leq n$.

Consider the case $|j - k| \geq 2$ and $j \geq 4$. Let $P' = u_2, u_3, \cdots, u_j$ and $P'' = u_{j+1}, u_{j+2}, \cdots, u_k$. Then P' and P'' are two paths with order at least 2. Since $\kappa(P' \circ H) \geq m$, from Lemma 2.1, there is a z, U'-fan in $P' \circ H$ and there is a t, U''-fan in $P'' \circ H$, respectively, $U' = V(H(u_2)) = \{(u_2, v_r) \,|\, 1 \leq r \leq m\}$ and $U'' = V(H(u_{j+1})) = \{(u_{j+1}, v_r) \,|\, 1 \leq r \leq m\}$. Thus there exist m pairwise internally disjoint paths $P_1', P_2', \cdots, P_m'$ such that each of P_r' $(1 \leq r \leq m)$ is a path connecting z and (u_2, v_r) and there exist m internally disjoint paths $P_1'', P_2'', \cdots, P_m''$ such that each of P_r'' $(1 \leq r \leq m)$ is a path connecting t and (u_{j+1}, v_r). Then the paths L_r induced by the edges in $\{x(u_2, v_{2r-1}), y(u_2, v_{2r-1}),$ $y(u_2, v_{2r}) \cup E(P_{2r}') \cup E(P_{2r}'')(1 \leq r \leq \lfloor \frac{m}{2} \rfloor)$ are $\lfloor \frac{m}{2} \rfloor$ internally disjoint S-paths.

Consider the case $|j - k| \geq 2$ and $j = 3$. Let $P' = u_3, u_4, \cdots, u_k$. Then P' is a path with order at least 2. Since $\kappa(P' \circ H) \geq m$, from Lemma 2.1, there is a t, U'-fan in $P' \circ H, U' = V(H(u_3)) = \{(u_3, v_r) \,|\, 1 \leq r \leq m\}$. Thus there exist m pairwise internally disjoint paths $P_1', P_2', \cdots, P_m'$ such that each of P_r' $(1 \leq r \leq m)$ is a path connecting t and (u_3, v_r). Let z', t' be the vertices corresponding to z, t in $H(u_1)$, x', y', z'' be the vertices corresponding to x, y, z in $H(u_2)$, x'', y'' be the vertices corresponding to x, y in $H(u_3)$.

If x, y, z', t' are distinct vertices in $H(u_1)$, without loss of generality, let $\{x, y, z', t'\} = \{(u_1, v_r) \,|\, m-3 \leq r \leq m\}$ and $\{x', y', z''\} = \{(u_2, v_r) \,|\, m-2 \leq r \leq m\}$ and $\{x'', y'', z\} = \{(u_3, v_r) \,|\, m-2 \leq r \leq m\}$. Then the paths L_r induced by the edges in $\{x(u_2, v_{3r-2}), y(u_2, v_{3r-2}), y(u_2, v_{3r-1}), z(u_2, v_{3r-1}),$

$z(u_2, v_{3r}), (u_2, v_{3r})(u_3, v_{3r})\} \cup E(P'_{3r})(1 \leq r \leq \lfloor \frac{m-4}{3} \rfloor)$ and the path L'_1 induced by the edges in $E(P) \cup \{xx', x'z\} \cup E(P'_m)$ and the path L'_2 induced by the edges in $\{xy', y'z, zz'', z''y, yx', x'x''\} \cup E(P'_{m-1})$ are $\lfloor \frac{m-4}{3} \rfloor + 2$ internally disjoint S-paths; see Figure 6.

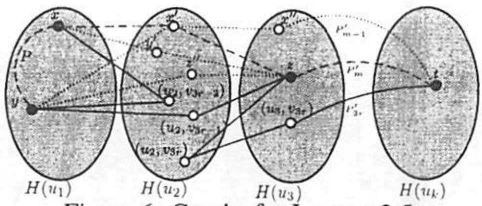


Figure 6: Graphs for Lemma 2.5.

Suppose that three of x, y, z', t' are the same vertices in $H(u_1)$. Without loss of generality, let $y = z' = t'$ and $\{x, y\} = \{(u_1, v_r) \,|\, m-1 \leq r \leq m\}$ and $\{x', z\} = \{(u_2, v_r) \,|\, m-1 \leq r \leq m\}$ and $\{x'', y''\} = \{(u_3, v_r) \,|\, m-1 \leq r \leq m\}$.Then the paths L_r induced by the edges in $\{x(u_2, v_{3r-2}), y(u_2, v_{3r-2}), y(u_2, v_{3r-1}), z(u_2, v_{3r-1}), z(u_2, v_{3r}), (u_2, v_{3r})(u_3, v_{3r})\} \cup E(P'_{3r})(1 \leq r \leq \lfloor \frac{m-2}{3} \rfloor)$ and the path L'_1 induced by the edges in $E(P) \cup \{xx', x'z\} \cup E(P'_m)$ are $\lfloor \frac{m-2}{3} \rfloor + 1$ internally disjoint S-paths.

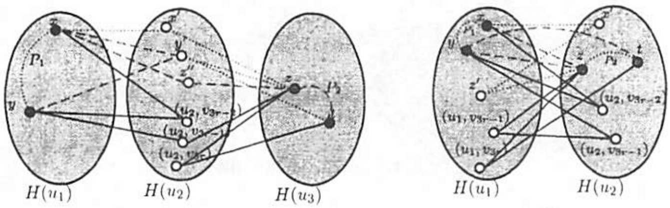
Suppose that two of x, y, z', t' are the same vertices in $H(u_1)$. Without loss of generality, let $z' = t'$ and $\{x, y, z'\} = \{(u_1, v_r) \,|\, m-2 \leq r \leq m\}$ and $\{x', y', z''\} = \{(u_2, v_r) \,|\, m-2 \leq r \leq m\}$ and $\{x'', y'', z\} = \{(u_3, v_r) \,|\, m-2 \leq r \leq m\}$. Then the paths L_r induced by the edges in $\{x(u_2, v_{3r-2}), y(u_2, v_{3r-2}), y(u_2, v_{3r-1}), z(u_2, v_{3r-1}), z(u_2, v_{3r}), (u_2, v_{3r})(u_3, v_{3r})\} \cup E(P'_{3r})(1 \leq r \leq \lfloor \frac{m-3}{3} \rfloor)$ and the path L'_1 induced by the edges in $E(P) \cup \{xx', x'z\} \cup E(P'_m)$and the path L'_2 induced by the edges in $\{xy', y'z, zz'', z''y, yx', x'x''\} \cup E(P'_{m-1})$ are $\lfloor \frac{m-3}{3} \rfloor + 2$ internally disjoint S-paths; see Figure 6.

Consider the case $|j - k| = 1$ and $j \geq 4$. Let $P' = u_2 u_3 \cdots u_{j-1}$. Since $\kappa(P' \circ H) \geq m$, from Lemma 2.2, if we add the vertex z to $P' \circ H$ and join an edge from z to each of (u_{j-1}, v_r) $(1 \leq r \leq m)$, then $\kappa((P' \circ H) \vee \{z, V(H(u_{j-1}))\}) \geq m$. Thus there exist m pairwise internally disjoint paths $P'_1, P'_2, \cdots, P'_m$ such that each of P'_r $(1 \leq r \leq m)$ is a path connecting z and (u_2, v_r). Then the paths L_r induced by the edges in $\{x(u_2, v_{2r-1}), y(u_2, v_{2r-1}), y(u_2, v_{2r})\} \cup E(P'_{2r}) \cup \{z(u_{j+1}, v_{2r}), t(u_j, v_{2r}), (u_j, v_{2r})(u_{j+1}, v_{2r})\}$ $(1 \leq r \leq \lfloor \frac{m}{2} \rfloor)$ are $\lfloor \frac{m}{2} \rfloor$ internally disjoint S-paths.

Consider the case $|j - k| = 1$ and $j = 3$. Without loss of generality, we may assume that $z \in V(H(u_3))$ and $t \in V(H(u_4))$. Let z', t' be the vertices corresponding to z, t in $H(u_1)$, x', y', z'', t'' be the vertices corresponding to x, y, z, t in $H(u_2)$, x'', y'', t''' be the vertices corresponding to x, y, t in $H(u_3)$.

If x, y, z', t' are distinct vertices in $H(u_1)$, without loss of generality, let $\{x, y, z', t'\} = \{(u_1, v_r) \,|\, m-3 \leq r \leq m\}$ and $\{x', y', z''\} = \{(u_2, v_r) \,|\, m-2 \leq r \leq m\}$ and $\{x'', y'', z\} = \{(u_3, v_r) \,|\, m-2 \leq r \leq m\}$ and $\{x''', t\} =$

$\{(u_4, v_r) \mid m-1 \le r \le m\}$. Then the paths L_r induced by the edges in $\{x(u_2, v_{3r-2}), y(u_2, v_{3r-2}), y(u_2, v_{3r-1}), z(u_2, v_{3r-1}), z(u_4, v_{3r-1}), (u_4, v_{3r-1})(u_3, v_{3r-1}), t(u_3, v_{3r-1})\}(1 \le r \le \lfloor \frac{m-4}{3} \rfloor)$ and the path L_1' induced by the edges in $\{xx', x'z, zt\} \cup E(P)$ and the path L_2' induced by the edges in $\{xy', yy', yz'', z''z, zx''', x'''x'', x''t\}$ are $\lfloor \frac{m-4}{3} \rfloor + 2$ internally disjoint S-paths; see Figure 7.



(a) (b)

Figure 7: Graphs for Lemma 2.5.

Suppose that three of x, y, z', t' are the same vertices in $H(u_1)$. Without loss of generality, let $y = z' = t'$ and $\{x, y\} = \{(u_1, v_r) \mid m-1 \le r \le m\}$ and $\{x', y'\} = \{(u_2, v_r) \mid m-1 \le r \le m\}$ and $\{x'', z\} = \{(u_3, v_r) \mid m-1 \le r \le m\}$ and $\{x''', t\} = \{(u_4, v_r) \mid m-1 \le r \le m\}$. Then the paths L_r induced by the edges in $\{x(u_2, v_{3r-2}), y(u_2, v_{3r-2}), y(u_2, v_{3r-1}), z(u_2, v_{3r-1}), z(u_4, v_{3r-1}), (u_4, v_{3r-1})(u_3, v_{3r-1}), t(u_3, v_{3r-1})\}(1 \le r \le \lfloor \frac{m-2}{3} \rfloor)$ and the path L_1' induced by the edges in $\{xx', x'z, zt\} \cup E(P)$ are $\lfloor \frac{m-2}{3} \rfloor + 1$ internally disjoint S-paths.

Suppose that two of x, y, z', t' are the same vertices in $H(u_1)$. Without loss of generality, let $z' = t'$ and $\{x, y, z'\} = \{(u_1, v_r) \mid m-2 \le r \le m\}$ and $\{x', y', z''\} = \{(u_2, v_r) \mid m-2 \le r \le m\}$ and $\{x'', y'', z\} = \{(u_3, v_r) \mid m-2 \le r \le m\}$ and $\{x''', y''', t\} = \{(u_4, v_r) \mid m-2 \le r \le m\}$.Then the paths L_r induced by the edges in $\{x(u_2, v_{3r-2}), y(u_2, v_{3r-2}), y(u_2, v_{3r-1}), z(u_2, v_{3r-1}), z(u_4, v_{3r-1}), (u_4, v_{3r-1})(u_3, v_{3r-1}), t(u_3, v_{3r-1})\}$ $(1 \le r \le \lfloor \frac{m-3}{3} \rfloor)$ and the path L_1' induced by the edges in $\{xx', x'z, zt\} \cup E(P)$ and the path L_2' induced by the edges in $\{xy', yy', yz'', z''z, zx''', x'''x'', x''t\}$ are $\lfloor \frac{m-3}{3} \rfloor + 2$ internally disjoint S-paths; see Figure 8.

Case 2. $x, y \in V(H(u_i))$, $z, t \in V(H(u_j))$, where $i < j$, $1 \le i \le n-1$, $2 \le j \le n$.

Without loss of generality, we may assume that $x, y \in V(H(u_1))$, $z, t \in V(H(u_j))$ and let $z = (u_j, v_m)$.

At first, we consider the case $j \ge 4$. Clearly, $H(u_1)$ is connected and so there exists a path P_1 connecting x and y in $H(u_1)$. For the same reason, $H(u_j)$ is connected and so there is a path P_2 connecting z and t in $H(u_j)$. Then the paths L_r induced by the edges in $\{x(u_2, v_{3r-2}), y(u_2, v_{3r-2}), y(u_2, v_{3r-1})\} \cup \{(u_i, v_{3r-1})(u_{i+1}, v_{3r-1}) \mid 2 \le i \le j-1\}(1 \le r \le m-2)$ and the path L_1'

induced by the edges in $E(P_1) \cup \{x(u_2, v_m)\} \cup \{ (u_i, v_m)(u_{i+1}, v_m) \,|\, 2 \leq i \leq j-2\} \cup \{ z(u_{j-1}, v_m)\} \cup E(P_2)$ are $m-1$ internally disjoint S-paths.

Next, we consider the case $j = 3$. We may assume that $x, y \in V(H(u_1))$ and $z, t \in V(H(u_3))$. Let z', t' be the vertices corresponding to z, t in $H(u_1)$, x', y', z'', t'' be the vertices corresponding to x, y, z, t in $H(u_2)$ and x'', y'' be the vertices corresponding to x, y in $H(u_3)$.

If x, y, z', t' are distinct vertices in $H(u_1)$, without loss of generality, let $\{x, y, z', t'\} = \{(u_1, v_r) \,|\, m-3 \leq r \leq m\}$ and $\{x', y', z'', t''\} = \{(u_2, v_r) \,|\, m-3 \leq r \leq m\}$ and $\{x'', y'', z, t\} = \{(u_3, v_r) \,|\, m-3 \leq r \leq m\}$. Then the paths L_r induced by the edges in $\{x(u_2, v_{3r-2}), y(u_2, v_{3r-2}), y(u_2, v_{3r-1}), z(u_2, v_{3r-1}), z(u_2, v_{3r}), t(u_2, v_{3r})\}$ $(1 \leq r \leq \lfloor \frac{m-4}{3} \rfloor)$ and the path L_1' induced by the edges in $\{xx', x'z, zy', y't\} \cup E(P_1)$ and the path L_2' induced by the edges in $\{zz'', z''x, xy', y'y\} \cup E(P_2)$are $\lfloor \frac{m-4}{3} \rfloor + 2$ internally disjoint S-paths; see Figure 8 (a).

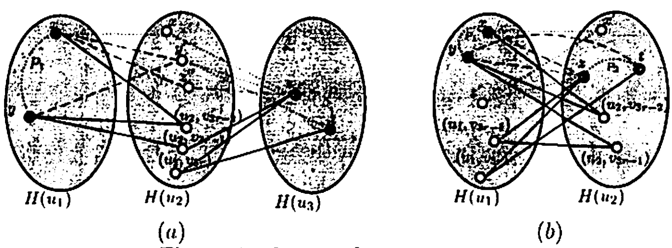


Figure 8: Graphs for Lemma 2.5.

Suppose that three of x, y, z', t' are the same vertices in $H(u_1)$, Without loss of generality, let $y = z' = t'$ and $\{x, y\} = \{(u_1, v_r) \,|\, m-1 \leq r \leq m\}$ and $\{x', y'\} = \{(u_2, v_r) \,|\, m-1 \leq r \leq m\}$ and $\{z, t\} = \{(u_3, v_r) \,|\, m-1 \leq r \leq m\}$. Then the paths L_r induced by the edges in $\{x(u_2, v_{3r-2}), y(u_2, v_{3r-2}), y(u_2, v_{3r-1}), z(u_2, v_{3r-1}), z(u_2, v_{3r}), t(u_j, v_{3r})(1 \leq r \leq \lfloor \frac{m-2}{3} \rfloor)$ and the path L_1' induced by the edges in $\{xx', x'z\} \cup E(P_1) \cup E(P_2)$ are $\lfloor \frac{m-2}{3} \rfloor + 1$ internally disjoint S-paths.

Suppose that two of x, y, z', t' are the same vertices in $H(u_1)$. If $y = t'$, without loss of generality, let $\{x, y, z'\} = \{(u_1, v_r) \,|\, m-2 \leq r \leq m\}$ and $\{x', y', z''\} = \{(u_2, v_r) \,|\, m-2 \leq r \leq m\}$ and $\{x'', z, t\} = \{(u_3, v_r) \,|\, m-2 \leq j \leq m\}$, then the paths L_r induced by the edges in $\{x(u_2, v_{3r-2}), y(u_2, v_{3r-2}), y(u_2, v_{3r-1}), z(u_2, v_{3r-1}), z(u_2, v_{3r}), t(u_j, v_{3r})(1 \leq r \leq \lfloor \frac{m-3}{3} \rfloor)$ and the path L_1' induced by the edges in $\{xx', x'z, zy', y't\} \cup E(P_1)$ and the path L_2' induced by the edges in $\{zz'', z''x, xy', y'y\} \cup E(P_2)$ are $\lfloor \frac{m-3}{3} \rfloor + 2$ internally disjoint S-paths.

Finally, we consider the case $j = 2$. Without loss of generality, we may assume that $x, y \in V(H(u_1))$ and $z, t \in V(H(u_2))$. Let z', t' be the vertices corresponding to z, t in $H(u_1)$, x', y' be the vertices corresponding to x, y in $H(u_2)$.

If x, y, z', t' are distinct vertices in $H(u_1)$, without loss of generality, let $\{x, y, z', t'\} = \{(u_1, v_r) \mid m-3 \le r \le m\}$ and $\{x', y', z, t\} = \{(u_2, v_r) \mid m-3 \le r \le m\}$, then the paths L_r induced by the edges in $\{x(u_2, v_{3r-2}), y(u_2, v_{3r-2}), y(u_2, v_{3r-1}), (u_2, v_{3r-1})(u_1, v_{3r-1}), z(u_1, v_{3r-1}), z(u_1, v_{3r}), t(u_1, v_{3r})\}$ $(1 \le r \le \lfloor\frac{m-4}{3}\rfloor)$ and the path L'_1 induced by the edges in $\{xx', x'z', z'z\} \cup E(P_1) \cup E(P_2)$ and the path L'_2 induced by the edges in $\{xz, zy, yt\}$ are $\lfloor\frac{m-4}{3}\rfloor + 2$ internally disjoint S-paths; see Figure 8 (b).

Suppose that three of x, y, z', t' are the same vertices in $H(u_1)$. Without loss of generality, let $y = z' = t'$ and $\{x, y\} = \{(u_1, v_r) \mid m-1 \le r \le m\}$ and $\{z, t\} = \{(u_2, v_r) \mid m-1 \le r \le m\}$. Then the paths L_r induced by the edges in $\{x(u_2, v_{3r-2}), y(u_2, v_{3r-2}), y(u_2, v_{3r-1}), (u_2, v_{3r-1})(u_1, v_{3r-1}), z(u_1, v_{3r-1}), z(u_1, v_{3r}), t(u_1, v_{3r})(1 \le r \le \lfloor\frac{m-2}{3}\rfloor)$ and the path L'_1 induced by the edges in $E(P_1) \cup \{xz\} \cup E(P_2)$ are $\lfloor\frac{m-2}{3}\rfloor + 1$ internally disjoint S-paths.

Suppose that two of x, y, z', t' are the same vertices in $H(u_1)$. Without loss of generality, let $y = t'$ and $\{x, y, z'\} = \{(u_1, v_r) \mid m-2 \le r \le m\}$ and $\{x', z, t\} = \{(u_2, v_r) \mid m-2 \le r \le m\}$. Then the paths L_r induced by the edges in $\{x(u_2, v_{3r-2}), y(u_2, v_{3r-2}), y(u_2, v_{3r-1}), (u_2, v_{3r-1})(u_1, v_{3r-1}), z(u_1, v_{3r-1}), z(u_1, v_{3r}), t(u_1, v_{3r})\}(1 \le r \le \lfloor\frac{m-3}{3}\rfloor)$ and the path L'_1 induced by the edges in $\{xx', x'z', z'z\} \cup E(P_1) \cup E(P_2)$ and the path L'_2 induced by the edges in $\{xz, zy, yt\}$ are $\lfloor\frac{m-3}{3}\rfloor + 2$ internally disjoint S-paths. ■

Lemma 2.6 *If x, y, z, t are contained in distinct $H(u_i)$s, then there exist $\lfloor\frac{m-2}{3}\rfloor + 1$ internally disjoint paths connecting S.*

Proof. We have the following cases to be considered.

Case 1. $d_{P_n \circ H}(x, y) = d_{P_n \circ H}(y, z) = d_{P_n \circ H}(z, t) = 1$.

Without loss of generality, we may assume that $x \in V(H(u_1))$, $y \in V(H(u_2))$, $z \in V(H(u_3))$ and $t \in V(H(u_4))$. Let y', z', t' be the vertices corresponding to y, z, t in $H(u_1)$, x', z'', t'' be the vertices corresponding to x, z, t in $H(u_2)$, x'', y'', t''' be the vertices corresponding to x, y, t in $H(u_3)$ and x''', y''', z''' be the vertices corresponding to x, y, z in $H(u_4)$.

If x, y', z', t' are distinct vertices in $H(u_1)$, without loss of generality, let $\{x, y', z', t'\} = \{(u_1, v_r) \mid m-3 \le r \le m\}$ and $\{x', y, z'', t''\} = \{(u_2, v_r) \mid m-3 \le r \le m\}$ and $\{x'', y'', z, t'''\} = \{(u_3, v_r) \mid m-3 \le r \le m\}$ and $\{x''', y''', z''', t\} = \{(u_4, v_r) \mid m-3 \le r \le m\}$. Then the paths L_r induced by the edges in $\{x(u_2, v_{3r-2}), (u_2, v_{3r-2})(u_1, v_{3r-2}), y(u_1, v_{3r-2}), y(u_3, v_{3r-2}), t(u_3, v_{3r-2}), t(u_3, v_{3r-1}), (u_3, v_{3r-1})(u_4, v_{3r-1}), z(u_4, v_{3r-1})\}$ $(1 \le r \le \lfloor\frac{m-4}{3}\rfloor)$ and the path L'_1 induced by the edges in $\{xy, yz, zt\}$ and the path L'_2 induced by the edges in $\{xx', x'y', y'y, yx'', x''t, ty'', y''x''', x'''z\}$ are $\lfloor\frac{m-4}{3}\rfloor + 2$ internally disjoint S-paths; see Figure 9.

Suppose that three of x, y', z', t' are the same vertices in $H(u_1)$. Without loss of generality, let $y' = z' = t'$ and $\{x, y'\} = \{(u_1, v_m), (u_1, v_{m-1})\}$ and $\{x', y\} = \{(u_2, v_m), (u_2, v_{m-1})\}$ and $\{x'', z\} = \{(u_3, v_m), (u_3, v_{m-1})\}$ and

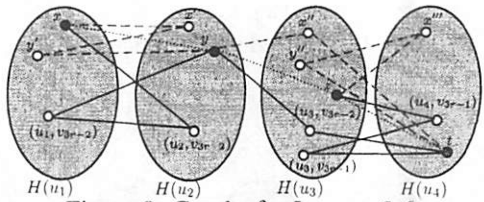


Figure 9: Graphs for Lemma 2.6.

$\{x''', t\} = \{(u_4, v_m), (u_4, v_{m-1})\}$. Then the paths L_r induced by the edges in $\{x(u_2, v_{3r-2}), (u_2, v_{3r-2})(u_1, v_{3r-2}), y(u_1, v_{3r-2}), y(u_3, v_{3r-2}), t(u_3, v_{3r-2}), t(u_3, v_{3r-1}), (u_3, v_{3r-1})(u_4, v_{3r-1}), z(u_4, v_{3r-1})\}$ $(1 \leq r \leq \lfloor \frac{m-2}{3} \rfloor)$ and the path L_1' induced by the edges in $\{xy, yz, zt\}$ are $\lfloor \frac{m-2}{3} \rfloor + 1$ internally disjoint S-paths.

Suppose that two of x, y', z', t' are the same vertices in $H(u_1)$. If $z' = t'$, without loss of generality, let $\{x, y', z'\} = \{(u_1, v_m), (u_1, v_{m-1}), (u_1, v_{m-2})\}$ and $\{x', y, z''\} = \{(u_2, v_m), (u_2, v_{m-1}), (u_2, v_{m-2})\}$ and $\{x'', y'', z\} = \{(u_3, v_m), (u_3, v_{m-1}), (u_3, v_{m-2})\}$ and $\{x''', y''', t\} = \{(u_4, v_m), (u_4, v_{m-1}), (u_4, v_{m-2})\}$. Then the paths L_r induced by the edges in $\{x(u_2, v_{3r-2}), (u_2, v_{3r-2})(u_1, v_{3r-2}), y(u_1, v_{3r-2}), y(u_3, v_{3r-2}), t(u_3, v_{3r-2}), t(u_3, v_{3r-1}), (u_3, v_{3r-1})(u_4, v_{3r-1}), z(u_4, v_{3r-1})\}$ $(1 \leq r \leq \lfloor \frac{m-3}{3} \rfloor)$ and the path L_1' induced by the edges in $\{xy, yz, zt\}$ and the path L_2' induced by the edges in $\{xx', x'y', y'y, yx'', x''t, ty'', y''x''', x'''z\}$ are $\lfloor \frac{m-3}{3} \rfloor + 2$ internally disjoint S-paths.

Suppose that x, y', z', t' are the same vertices in $H(u_1)$. Without loss of generality, let $x = (u_1, v_m)$, $y = (u_2, v_m)$ and $z = (u_3, v_m)$and $t = (u_4, v_m)$. Then the paths L_r induced by the edges in $\{x(u_2, v_{3r-2}), (u_2, v_{3r-2})(u_1, v_{3r-2}), y(u_1, v_{3r-2}), y(u_3, v_{3r-1}), (u_3, v_{3r-1})(u_2, v_{3r-1}), z(u_2, v_{3r-1}), z(u_4, v_{3r}), (u_4, v_{3r})(u_3, v_{3r})t(u_3, v_{3r})\}$ $(1 \leq r \leq \lfloor \frac{m-1}{3} \rfloor)$ and the path L_1' induced by the edges in $\{xy, yz, zt\}$ are $\lfloor \frac{m-1}{3} \rfloor + 1$ internally disjoint S-paths.

Case 2. $d_{P_n \circ H}(x, y) = d_{P_n \circ H}(y, z) = 1$ and $d_{P_n \circ H}(z, t) \geq 2$.

Without loss of generality, we may assume that $x \in V(H(u_1))$, $y \in V(H(u_2))$, $z \in V(H(u_3))$ and $t \in V(H(u_i))$ $(5 \leq i \leq n)$. Let y', z', t' be the vertices corresponding to y, z, t in $H(u_1)$, x', z'', t'' be the vertices corresponding to x, z, t in $H(u_2)$,x'', y'', t''' be the vertices corresponding to x, y, t in $H(u_3)$ and $P' = u_4 u_5 \cdots u_i$. Clearly, $\kappa(P' \circ H) \geq m$. From Lemma 2.1, there is a t, U-fan in $P' \circ H$, where $U = V(H(u_4)) = \{(u_4, v_r) \,|\, 1 \leq r \leq m\}$. Thus there exist m pairwise internally disjoint paths $P_1', P_2', \cdots, P_m'$ such that each P_r' $(1 \leq r \leq m)$ is a path connecting t and (u_4, v_r).

If x, y', z', t' are distinct vertices in $H(u_1)$, without loss of generality, we let $\{x, y', z', t'\} = \{(u_1, v_r) \,|\, m-3 \leq r \leq m\}$ and $\{x', y, z'', t''\} = \{(u_2, v_r) \,|\, m-3 \leq r \leq m\}$ and $\{x'', y'', z, t'''\} = \{(u_3, v_r) \,|\, m-3 \leq r \leq m\}$. Then the paths L_r induced by the edges in $\{y(u_3, v_{3r-2}), y(u_3, v_{3r-1}), (u_2, v_{3r-1})(u_3, v_{3r-1}),$

$z(u_2, v_{3r-1}), z(u_2, v_{3r}), x(u_2, v_{3r}), (u_3, v_{3r-2})(u_4, v_{3r-2})\} \cup E(P'_{3r-2})$ $(1 \leq r \leq \lfloor \frac{m-4}{3} \rfloor)$ and the path L'_1 induced by the edges in $\{xy, yz, z(u_4, v_m)\} \cup E(P'_m)$ and the path L'_2 induced by the edges in $\{xx', x'y', y'y, yx'', x''z'', z''z, z(u_4, v_{m-1})\} \cup E(P'_{m-1})$ are $\lfloor \frac{m-4}{3} \rfloor + 2$ internally disjoint S-paths; see Figure 10.

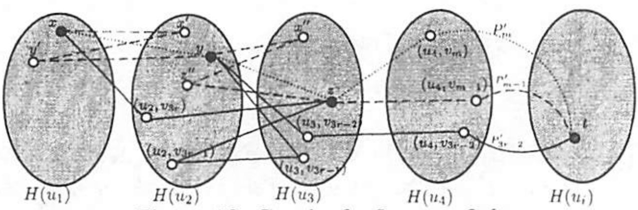


Figure 10: Graphs for Lemma 2.6.

Suppose that three of x, y', z', t' are the same vertices in $H(u_1)$. Without loss of generality, let $y' = z' = t'$ and $\{x, y'\} = \{(u_1, v_m), (u_1, v_{m-1})\}$ and $\{x', y\} = \{(u_2, v_m), (u_2, v_{m-1})\}$ and $\{x'', z\} = \{(u_3, v_m), (u_3, v_{m-1})\}$. Then the paths L_r induced by the edges in $\{y(u_3, v_{3r-2}), y(u_3, v_{3r-1}), (u_2, v_{3r-1})(u_3, v_{3r-1}), z(u_2, v_{3r-1}), z(u_2, v_{3r}), x(u_2, v_{3r}), (u_3, v_{3r-2})(u_4, v_{3r-2})\} \cup E(P'_{3r-2})$ $(1 \leq r \leq \lfloor \frac{m-2}{3} \rfloor)$ and the path L'_1 induced by the edges in $\{xy, yz, z(u_4, v_m)\} \cup E(P'_m)$ are $\lfloor \frac{m-2}{3} \rfloor + 1$ internally disjoint S-paths.

Suppose that two of x, y', z', t' are the same vertices in $H(u_1)$. Without loss of generality, let $z' = t'$ and $\{x, y', z'\} = \{(u_1, v_m), (u_1, v_{m-1}), (u_1, v_{m-2})\}$ and $\{x', y, z''\} = \{(u_2, v_m), (u_2, v_{m-1}), (u_2, v_{m-2})\}$ and $\{x'', y'', z\} = \{(u_3, v_m), (u_3, v_{m-1}), (u_3, v_{m-2})\}$. Then the paths L_r induced by the edges in $\{y(u_3, v_{3r-2}), y(u_3, v_{3r-1}), (u_2, v_{3r-1})(u_3, v_{3r-1}), z(u_2, v_{3r-1}), z(u_2, v_{3r}), x(u_2, v_{3r}), (u_3, v_{3r-2})(u_4, v_{3r-2})\} \cup E(P'_{3r-2})$ $(1 \leq r \leq \lfloor \frac{m-3}{3} \rfloor)$ and the path L'_1 induced by the edges in $\{xy, yz, z(u_4, v_m)\} \cup E(P'_m)$ and the path L'_2 induced by the edges in $\{xx', x'y', y'y, yx'', x''z'', z''z, z(u_4, v_{m-1})\} \cup E(P'_{m-1})$ are $\lfloor \frac{m-3}{3} \rfloor + 2$ internally disjoint S-paths; see Figure 10.

Suppose that x, y', z', t' are the same vertices in $H(u_1)$. Without loss of generality, let $x = (u_1, v_m)$, $y = (u_2, v_m)$ and $z = (u_3, v_m)$and $t = (u_4, v_m)$. Then the paths L_r induced by the edges in $\{y(u_3, v_{3r-2}), y(u_3, v_{3r-1}), (u_2, v_{3r-1})(u_3, v_{3r-1}), z(u_2, v_{3r-1}), z(u_2, v_{3r}), x(u_2, v_{3r})\} \cup E(P'_{3r-2})$ $(1 \leq r \leq \lfloor \frac{m-1}{3} \rfloor)$ and the path L'_1 induced by the edges in $\{xy, yz\} \cup E(P'_m)$ are $\lfloor \frac{m-1}{3} \rfloor + 1$ internally disjoint S-paths.

The other cases $d_{P_n \circ H}(y, z) = d_{P_n \circ H}(z, t) = 1$ or $d_{P_n \circ H}(x, y) \geq 2$ can be proved with similar arguments.

Case 3. $d_{P_n \circ H}(x, y) = 1$, $d_{P_n \circ H}(y, z) \geq 2$ and $d_{P_n \circ H}(z, t) \geq 2$.

Without loss of generality, We may assume that $x \in V(H(u_1))$, $y \in V(H(u_2))$, $z \in V(H(u_i))$ and $t \in V(H(u_j))$, where $3 < i < j$, $|j - i| \geq 2$, $4 \leq i \leq n-2$, $6 \leq j \leq n$. Let y', z', t' be the vertices corresponding to y, z, t

in $H(u_1)$, x', z'', t'' be the vertices corresponding to x, z, t in $H(u_2)$ and $P' = u_3, u_4, \cdots, u_i$ and $P'' = u_{i+1}, u_{i+2}, \cdots, u_j$. Then P' and P'' are two paths with order at least 2. Since $\kappa(P' \circ H) \geq m$, from Lemma 2.2, if we add the vertex z to $P' \circ H$ and join an edge from z to each of (u_3, v_r) $(1 \leq r \leq m)$, then $\kappa((P' \circ H) \vee \{y, V(H(u_3))\}) \geq m$. By the same reason, $\kappa((P'' \circ H) \vee \{t, V(H(u_{i+1}))\}) \geq m$. From Menger's Theorem, there exist m internally disjoint paths connecting y and z in $(P' \circ H) \vee \{y, V(H(u_3))\}$, and we say $P'_1, P'_2, \cdots, P'_m$. Also there exist m internally disjoint paths connecting z and t in $(P'' \circ H) \vee \{z, V(H(u_{i+1}))\}$, and we say $P''_1, P''_2, \cdots, P''_m$. Suppose $x = (u_1, v_m) \in V(H(u_1))$ and $y = (u_2, v_m) \in V(H(u_2))$. Then the paths L_r induced by the edges in $\{x(u_2, v_r), (u_1, v_r)(u_2, v_r), y(u_1, v_r)\} \cup E(P'_r) \cup E(P''_r)$ $(2 \leq r \leq m)$ are $m-1$ internally disjoint S-paths, as desired.

The other cases $d_{P_n \circ H}(y, z) = 1$, $d_{P_n \circ H}(x, y) \geq 2$ and $d_{P_n \circ H}(z, t) \geq 2$ or $d_{P_n \circ H}(z, t) = 1$, $d_{P_n \circ H}(x, y) \geq 2$ and $d_{P_n \circ H}(y, z) \geq 2$ can be discussed similarly.

Case 4. $d_{P_n \circ H}(x, y) \geq 2$, $d_{P_n \circ H}(y, z) \geq 2$ and $d_{P_n \circ H}(z, t) \geq 2$.

Without loss of generality, we may assume that $x \in V(H(u_1)), y \in V(H(u_i))$, $z \in V(H(u_j))$ and $t \in V(H(u_k))$, where $i < j < k$, $|j - i| \geq 2$, $|k - j| \geq 2$, $1 \leq i \leq n-7$, $3 \leq j \leq n-4$ and $7 \leq k \leq n$. Let $P' = u_1, u_2, \cdots, u_{i-1}$ and $P'' = u_i, u_{i+1}, \cdots, u_{j-1}$ and $P''' = u_j, u_{j+1}, \cdots, u_k$. Then P' and P'' and P''' are three paths with order at least 2. Since $\kappa(P' \circ H) \geq m$, it follows from Lemma 2.2, if we add the vertex y to $P' \circ H$ and join an edge from y to each of (u_{i-1}, v_r) $(1 \leq r \leq m)$, then $\kappa((P' \circ H) \vee \{y, V(H(u_{i-1}))\}) \geq m$. By the same reason, if we add the vertex z to $P'' \circ H$ and join an edge from z to each of (u_{j-1}, v_r) $(1 \leq r \leq m)$, then $\kappa((P'' \circ H) \vee \{z, V(H(u_{j-1}))\}) \geq m$ and if we add the vertex t to $P''' \circ H$ and join an edge from t to each of (u_{k-1}, v_r) $(1 \leq r \leq m)$, then $\kappa((P''' \circ H) \vee \{t, V(H(u_{k-1}))\}) \geq m$. From Menger' s Theorem, there exist m internally disjoint paths connecting x and y in $(P' \circ H) \vee \{y, V(H(u_{i-1}))\}$, and we say $P'_1, P'_2, \cdots, P'_m$. Also, there exist m internally disjoint paths connecting y and z in $(P'' \circ H) \vee \{z, V(H(u_{j-1}))\}$, and we say $P''_1, P''_2, \cdots, P''_m$ and there exist m internally disjoint paths connecting z and t in $(P''' \circ H) \vee \{t, V(H(u_{k-1}))\}$, and we say $P'''_1, P'''_2, \cdots, P'''_m$. Note that the union of any path in $\{P'_i \mid 1 \leq i \leq m\}$ with any path in $\{P''_j \mid 1 \leq j \leq m\}$ with any path in $\{P'''_k \mid 1 \leq k \leq m\}$ is a path connecting S. Then the paths Q_r induced by the edges in $E(P'_r) \cup E(P''_r) \cup E(P'''_r)$ $(1 \leq r \leq m)$ are m internally disjoint S-paths, as desired. ∎

From the proof of Proposition 2.1, the following proposition is easily seen.

Proposition 2.2 *Let H be a graph and P_n be a path with n vertices. Then* $\pi_4(P_n \circ H) \geq \lfloor \frac{|V(H)|-2}{3} \rfloor$.

2.2 The Lexicographic product of two general graphs

After the above preparations, we are ready to prove Theorem 1.2 in this subsection.

Proof of Theorem 1.2: Without loss of generality, we set $\pi_4(G) = \ell$. Recall that $V(G) = \{u_1, u_2, \ldots, u_n\}$, $V(H) = \{v_1, v_2, \ldots, v_m\}$. From the definition of $\pi_4(G \circ H)$, we need to prove that $\pi_{G \circ H}(S) \geq \ell\lfloor\frac{m-2}{3}\rfloor + 1$ for any $S = \{x, y, z, t\} \subseteq V(G \circ H)$. Furthermore, it suffices to show that there exist $\ell\lfloor\frac{m-2}{3}\rfloor + 1$ internally disjoint paths connecting S in $G \circ H$. Clearly, $V(G \circ H) = \bigcup_{i=1}^{n} V(H(u_i))$. Without loss of generality, let $x \in V(H(u_i))$, $y \in V(H(u_j))$, $z \in V(H(u_k))$ and $t \in V(H(u_w))$.

Suppose that x, y, z, t belong to the same $V(H(u_i))$ $(1 \leq i \leq n)$. Without loss of generality, let $x, y, z, t \in V(H(u_1))$. Since $\delta(G) \geq \pi_4(G) = \ell$, it follows that the vertex u_1 has ℓ neighbors in G, and we say $u_2, u_3, \cdots, u_{\ell+1}$. From Proposition 2.1, there exist $\lfloor\frac{m-2}{3}\rfloor + 1$ internally disjoint paths connecting S in $P_i \circ H$, where $P_i = u_1u_i$ $(2 \leq i \leq \ell+1)$, which occupy at most one path in each of $H(u_j)$ $(2 \leq j \leq m)$. These paths together with the paths $P_{i,j}$ induced by the edges in $\{x(u_i, v_{3j-2}), y(u_i, v_{3j-2}), y(u_i, v_{3j-1}), z(u_i, v_{3j-1}), z(u_i, v_{3j}), t(u_i, v_{3j})\}(2 \leq i \leq \ell+1,\ 1 \leq j \leq \lfloor\frac{m-2}{3}\rfloor)$ are $(\lfloor\frac{m-2}{3}\rfloor + 1) + (\ell-1)\lfloor\frac{m-2}{3}\rfloor = \ell\lfloor\frac{m-2}{3}\rfloor + 1$ internally disjoint paths connecting S in $G \circ H$, as desired.

Suppose that three of $\{x, y, z, t\}$ belong to some copy $H(u_i)$ $(1 \leq i \leq n)$. Without loss of generality, let $x, y, z \in H(u_1)$ and $t \in H(u_2)$. Observe that $\kappa(G) \geq \pi_4(G) = \ell$. Therefore, there exist ℓ internally disjoint paths connecting u_1 and u_2 in G, and we say $P_1, P_2, \cdots, P_\ell$. From Proposition 2.1, there exist $\lfloor\frac{m-2}{3}\rfloor + 1$ internally disjoint paths connecting S in $P_1 \circ H$, which occupy at most one path in each $H(u_j)$. For P_k $(2 \leq k \leq \ell)$, there exist $\lfloor\frac{m-2}{3}\rfloor$ internally disjoint paths connecting S in $P_k \circ H$ by Proposition 2.2, which occupy no edge in $H(u_r)$ $(1 \leq r \leq n)$. Observe that $\bigcup_{i=1}^{\ell} P_i$ is a subgraph of G and $(\bigcup_{i=1}^{\ell} P_i) \circ H$ is a subgraph of $G \circ H$, so the total number of the internally disjoint paths connecting S is $(\lfloor\frac{m-2}{3}\rfloor + 1) + (\ell-1)\lfloor\frac{m-2}{3}\rfloor = \ell\lfloor\frac{m-2}{3}\rfloor + 1$, as desired.

Suppose that two of $\{x, y, z, t\}$ belong to some copy $H(u_i)$ $(1 \leq i \leq n)$. At first, we consider the case that $x, y \in V(H(u_i))$, $z \in V(H(u_j))$ and $t \in V(H(u_k))$. Without loss of generality, let $x, y \in H(u_1)$, $z \in H(u_2)$ and $t \in H(u_3)$. Observe that $\kappa(G) \geq \pi_4(G) = \ell$. Therefore, there exist ℓ internally disjoint paths connecting u_1 and u_2 in G, and we say $P_1, P_2, \cdots, P_\ell$. From Proposition 2.1, there exist $\lfloor\frac{m-2}{3}\rfloor + 1$ internally disjoint paths connecting S in $P_1 \circ H$, which occupy at most one path in each of $H(u_j)$. For P_i $(2 \leq i \leq \ell)$, there exist $\lfloor\frac{m-2}{3}\rfloor$ internally disjoint paths connecting S in $P_i \circ H$ by Proposition 2.2, which occupy no edge in $H(u_j)$ $(1 \leq j \leq n)$. Observe that $\bigcup_{i=1}^{\ell} P_i$ is a subgraph of G and $(\bigcup_{i=1}^{\ell} P_i) \circ H$ is a subgraph of $G \circ H$,so the total number of the internally disjoint paths connecting S is $(\lfloor\frac{m-2}{3}\rfloor + 1) + (\ell-1)\lfloor\frac{m-2}{3}\rfloor = \ell\lfloor\frac{m-2}{3}\rfloor + 1$, as desired.

Next, we consider the case that $x, y \in V(H(u_i))$ and $z, t \in V(H(u_j))$. Without loss of generality, let $x, y \in H(u_1)$, $z, t \in H(u_2)$. Observe that $\kappa(G) \geq \pi_4(G) = \ell$. Therefore, there exist ℓ internally disjoint paths connecting u_1 and u_2 in G, and we say $P_1, P_2, \cdots, P_\ell$. From Proposition 2.1, there exist $\lfloor\frac{m-2}{3}\rfloor + 1$

internally disjoint paths connecting S in $P_1 \circ H$, which occupy at most one path in each $H(u_j)$. For P_i $(2 \leq i \leq \ell)$, there exist $\lfloor\frac{m-2}{3}\rfloor$ internally disjoint paths connecting S in $P_i \circ H$ by Proposition 2.2, which occupy no edge in $H(u_j)$ $(1 \leq j \leq n)$. Observe that $\bigcup_{i=1}^{\ell} P_i$ is a subgraph of G and $(\bigcup_{i=1}^{\ell} P_i) \circ H$ is a subgraph of $G \circ H$, so the total number of the internally disjoint paths connecting S is $(\lfloor\frac{m-2}{3}\rfloor + 1) + (\ell - 1)\lfloor\frac{m-2}{3}\rfloor = \ell\lfloor\frac{m-2}{3}\rfloor + 1$, as desired.

Suppose that x, y, z, t are contained in distinct $H(u_i)$. Without loss of generality, let $x \in H(u_1)$, $y \in H(u_2)$, $z \in H(u_3)$ and $t \in H(u_4)$. Since $\pi_4(G) = \ell$, it follows that there exist ℓ internally disjoint paths connecting $\{u_1, u_2, u_3, u_4\}$ in G, and we say $P_1, P_2, \cdots, P_\ell$. From Proposition 2.1, there exist $\ell\lfloor\frac{m-2}{3}\rfloor + 1$ internally disjoint paths connecting S in $P_1 \circ H$, which occupy at most one path in some $H(u_j)$. For P_i $(2 \leq i \leq \ell)$, there exist $(\ell - 1)\lfloor\frac{m-2}{3}\rfloor$ internally disjoint paths connecting S in $P_i \circ H$ by Proposition 2.2, which occupy no edge in $H(u_j)$ $(1 \leq j \leq n)$. Observe that $\bigcup_{i=1}^{\ell} P_i$ is a subgraph of G and $(\bigcup_{i=1}^{\ell} P_i) \circ H$ is a subgraph of $G \circ H$. Therefore, the total number of the internally disjoint paths connecting S is $(\lfloor\frac{m-2}{3}\rfloor + 1) + (\ell - 1)\lfloor\frac{m-2}{3}\rfloor = \ell\lfloor\frac{m-2}{3}\rfloor + 1$, as desired.

From the above argument, we conclude that $\pi_{G \circ H}(S) \geq \pi_{(\bigcup_{i=1}^{\ell} P_i) \circ H}(S) \geq \ell\lfloor\frac{m-2}{3}\rfloor + 1$ for any $S \subseteq V(G \circ H)$, which implies that $\pi_4(G \circ H) \geq \ell\lfloor\frac{m-2}{3}\rfloor + 1 = \pi_4(G)\lfloor\frac{|V(G)|-2}{3}\rfloor + 1$. The proof is complete.

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