Super 3-restricted edge connectivity of triangle-free graphs

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Abstract: Let $G = (V, E)$ be a connected graph. An edge set $S \subset E$ is a k-restricted edge cut, if $G - S$ is disconnected and every component of $G - S$ has at least k vertices. The k-restricted edge connectivity $\lambda_k(G)$ of G is the cardinality of a minimum k-restricted edge cut of G. A graph G is λ_k-connected, if k-restricted edge cuts exist. A graph G is called λ_k-optimal, if $\lambda_k(G) = \xi_k(G)$, where

$$\xi_k(G) = \min\{|[X, Y]| : X \subset V, |X| = k \text{ and } G[X] \text{ is connected}\};$$

$G[X]$ is the subgraph of G induced by the vertex subset $X \subseteq V$, and $Y = V \backslash X$ is the complement of X; $[X, Y]$ is the set of edges with one end in X and the other in Y. G is said to be super-λ_k, if each minimum k-restricted edge cut isolates a

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connected subgraph of order k. In this paper, we give some sufficient conditions for triangle-free graphs to be super-λ_3.

Key words: Edge connectivity; 3-Restricted edge connectivity; Triangle-free; Degree sequence; Fault tolerance

1 Introduction

We only consider undirected simple connected graphs. Unless stated otherwise, we follow Bondy and Murty [1] for terminology and definitions.

Let $G = (V, E)$ be a connected graph. We define the *order* of G by $n(G) = |V|$ and the *size* by $m(G) = |E|$. The *vertex degree* $d(v)$ of a vertex $v \in V$ of a graph G is the number of vertices adjacent to v, $\delta(G)$ is the minimum degree of G. The *degree sequence* of a graph G is defined as the non-increasing sequence of the degrees of the vertices of G. For a vertex $v \in V$, $N(v)$ is the set of all vertices adjacent to v. More generally for $S \subset V$, $N_G(S) = \{x \mid x \in V \setminus S, x \text{ is adjacent to a vertex in } S\}$ denotes the neighbor set of S in G. We denote the diameter of G by $d(G)$, and write $G - v$ for $G - \{v\}$.

A network can be conveniently modelled as a graph G. A classic measure of the fault tolerance of a network is the edge-connectivity $\lambda(G)$. In general, the larger $\lambda(G)$ is, the more reliable the network is. It is known that $\lambda(G) \leq \delta(G)$. If G satisfies $\lambda(G) = \delta(G)$, then it is said to be *maximally edge connected*, or λ-*optimal* for simplicity.

In the definitions of $\lambda(G)$, no restrictions are imposed on the components of $G - S$, where S is an edge cut. To compensate for this shortcoming, it would seem natural to generalize the notion of the classical connectivity by imposing some conditions or restrictions on the components of $G - S$. Hence, k-restricted edge connectivity were proposed [3]. An edge set $S \subset E$ is said to be a *k-restricted edge cut*, if $G - S$ is disconnected and every component of $G - S$ has at least k vertices. The *k-restricted edge connectivity* $\lambda_k(G)$ of G is the cardinality of a minimum k-restricted edge cut of G. If S is a k-restricted edge cut and $|S| = \lambda_k(G)$, then we call S a *λ_k-cut*. Not all graphs have k-restricted edge cuts. A connected graph G is called *λ_k-connected*, if it has a k-restricted edge cut. If S is a *λ_k-cut*, then $G - S$ has only two connected components. We can see that if G is λ_k-connected ($k \geq 2$), then it is also λ_{k-1}-connected and $\lambda_{k-1}(G) \leq \lambda_k(G)$. It seems that the larger $\lambda_k(G)$ is, the more reliable the network is [5, 6, 10]. So, we expect $\lambda_k(G)$ to be as large as possible. Let

$$\xi_k(G) = min\{|[X, Y]| : X \subseteq V, |X| = k, G[X] \text{ is connected}\},$$

where $[X, Y]$ the set of edges of G with one end in X and the other in Y and $Y = V - X$. It has been shown that $\lambda_k(G) \leq \xi_k(G)$ holds for many graphs [7, 11]. G is said to be *λ_k-optimal*, if $\lambda_k(G) = \xi_k(G)$. Furthermore, G is called *super k-restricted edge connected* or *super-λ_k*, if every λ_k-cut of G isolates one connected subgraph of order k, that is, every λ_k-cut is a set of edges adjacent to a certain connected subgraph of order k. Clearly, $\lambda_1 = \lambda$, $\lambda_2 = \lambda'$, $\xi_1 = \delta$ and $\xi_2 = \xi$ is the minimum edge degree. If G is super-λ_k, then it is λ_k-optimal. However, the converse is not true. The cycle of length $n \geq 2k+2$ is a counterexample.

Esfahanian and Hakimi proved the existence of 2-restricted edge cuts and upper bound for the 2-restricted edge connectivity:

Theorem 1.1. *[2] For any connected graph G with at least four vertices which is not isomorphic to the star $K_{1,n-1}$, $\lambda'(G)$ is well defined. Furthermore, $\lambda'(G) \leq \xi(G)$.*

For $\lambda_3(G)$, It has been shown by Meng et al. that

Theorem 1.2. *(Meng, Ji [5]) If G is a λ_3-connected graph, then $\lambda_3(G) \leq \xi_3(G)$.*

If a graph G is triangle-free, then a connected subgraph of G with three vertices is a path xyz of length two. Thus,

$$\xi_3(G) = \min\{d(x) + d(y) + d(z) - 4 : xyz \text{ is a path of length two in } G\}.$$

In this paper, we give some sufficient conditions for graphs to be super-λ_3.

2 Sufficient conditions for graphs to be super-λ_3

We start this section with the following lemma.

Lemma 2.1. *[8] Let G be a λ_3-connected graph. G is super-λ_3 if and only if G is not λ_4-connected, or G is λ_4-connected and $\lambda_4(G) > \xi_3(G)$.*

Lemma 2.2. *Let G be a λ_4-connected triangle-free graph. If there is a λ_4-cut $S = [X, Y]$ with the vertex sets X and Y of the* *two components of $G-S$ such that there exists a path xyz in $G[X]$ with the property that*

$$\begin{aligned}|[X\backslash\{x,y,z\},Y]| &> |(N(x)\cap X)\backslash N(z)|+|(N(z)\cap X)\backslash N(x)|+\\ &\quad 2|(N(x)\cap N(z)\cap X)\backslash\{y\}|+\\ &\quad |(N(y)\cap X)\backslash\{x,z\}|,\end{aligned}$$

then G is super-λ_3.

Proof. Suppose G is not super-λ_3. The hypotheses imply

$$\begin{aligned}\lambda_4(G)=|[X,Y]| &= |[\{x,y,z\},Y]|+|[X\backslash\{x,y,z\},Y]|\\ &> |[\{x,y,z\},Y]|+|(N(y)\cap X)\backslash\{x,z\}|+\\ &\quad |(N(x)\cap X)\backslash N(z)|+|(N(z)\cap X)\backslash N(x)|+\\ &\quad 2|(N(x)\cap N(z)\cap X)\backslash\{y\}|\\ &= |(N(x)\cap X)\backslash\{y\}|+|N(x)\cap Y|+\\ &\quad |(N(y)\cap X)\backslash\{x,z\}|+|N(y)\cap Y|+\\ &\quad |(N(z)\cap X)\backslash\{y\}|+|N(z)\cap Y|\\ &= |N(x)\backslash\{y\}|+|N(y)\backslash\{x,z\}|+|N(z)\backslash\{y\}|\\ &= d(x)+d(y)+d(z)-4\geq \xi_3(G)\end{aligned}$$

By Lemma 2.1, we deduce a contradiction, and thus G is super-λ_3. □

Corollary 2.3. *Let G be a λ_4-connected triangle-free graph. If there is a λ_4-cut S with the vertex sets X and Y of the two components of $G-S$ with the property that each vertex in $[(N(x)\cap X)\backslash\{y\}]\cup[(N(y)\cap X)\backslash\{x,z\}]\cup[(N(z)\cap X)\backslash\{y\}](\neq\varnothing)$ has at least two neighbor in Y, then G is super-λ_3.*

Corollary 2.4. *Let G be a λ_4-connected triangle-free graph. If there is a λ_4-cut S with the vertex sets X and Y of the two* *components of $G-S$ such that each vertex in X has at least three neighbors in Y, then G is super-λ_3.*

Let G be a λ_3-connected triangle-free graph and let $S=[X,Y]$ be an arbitrary λ_3-cut, where X and Y are the vertex sets of the two components of $G-S$. We assume that $S(x)$ is the number of edges of S incident to x. Set $X_i=\{x : x\in X, S(x)=i\}, i=0,1,2$; $X_3=\{x : x\in X, S(x)\geq 3\}$, and $Y_i=\{x : x\in Y, S(x)=i\}, i=0,1,2$; $Y_3=\{x : x\in Y, S(x)\geq 3\}$. We will study the property of X and Y. By symmetry, we will consider X.

Lemma 2.5. *If $|X|\geq 4$, then there is a vertex $x\in X$ such that $S(x)\leq 3$.*

Proof. Suppose that for any $x\in X$, $S(x)\geq 4$. Take a path xyz in $G[X]$, we have

$$\begin{aligned}\xi_3(G) &\leq d(x)+d(y)+d(z)-4\\ &= S(x)+S(y)+S(z)+|N_X(x)\backslash\{y\}|+|N_X(y)\backslash\{x,z\}|\\ &\quad +|N_X(z)\backslash\{y\}|\\ &\leq S(x)+S(y)+S(z)+\\ &\quad 3|N_X(x)\cup N_X(y)\cup N_X(z)\backslash\{x,y,z\}|\\ &\leq S(x)+S(y)+S(z)+3|X\backslash\{x,y,z\}|\\ &< S(x)+S(y)+S(z)+4|X\backslash\{x,y,z\}|\\ &\leq S(x)+S(y)+S(z)+\sum_{u\in X\backslash\{x,y,z\}} S(u)=|S|=\lambda_3(G),\end{aligned}$$

which is a contradiction. □

Lemma 2.6. *If $|X|\geq 4$ and $X_i=\varnothing, i=0,1,2$, then for any $x\in X$, $S(x)=3$.*

Proof. Since $X_i = \varnothing, i = 0, 1, 2$, for any $x \in X$, $S(x) \geq 3$. Assume that there is a vertex $u \in X$ with $S(u) \geq 4$. We claim that there is a path xyz in $G[X]$ such that $x \neq u, y \neq u, z \neq u$. If not, because $G[X]$ is connected, we have $G[X]$ is a star or is isomorphic to the following graph

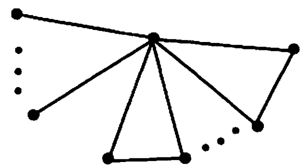

Fig. 1
The isomorphic graph of G[X]

If $G[X]$ is a star, and $|X| \geq 4$, then there is a vertex $x \in X - u$ and $G[X - x]$ is connected, $V(G[X - x]) \geq 3$. Because $S(x) \neq 0$ and $G[Y \cup \{x\}]$ is connected. Hence $S' = [X - x, Y \cup \{x\}]$ is a 3-restricted edge cut with $|S'| = |S| - S(x) + 1 \leq |S| - 3 + 1 = |S| - 2$, contradicting to the minimum of S.

If $G[X]$ is is isomorphic to Fig.1, then G has a triangle, a contradiction.

Hence there is a path xyz in $G[X]$ such that $x \neq u, y \neq u, z \neq u$. We have

$$\begin{aligned}
\xi_3(G) &\leq d(x) + d(y) + d(z) - 4 \\
&= S(x) + S(y) + S(z) + |N_X(x)\backslash\{y\}| + |N_X(y)\backslash\{x, z\}| \\
&\quad +|N_X(z)\backslash\{y\}| \\
&\leq S(x) + S(y) + S(z) + \\
&\quad 3|N_X(x) \cup N_X(y) \cup N_X(z)\backslash\{x, y, z\}| \\
&\leq S(x) + S(y) + S(z) + 3|X\backslash\{x, y, z\}|
\end{aligned}$$

$$
\begin{aligned}
&< \quad S(x) + S(y) + S(z) + \sum_{v \in X \setminus \{x,y,z\}} S(v) \quad (S(u) \geq 4) \\
&= \quad |S| = \lambda_3(G),
\end{aligned}
$$

which is a contradiction. □

Lemma 2.7. *If* $|X|, |Y| \geq 4$ *and* $|N(u) \cap N(v)| \geq 4$ *for all pairs* u, v *of nonadjacent vertices, then* $X_0, Y_0 = \varnothing$.

Proof. Since $|N(u) \cap N(v)| \geq 4$ for all pairs u, v of nonadjacent vertices, $d(G) \leq 2$. So $X_0 = \varnothing$ or $Y_0 = \varnothing$, assume $X_0 = \varnothing$. Take a vertex $y \in Y_0$. If $X_1 \neq \varnothing$, then $|N(x) \cap N(y)| \leq 1$ for some $x \in X_1$. Hence $X_1 = \varnothing$. Similarly, $X_2 = \varnothing$. By Lemma 2.6 for any $x \in X$, $S(x) = 3$. Then $|N(x) \cap N(y)| \leq 3$ for any $x \in X$, which is a contradiction. □

Lemma 2.8. *[4] Let* G *be a* λ_3*-connected triangle-free graph. If* $|N(u) \cap N(v)| \geq 3$ *for all pairs* u, v *of nonadjacent vertices, then* G *is* λ_3*-optimal.*

Theorem 2.9. *Let* G *be a* λ_3*-connected triangle-free graph. If* $|N(u) \cap N(v)| \geq 4$ *for all pairs* u, v *of nonadjacent vertices, then* G *is super-*λ_3.

Proof. By Lemma 2.8 G is λ_3-optimal. Suppose that G is not super-λ_3. Let $S = [X, Y]$ be an arbitrary λ_3-cut and $|X|, |Y| \geq 4$. By Lemma 2.7, $X_0, Y_0 = \varnothing$.

Case 1. $X_1, Y_1 \neq \varnothing$.

Let $x \in X_1, y \in Y_1$, then $S(x) = S(y) = 1$, that is $|N(x) \cap N(y)| \leq 2$. Hence we can get $xy \in E(G)$ and $X_1 = \{x\}, Y_1 = \{y\}$. For any $u \in X - x$, $|N(u) \cap Y| \geq 2$. Take a path xuv in

$G[X]$. We have

$$\begin{aligned}
\xi_3(G) &\leq d(x)+d(u)+d(v)-4 \\
&= |N(x)\backslash\{u\}|+|N(u)\backslash\{x,v\}|+|N(v)\backslash\{u\}| \\
&= |(N(x)\cap X)\backslash\{u\}|+|N(x)\cap Y|+|(N(u)\cap X)\backslash\{x,v\}| \\
&\quad +|N(u)\cap Y|+|(N(v)\cap X)\backslash\{u\}|+|N(v)\cap Y| \\
&= |[\{x,u,v\},Y]|+|(N(u)\cap X)\backslash\{x,v\}|+ \\
&\quad |(N(x)\cap X)\backslash N(v)|+|(N(v)\cap X)\backslash N(x)|+ \\
&\quad 2|(N(x)\cap N(v)\cap X)\backslash\{u\}| \\
&\leq |[\{x,u,v\},Y]|+|[X\backslash\{x,u,v\},Y]| \\
&= |[X,Y]| = \lambda_3(G).
\end{aligned}$$

Since $\xi_3(G)=\lambda_3(G)$, we have $[(N(x)\cap X)\backslash N(v)]\cup[(N(u)\cap X)\backslash\{x,v\}]\cup[(N(v)\cap X)\backslash N(x)]=\varnothing$, then we can get $N(x)\cap X = N(v)\cap X, N(u)\cap X=\{x,v\}$ and $X=(N(x)\cup N(u)\cup N(v))\cap X$. For any $w\in N(x)\cap X, wy\overline{\in}E$, then $|N(w)\cap N(y)|\geq 3$, that is $|N(w)\cap Y|\geq 3$. Hence

$$\begin{aligned}
|[X\backslash\{x,u,v\},Y]| &\geq 3|(N(x)\cap N(v)\cap X)\backslash\{u\}| \\
&> 2|(N(x)\cap N(v)\cap X)\backslash\{u\}|.
\end{aligned}$$

By Lemma 2.2, G is super-λ_3, a contradiction.

Case 2. X_1 or $Y_1=\varnothing$. Set $Y_1=\varnothing$.

Let $u\in X_1$ and $N(u)\cap Y=\{x\}$. Choose a path xyz in $G[Y]$. It is analogous to Case 1, we can get G is super-λ_3, which is a contradiction.

Case 3. $X_1=Y_1=\varnothing$.

If $X_2\neq\varnothing$, then take $u\in X_2, N(u)\cap Y=\{x,x'\}$. Choose a path xyz in $G[Y]$. It is analogous to Case 1, we can get $N(x)\cap Y=N(z)\cap Y, N(y)\cap X=\{x,z\}$ and $Y=(N(x)\cup$

$N(y) \cup N(z)) \cap Y$. Hence $x' \in N(x)$ and uxx' is a K_3, which is a contradiction again.

Hence $X_2 = \varnothing$, by Corollary 2.4 G is super-λ_3, again a contradiction. □

Using Turán's bound $2m(G) \leq \frac{n(G)^2}{2}$ for triangle-free graphs G [9], we obtain the following theroem.

Theorem 2.10. *Let G be a λ_3-connected triangle-free graph of order $n \geq 6$ and degree sequence $d_1 \geq d_2 \geq \ldots \geq d_n = \delta$. If*

$$\sum_{i=1}^{max\{1,\delta-4\}} d_{n-i} \geq max\{1, \delta-4\}\frac{1}{2}(\left\lfloor \frac{n}{2} \right\rfloor + 3 - \frac{2}{n-5}) + 1,$$

then G is super-λ_3.

Proof. If G is not λ_4-connected, then we are done. Assume that G is λ_4-connected. Let $S = [X, Y]$ be an arbitrary λ_4-cut. Assume, without loss of generality that $|X| \leq |Y|$, then $|X| \leq \lfloor n/2 \rfloor$ and $|X|, |Y| \geq 4$. If $|X| \leq \delta - 2$, then every vertex in X has at least three neighbors in Y, Corollary 2.4 leads to the desired result in this case. For $|X| \geq \delta - 1$, let $x \in X$ such that $d(x) = \min\{d(u) : u \in X\}$. Choose a path xyz in $G[X]$, using Turán's bound and the inequality $\max\{1, \delta-4\} \leq |X|-3$, the hypothesis yields

$$\begin{aligned}
|[X,Y]| &\geq \sum_{u \in X} d(u) - \frac{|X|^2}{2} \\
&= d(x) + d(y) + d(z) - 4 + 4 + \sum_{u \in X \setminus \{x,y,z\}} d(u) - \frac{|X|^2}{2} \\
&\geq \xi_3(G) + 4 + \sum_{i=1}^{max\{1,\delta-4\}} d_{n-i} + \sum_{i=max\{2,\delta-3\}}^{|X|-3} d_{n-i} -
\end{aligned}$$

$$\begin{aligned}
&\frac{(|X|+3)(|X|-3)}{2} - \frac{9}{2} \\
\geq\ & \xi_3(G) - \frac{1}{2} + \frac{1}{2}(|X|-3)(\lfloor\frac{n}{2}\rfloor + 3 - \frac{2}{n-5} - |X| - 3) + 1 \\
\geq\ & \xi_3(G) - (\frac{1}{2} + \frac{|X|-3}{n-5}) + 1.
\end{aligned}$$

Since $\lambda_4(G)$ and $\xi_3(G)$ are integers and $1/2+(|X|-3)/(n-5) < 1$, it follows that $\lambda_4(G) \geq \xi_3(G)+1$. By Lemma 2.1 G is super-λ_3. □

Theorem 2.11. *Let G be triangle-free graph of order $n \geq 10$. If $d(x)+d(y) \geq n-1$ for all nonadjacent vertices x, y in G, then G is super-λ_3.*

Proof. If there is a vertex v with $d(v) = 1$, then let $N(v) = u$. For any $w \in V \setminus \{u, v\}$, w is not adjacent to v and $d(w) \geq n-2$. Hence G contains a clique of order $n-2 > 3$, contradicting G is triangle-free. Then $\delta(G) \geq 2$. Suppose G is not super-λ_3. Let $S = [X, Y]$ be an λ_3-cut with $|X|, |Y| \geq 4$. If for any $x \in X, S(x) \geq 1$, then we say S saturates X. The hypotheses imply $d(G) \leq 2$.

Claim 1. S saturates X or Y. If not, there are $x \in X, y \in Y$ such that $S(x) = 0, S(y) = 0$. Hence $d(G) \geq 3$, a contradiction.

Let S saturate X.

Claim 2. There is a vertex $x \in X$ such that $S(x) = 1$. Otherwise, for any $u \in X$ such that $S(u) \geq 2$. Take a path xyz in $G[X]$, then $X - \{x, y, z\} \neq \varnothing$.

$$\begin{aligned}
\xi_3(G) &\leq d(x)+d(y)+d(z)-4 \\
&= S(x)+S(y)+S(z)+|N_X(x)\backslash N_X(z)|+|N_X(y)\backslash \\
&\quad \{x,z\}|+|N_X(z)\backslash N_X(x)|+2|N_X(z)\cap N_X(x)-y|.
\end{aligned}$$

If $(N_X(x)\backslash N_X(z))\cup(N_X(y)\backslash\{x,z\})\cup(N_X(z)\backslash N_X(x)) \neq \varnothing$, then

$$\begin{aligned}\xi_3(G) &< S(x)+S(y)+S(z)+\sum_{v\in X\backslash\{x,y,z\}} S(v)\\ &= |S| = \lambda_3(G),\end{aligned}$$

which is a contradiction.

If $(N_X(x)\backslash N_X(z)) \cup (N_X(y)\backslash\{x,z\}) \cup (N_X(z)\backslash N_X(x)) = \varnothing$, then

$$\begin{aligned}\xi_3(G) &\leq S(x)+S(y)+S(z)+2|N_X(z)\cap N_X(x)-y|\\ &\leq S(x)+S(y)+S(z)+\sum_{v\in X\backslash\{x,y,z\}} S(v)\\ &= |S| = \lambda_3(G).\end{aligned}$$

Hence each vertex of $X-\{x,y,z\}$ has two neighbors in Y and $X = N_X(x)\cup N_X(y)\cup N_X(z)$. Take $u \in N_X(z)\cap N_X(x)-y$, for the path yxu, we use the similar methods as above. We can get z also has two neighbors in Y. Similarly, x and y have two neighbors in Y, respectively. Hence $d(y)+d(u) \leq 4+4 < n-1$, a contradiction.

Claim 3. S saturates Y. If not, there is $y \in Y$ such that $S(y) = 0$. And y is not adjacent to x of the vertex in Claim 2. Then $n-1 \leq d(x)+d(y) \leq (|X|-1)+1+(|Y|-1) \leq n-1$, that is x is adjacent to each vertex of X and y is adjacent to each vertex of Y. Let $y_1 = N(x)\cap Y$, then $d(y_1) = 2$. Take $x_1 \in X-x$, then x_1 is not adjacent to y_1, y and each vertex of $X-x$. Hence we have $d(x_1)+d(y_1) \leq (n-4)+2 \leq n-2$, a contradiction.

Claim 4. There is a vertex $y \in Y$ such that $S(y) = 1$. The proof is similar to Claim 2.

Claim 5. The vertex x in Claim 2 is adjacent to the vertex y in Claim 4. Otherwise, $n-1 \leq d(x)+d(y) \leq (|X|-1)+1+(|Y|-1)+1 \leq n$, we have $d(x)+d(y)=n$ or $d(x)+d(y)=n-1$.

If $d(x)+d(y)=n$, then x is adjacent to each vertex of X and y is adjacent to each vertex of Y. Let $y_1 = N(x)\cap Y, x_1 = N(y)\cap X$, then $d(y_1)=d(x_1)=2$. Hence we have $d(x_1)+d(y_1) = 4 < n-1$, a contradiction.

If $d(x)+d(y)=n-1$, then $d(x)=(|X|-1)+1, d(y) = (|Y|-2)+1$ or $d(x)=(|X|-2)+1, d(y)=(|Y|-1)+1$. Without loss of generality, let $d(x)=(|X|-1)+1, d(y)=(|Y|-2)+1$. Hence x is adjacent to each vertex of X and y is adjacent to each vertex of $Y-u$ for some u. If $u=y_1$, then $x_1y_1\overline{\in}E$ and $d(x_1)=2$. We have $d(y_1)+d(x_1)\leq 2+(|Y|-2)+1 \leq n-|X|+1 \leq n-3$, a contradiction. If $u \neq y_1$ and $x_1y_1\overline{\in}E$, then $d(x_1)\leq 3, d(y_1)\leq 3$. We have $d(y_1)+d(x_1)\leq 6 < n-1$, a contradiction.

Claim 6. There is only one $x \in X$ such that $S(x)=1$. Otherwise, there is a vertex $x'(\neq x)\in X$ such that $S(x')=1$. Then x' is adjacent to y of the vertex in Claim 4, that is $S(y)\geq 2$, a contradiction. Hence for any $u \in X-x, S(u)\geq 2$. Choose a path xyz in $G[X]$, and $X\setminus\{x,y,z\}\neq \varnothing$. it is similar to Claim 2, we can get a contradiction. □

Remark . The bound is sharp. Let $H=Q_3, x, y$ be two vertices, and G be the union of H and x, y. $V(G)=V(Q_3)\cup\{x,y\}$, $E(G)=E(Q_3)\cup\{x(011), x(110), x(000), x(101)\}\cup\{y(100), y(001), y(010), y(111)\}$. For all nonadjacent vertices u, v, $d(u)+d(v)\geq 8$ but G is not super-λ_3. G is the following graph.

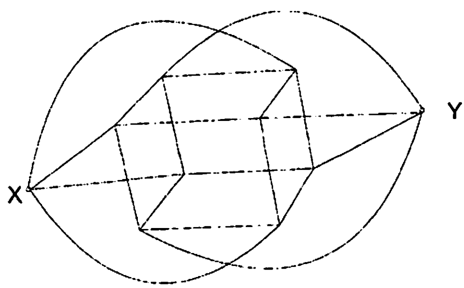

Fig. 2 The counterexample G

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