λ'-optimal regular graphs with two orbits*

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Abstract An edge set F is called a restricted edge-cut if $G-F$ is disconnected and contains no isolated vertices. The minimum cardinality over all restricted edge-cuts is called restricted edge-connectivity of G, and denoted by $\lambda'(G)$. A graph G is called λ'-optimal if $\lambda'(G)=\xi(G)$, where $\xi(G)=\min\{d_G(u)+d_G(v)-2 : uv\in E(G)\}$. In this note, we obtain a sufficient condition for a $k(\geq 3)$-regular connected graph with two orbits to be λ'-optimal.

Keywords: Orbit; Atom; Restricted edge connectivity

1 Introduction

For graph-theoretical terminology and notation not given here, we follow Bondy [1]. We consider finite, undirected and simple connected graphs with vertex set $V(G)$ and edge set $E(G)$. We use $d_G(v)$ and $\delta(G)$ to denote the degree of vertex $v\in V(G)$ and the minimum degree of G, respectively. For $X\subset V(G)$, we use $G[X]$ to denote the subgraph induced by X. Let G_1 and G_2 be two graphs. The union $G_1\cup G_2$ of G_1 and G_2 is the graph with vertex set $V(G_1)\cup V(G_2)$ and edge set $E(G_1)\cup E(G_2)$. An edge set F of G is called an *edge-cut* if $G-F$ is disconnected. The *edge-connectivity* $\lambda(G)$ of a graph G is the minimum cardinality over all edge-cuts of G. A graph G

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is called *maximally edge connected* or simply *max-*λ if $\lambda(G) = \delta(G)$, see [5] for the details. Furthermore, we call a graph G is *super edge connected* or simply *super-*λ if G is max-λ and every minimum edge-cut set of G isolates one vertex. For the studies of super-λ graphs, we suggest readers to refer to [5, 11].

Esfahanian and Hakimi [3] introduced the concept of restricted edge-connectivity. The concept of restricted edge-connectivity is also one kind of conditional edge-connectivity proposed by Harary in [4]. Call an edge set F a *restricted edge-cut* of G if $G - F$ is disconnected and contains no isolated vertices. The minimum cardinality over all restricted edge-cuts is the *restricted edge-connectivity* of G, denoted by $\lambda'(G)$. Esfahanian in [3] proved that if a connected graph G with $|V(G)| \geq 4$ is not a star $K_{1,n-1}$, then $\lambda'(G)$ exists and $\lambda(G) \leq \lambda'(G) \leq \xi(G)$, where $\xi(G) = \min\{d_G(u) + d_G(v) - 2 : uv \in E(G)\}$ is the minimum edge degree of G(for $uv \in E(G)$, we call $d_G(u) + d_G(v) - 2$ the *edge degree* of uv). A graph G with $\lambda'(G) = \xi(G)$ is called λ'*-optimal*. For the studies of λ'-optimal graphs, see [8, 11, 15, 14] for examples.

A graph G is said to be *vertex transitive* if for every two vertices u and v of G, there exists an automorphism $g \in Aut(G)$, such that $g(u) = v$. For the studies of the connectedness of vertex transitive graphs, we suggest readers to refer to [10, 12, 13]. Let $x \in G$, we call the set $\{x^g : g \in Aut(G)\}$ an orbit of $Aut(G)$. If no confusion, we directly call an orbit of $Aut(G)$ an orbit of G. Let W be a subgroup of the symmetric group over a set S. We say that W acts transitively on a subset T of S if for any $h, l \in T$, there exists a permutation $\varphi \in W$ with $\varphi(h) = l$. Clearly, the automorphism group $Aut(G)$ acts transitively on each orbit of G.

Let H be a connected graph with vertex set $\{v_1, v_2, \cdots, v_m\}$, $d(v_1) = \cdots = d(v_l) = k-1$ and $d(v_{l+1}) = \cdots = d(v_m) = k$. The graph $G_2(H)$ is constructed by taking two copies of H, H_1, H_2, and placing edges $\{v_{i1}v_{i2} | v_{i1} \in V(H_1), v_{i2} \in V(H_2), i = 1, 2, \cdots, l\}$ between H_1 and H_2. Clearly, $G_2(H)$ is a k-regular l-edge-connected graph. In Fig. 1, we give seven graphs, and they are useful in the rest of the paper. For $i = 2, 3, 4, 5$, the connected 3-regular graph $G_n(N_i)$ is constructed by taking n copies of N_i and join these N_i's by adding edges on the vertices of degree 2. For simplicity, only $G_n(N_4)$ is shown in Fig. 2.

Mader [10] proved that the edge-connectivity of a vertex transitive graph attains its regular degree. It is then natural to consider the relation between

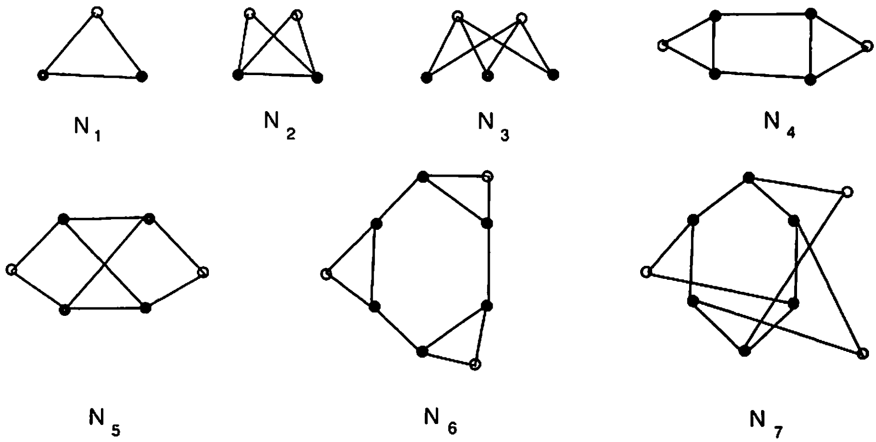

Figure. 1

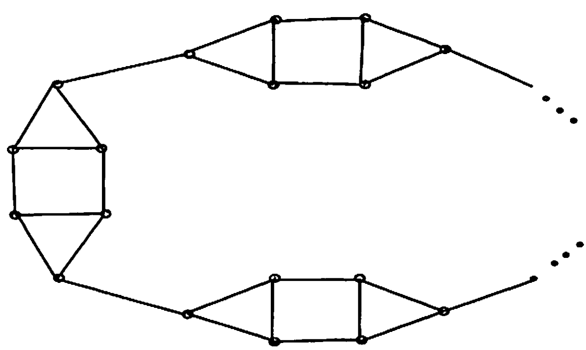

Figure. 2

the edge connectivity and the number of orbits. In [6], Liu and Meng characterized the max-λ 3-regular and 4-regular double-orbit graphs and also reported a sufficient condition for k-regular double-orbit graphs to be max-$\lambda, k > 4$. In this note, we study the λ'-optimality of k-regular double-orbit graphs in this note for $k \geq 3$.

2 Main results

We use $\omega(A)$ to denote the set of edges with exactly one end vertex in A and the other end vertex in $V\backslash A$. A restricted edge-cut F of G is called a *λ'-cut* if $|F| = \lambda'(G)$. It is easy to see that for any λ'-cut F, $G - F$ has exactly two connected non-trivial components. Let A be a proper subset of $V(G)$. If $\omega(A)$ is a λ'-cut of G, then A is called a *restricted edge fragment* of G, or simply *λ'-fragment* of G. It is clear that if A is a restricted edge fragment of G, then so is $V \setminus A$. A restricted edge fragment with the least cardinality is called a *restricted edge atom*, or simply *λ'-atom* of G. By the minimality of $\omega(A)$, $G[A]$ is connected. Note that $\lambda'(G) \leq \xi(G)$, then $\lambda'(G) < \xi(G)$ if G is not λ'-optimal.

J. Xu and K. Xu in [15] reported the following useful lemma.

Lemma 2.1. *If G is not λ'-optimal, then any two distinct λ'-atoms of G are disjoint.*

Let G be a k-regular connected double-orbit graph with two orbits V_1, V_2, A be an edge atom of G and $A_1 = A \cap V_1$ and $A_2 = A \cap V_2$.

Lemma 2.2. *Let G be a connected non-λ'-optimal graph with two orbits. Assume that A is a λ'-atom of G, $Y = G[A]$ and $Y_i = G[A_i]$ for $i = 1, 2$. Then $G[A]$ is a double-orbit graph and $G[A_i]$, $i = 1, 2$ are vertex transitive.*

Proof. Given $u, v \in A_1$, there is an automorphism φ of G with $\varphi(u) = v$, and so $\varphi(A) \cap A \neq \varnothing$. It is clear that $\varphi(A)$ is a λ'-atom of G, then by Lemma 2.1, $\varphi(A) = A$. The automorphism φ induces an automorphism of $G[A]$. Note that G is a double-orbit graph. Thus, the automorphism group $Aut(Y)$ acts transitively on A_1. Similarly, $Aut(Y)$ acts transitively on A_2. Since A_1 and A_2 are contained in different orbits, we have $\varphi(A_1) = A_1$ and $\varphi(A_2) = A_2$ for any automorphism φ of Y. That is, the automorphism φ induces an automorphism of $G[A_1]$. Thus, the automorphism group $Aut(Y_1)$ acts transitively on A_1. Similarly, the automorphism group $Aut(Y_2)$ acts transitively on A_2. The result follows. □

We use $E(x, A)$ to denote the set of edges which have one end vertex x and the other in A. In the following, we can assume that $G[A_i]$ is a r_i-regular graph by Lemma 2.2, and $k_i = |E(x, A_{3-i})|$ for $x \in A_i$. Thus, $k_i \leq |A_{3-i}|$, $r_i < |A_i|$, $|E(x, V \backslash A)| = k - k_i - r_i \geq 0$, $i = 1, 2$ and $k_1|A_1| = k_2|A_2|$.

Lemma 2.3. *Let G be a k-regular connected non-λ'-optimal double-orbit graph, $k \geq 3$. Then $\lambda'(G) \geq 2$.*

Proof. By contradiction. Assume that $\lambda'(G) = 1$. Let $A = A_1 \cup A_2$ be a λ'-atom of G. Then

$$0 < \lambda' = |A_1|(k - k_1 - r_1) + |A_2|(k - k_2 - r_2) = 1.$$

It implies that $|A_1|(k - k_1 - r_1) = 1$ or $|A_2|(k - k_2 - r_2) = 1$. Without loss of generality, we assume that $|A_1|(k - k_1 - r_1) = 1$, then $|A_1| = 1, k - k_1 - r_1 = 1$ and $k - k_2 - r_2 = 0$. We have that $r_1 = 0, k_2 = 1, k_1 = k - 1$ and $r_2 = k - 1$. Since $k_1|A_1| = k_2|A_2|$, we have $|A_2| = k_1 = k - 1$. On the other hand, $|A_2| \geq r_2 + 1 = k > |A_2|$, a contradiction. □

Lemma 2.4. *Let G be a 3-regular connected non-λ'-optimal double-orbit graph and $A = A_1 \cup A_2$ be a λ'-atom of G. Then the λ'-atom of G is isomorphic to a graph of $N_i, i = 1, 2, \cdots, 7$ (see Figure 1).*

Proof. Since G is not λ'-optimal, we have $|A| \geq 3$ and either $\lambda'(G) = 2$ or $\lambda'(G) = 3$ by Lemma 2.3. If A is contained in an orbit of G, then $G[A]$ is vertex transitive by Lemma 2.2. It is not difficult to see that $|A| = 3$ and thus $G[A] \cong K_3 \cong N_1$. We next assume that $A_i \neq \emptyset, i = 1, 2$. Note that $\lambda' = |\omega(A)| = |A_1|(3 - k_1 - r_1)| + |A_2|(3 - k_2 - r_2)$, then either $|A_1| \leq 3$ or $|A_2| \leq 3$. Without loss of generality, assume that $|A_1| \leq |A_2|$.

Case 1. $|A_1| = 1$.

Since $G[A_i]$ is vertex transitive, we can see that $r_1 = 0, k_2 = 1, k_1 = |A_2|$. Note that $|A| \geq 3$ and G is 3-regular, then we have $2 \leq k_1 \leq 3$. If $k_1 = 2$, then $3 - k_1 - r_1 = 1, 3 - k_2 - r_2 = 1$ and thus $A \cong N_1$. If $k_1 = 3$, then $3 - k_2 - r_2 = 1$, that is, $G[A_2]$ is vertex transitive with 3 vertices and the sum of degree of all vertices is 3, which is impossible.

Case 2. $|A_1| = 2$.

We first assume that $|A_1| = |A_2| = 2$, then $k_1 = k_2$. If $3 - k_1 - r_1 = 0$, then $3 - k_2 - r_2 = 1$ and $r_1 - r_2 = 1$, that is, $r_1 = 1, r_2 = 0$, and $k_1 = k_2 = 2$.

Thus, $A \cong N_2$. If $3 - k_1 - r_1 = 1, 3 - k_2 - r_2 = 0$, then we can deduce that $A \cong N_2$ similarly.

If $|A_1| = 2, |A_2| = 3$, then $k_1 = 3, k_2 = 2$ and thus $3 - k_1 - r_1 = 0$, $3 - k_2 - r_2 = 1$, $r_1 = r_2 = 0$. Clearly, $A \cong N_3$.

If $|A_1| = 2, |A_2| \geq 4$, then $3 - k_1 - r_1 = 1, 3 - k_2 - r_2 = 0$, $k_1 = 2, k_2 = 1$, $|A_2| = 4$, and $r_1 = 0, r_2 = 2$. Thus, we have either $A \cong N_4$ or $A \cong N_5$.

Case 3. $|A_1| = 3$.

If $|A_2| = 3$, then $k_1 = k_2$, $3 - k_1 - r_1 = 1, 3 - k_2 - r_2 = 0$, $r_2 - r_1 = 1$, that is, $r_1 = 1$ or $r_2 = 1$ which is impossible. If $|A_2| > 3$, then $3 - k_1 - r_1 = 1, 3 - k_2 - r_2 = 0$. It is easy to see that $r_1 = 0$ and $k_1 = 2$. Thus, $k_2 = 1$ or 2. Note that $|A_2| > 3$, then $k_2 = 2$ is impossible. As $k_2 = 1$, we have $|A_2| = 6$ and $r_1 = 0, r_2 = 2$. Thus, $A \cong N_6$ or N_7. □

An *imprimitive block* of G is a proper nonempty subset A of $V(G)$ such that for any automorphism $\phi \in Aut(G)$, either $\phi(A) = A$ or $\phi(A) \cap A = \emptyset$. The following lemma is well-known (see [12]):

Lemma 2.5. *Let G be a graph with two orbits V_1 and V_2. Suppose that A is a λ'-atom of G. Then we have*

(i) If $A \subseteq V_1$ (or V_2), then V_1 (or V_2) is a disjoint union of distinct λ'-atoms;

(ii) If $A_i = A \cap V_i \neq \emptyset$ for $i = 1, 2$, then $V(G)$ is a disjoint union of distinct λ'-atoms.

Proof. Clearly, A is an imprimitive block of G by Lemma 2.1.

(i) If $A \subseteq V_1$ (or V_2), then V_1 (or V_2) is a disjoint union of distinct λ'-atoms by Theorem 2.1.

(ii) Assume $A_1 = A \cap V_1$ and $A_2 = A \cap V_2$. By Lemma 2.1, A_1 and A_2 are imprimitive blocks of $G[V_1]$ and $G[V_2]$, respectively. By the property of imprimitive blocks, we have that $V(G)$ is a disjoint union of distinct λ'-atoms. □

From the above lemma, we have the following theorem:

Theorem 2.1. *Let G be a 3-regular connected graph with two orbits. We have:*

(1) if $\lambda'(G) = 2$, then $G \cong G_n(N_2)$, $G_n(N_3)$, $G_n(N_4)$ or $G_n(N_5)$;

(2) if $\lambda'(G) = 3$, then the 3-regular graph G is constructed based on some disjoint copies of N_i, i.e. $V(G)$ is the disjoint union of the vertex sets of these copies, $i = 1, 6, 7$;

(3) otherwise G is λ'-optimal.

To prove another main result, a known result is needed:

Lemma 2.6 ([7]). *If G is a k-regular graph with girth g, then*

$$|V(G)| \geq n(k,g) = \begin{cases} 1 + k + k(k-1) + \cdots + k(k-1)^{\frac{g-3}{2}} & \text{if } g \text{ is odd,} \\ 2(1 + k - 1 + \cdots + (k-1)^{\frac{g}{2}-1}) & \text{if } g \text{ is even.} \end{cases}$$

Clearly, if the degree of every vertex of graph G is at least k, and the girth of G is g, then $|V(G)| \geq n(k,g)$. If G is a k-regular connected graph and $|V(G)| < n(k, g_0)$, then $g(G) < g_0$.

Lemma 2.7. *Let G be a k-regular connected graph with two orbits V_1, V_2, $k \geq 3$, girth $g(G) \geq 6$ and $\lambda'(G) < 2k-2$. If $A = A_1 \cup A_2$ is a λ'-atom of G, $A_i \subset V_i, i = 1, 2$, then $|A_i| \geq 2$.*

Proof. Without loss of generality, we assume that $|A_1| \leq |A_2|$.

Case 1. $|A_1| = 0$.

If $|A_1| = 0$, then $k_2 = 0$, $k - r_2 \geq 1$ and $|A_2| \geq 3$. Thus,

$$\lambda'(G) = \omega(G) = |A_2|(k - k_2 - r_2) = |A_2|(k - r_2) < 2k - 2.$$

Thus, $|A_2|(k - r_2) < 2(k - r_2) + 2(r_2 - 1)$. Note that $|A_2| \geq 3$ and $|A_2| \geq r_2 + 1$, then we have $k - r_2 < \frac{2(r_2-1)}{|A_2|-2} \leq 2$. Hence $k - r_2 = 1$, that is, A_2 is $(k-1)$-regular. By Lemma 2.5, $|A_2| \geq n(k-1, 6) \geq 2(1 + (k-2) + (k-2)^2) > 2k-2$, contradicting the fact that $|A_2| \leq \lambda' < 2k-2$.

Case 2. $|A_1| = 1$.

If $|A_1| = 1$, then $k_2 = 1, r_1 = 0$ and $k_1 = |A_2| \geq 2$. Note that $g(G) \geq 6$, then $r_2 = 0$ (otherwise, $G[A]$ must contain a 3-cycle).

$$\lambda' = |\omega(A)| = |A_1|(k - k_1) + |A_2|(k - k_2) \geq k - k_1 + 2(k-1).$$

It can be seen that $\lambda' \geq 2(k-1)$, a contradiction. Therefore, we have $|A_i| \geq 2$. ☐

Now, we give our main result.

Theorem 2.2. *Let G be a k-regular connected graph with two orbits, $k \geq 3$ and girth $g(G) \geq 6$. Then G is λ'-optimal.*

Proof. We first note by Theorem 2.1 that the girth of the the non-λ'-optimal 3-regular graphs are at most 5, then we may assume $k \geq 4$ in the following.

By contradiction, suppose $\lambda'(G) < 2k-2$. Let $A = A_1 \cup A_2$ be a λ'-atom of G.

$$\lambda' = |\omega(A)| = |A_1|(k - k_1 - r_1)| + |A_2|(k - k_2 - r_2) < 2k - 2.$$

By Lemma 2.6, we know that $|A_i| \geq 2$. Note that $Aut(G[A])$ acts transitively on A_i, the number of neighbors in $V \backslash A$ of each vertex in A_i is constant $k - k_i - r_i$. Thus, if one of the vertices in A_i is adjacent to vertices in $V \backslash A$, then every vertex in A_i is adjacent to vertices in $V \backslash A$. We consider two cases:

Case 1. Each vertex in A_1 and A_2 is adjacent to some vertices of $V \backslash A$.

In this case, $|A_1| + |A_2| \leq \lambda'(G)$.

Subcase 1.1. If $|A_1|+|A_2| = \lambda'(G) < 2k-2$, then $k-k_1-r_1 = k-k_2-r_2 = 1$, that is $k_1+r_1 = k_2+r_2 = k-1$. By Lemma 2.5, $|V(G[A])| \geq n(k-1, 6) = 2(1 + (k-2) + (k-2)^2) \geq 2k > 2k-2$, a contradiction.

Subcase 1.2. $|A_1| + |A_2| < \lambda'(G) < 2k - 2$.

Claim 1. $k - k_i - r_i \geq 3$, for $i = 1, 2$.

Note that $|A_1| + |A_2| < \lambda'(G) < 2k - 2$, then at least one of $k - k_1 - r_1$ and $k - k_2 - r_2$ more than 2. Without loss of generality we assume that $k - k_1 - r_1 \leq k - k_2 - r_2$.

If $k - k_1 - r_1 = 1$ and $k - k_2 - r_2 \geq 2$. Let $v \in A_1$ and $\{v_1, \cdots, v_{r_1}\} \subset N_{G[A_1]}(v)$. Note that $g(G) \geq 6$ and then the neighbors of the vertices of $N_{G[A_1]}(v) \cup \{v\}$ are distinct, thus $|A_2| \geq (1+r_1)k_1$. We have $\lambda' = |\omega(A)| = |A_1|(k-k_1-r_1)+|A_2|(k-k_2-r_2) \geq 1+r_1+2|A_2| \geq 1+r_1+2(1+r_1)k_1 = 1 + r_1 + 2r_1k_1 + 2k_1 \geq 1 + r_1 + 2(k_1 + r_1) \geq 1 + 2(k-1)$, a contradiction.

If $k - k_1 - r_1 = 2$ and $k - k_2 - r_2 \geq 2$, then $\lambda'(G) \geq 2|A_1| + 2|A_2| \geq 2[1 + r_1 + k_1(1 + r_1)] \geq 2 + 2(k-2) = 2k-2$, a contradiction. Therefore $k - k_i - r_i \geq 3$, for $i = 1, 2$.

Now, we assume that $2 \leq |A_1| \leq |A_2|$. Since $\lambda' = |A_1|(k - k_1 - r_1) + |A_2|(k - k_2 - r_2) \geq 3(|A_1| + |A_2|)$, we have $|A_1| + |A_2| \leq \frac{\lambda'}{3} \leq \frac{2k-3}{3}$. By $k_i + r_i \leq |A_1| + |A_2| - 1 \leq \frac{2k-6}{3}$, we have $k - k_i - r_i \geq \frac{k+6}{3}$. Thus,

$$\lambda' = |\omega(A)| = |A_1|(k - k_1 - r_1) + |A_2|(k - k_2 - r_2) \geq \frac{k+6}{3}(|A_1| + |A_2|).$$

Since $\lambda' < 2k-2$, we have $|A_1|+|A_2| < 6$.

Note that $g(G) \geq 6$, then if $|A_1| = 2, |A_2| = 2$, then $A \cong P_4$ and we have $\lambda' = |\omega(A)| = 2(k-2)+2(k-1) > 2k-2$, a contradiction. Similarly, the case $|A_1| = 2, |A_2| = 3$ is not difficult to obtain a contradiction.

The argument of the case $|A_1| > |A_2|$ is similar to that above mentioned.

Case 2. One of A_1 and A_2 has neighbors in $V \backslash A$.

Without loss of generality, assume that the vertices in A_1 has neighbors in $V \backslash A$, then $\lambda' = |A_1|(k-k_1-r_1)$ and $k_2+r_2 = k$.

Subcase 2.1. If $|A_1| = \lambda' < 2k-2$, then $k-k_1-r_1 = 1$ and thus $k_1+r_1 = k-1$.

Subcase 2.1.1. If $k_2 = 1$ and $r_1 \neq 0$, then A_2 is $(k-1)$-regular. By Lemma 2.6, we have $|A_2| \geq n(k-1,6) = 2(k^2-3k+3) = 2k^2-6k+6$. On the other hand, we have $k_1|A_1| = k_2|A_2|$ and $k_1 = k-1-r_1 \leq k-2$, then $|A_2| = \frac{k_1}{k_2}|A_1| = k_1|A_1| \leq (2k-3)(k-2) = 2k^2-7k+6$, a contradiction.

Subcase 2.1.2. If $k_2 = 1$ and $r_1 = 0$, then $r_2 = k-1$ and $k_1 = k-1$. By Lemma 2.6, we have $|A_2| \geq n(k-1,6) = 2k^2-6k+6$. Since $k_1|A_1| = k_2|A_2|$, $|A_2| = (k-1)|A_1|, |A_1| = \frac{1}{k-1}|A_2| \geq \frac{2k^2-6k+6}{k-1} = 2k-4+\frac{2}{k-1}$. On the other hand, $|A_1| \leq 2k-3$ by the assumption. Note that $\frac{2}{k-1} \leq 1$ for $k \geq 3$, then $|A_1| = 2k-3, |A_2| = (2k-3)(k-1)$. It is easy to see that $G[A]$ is the union of $2k-3$ $K_{1,k-1}$'s. Pick an edge (x,y) of $G[A_2]$ and assume that $N_{G[A_2]}(x) = \{y, x_1, \cdots, x_{k-2}\}$ and $N_{G[A_2]}(y) = \{x, y_1, \cdots, y_{k-2}\}$. Note that $g(A) \geq 6$, it can be seen that any two vertices of $N_{G[A_2]}(x) \cup N_{G[A_2]}(y)$ are in two distinct stars. But there only has $2k-3$ stars, thus, $2k-3 \geq 2k-2$, a contradiction.

Subcase 2.1.3. Assume $k_2 = 2$. Note that the degree of vertices of A_1 in $G[A]$ is $k-1$, and vertex of A_2 in $G[A]$ is k. By Lemma 2.5, we have $|V(G[A])| \geq n(k-1,6) = 2k^2-6k+6$. On the other hand, $|A| = |A_1| + |A_2| = \frac{k_1+k_2}{k_2}|A_1| \leq \frac{2+k-1}{2}(2k-3) = k^2 - \frac{1}{2}k - \frac{3}{2} < 2k^2-6k+6 \leq |A|$ for $k \geq 4$, a contradiction.

Subcase 2.1.4. Assume $k_2 \geq 3$. Note that the degree of vertices of A_1 in $G[A]$ is $k-1$, and vertex of A_2 in $G[A]$ is k. By Lemma 2.6, we have $|V(G[A])| \geq n(k-1,6) = 2k^2-6k+6$. On the other hand, since $3 \leq k_2 \leq k-1, k_1 = k-1-r_1 \leq k-1$, we have $|A| = V(G[A]) = |A_1|+|A_2| = \frac{k_1+k2}{k_2}|A_1| \leq \frac{(2k-2)(2k-3)}{3} = \frac{4k^2-10k+6}{3} < 2k^2-6k+6 \leq |A|$, a contradiction.

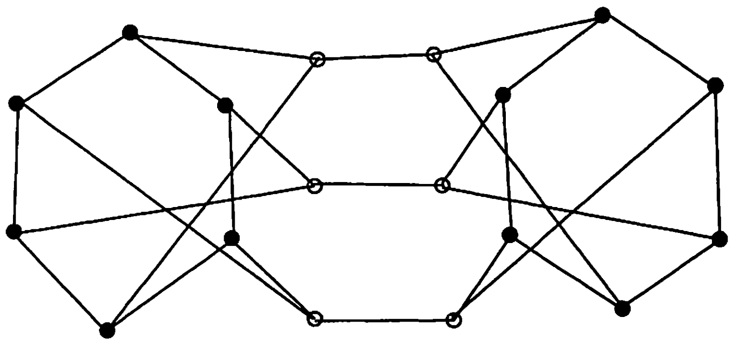

Figure.3

Subcase 2.2. If $|A_1| < \lambda'$, then $k - k_1 - r_1 \geq 2$, $|A_1| \leq \frac{\lambda'}{2} < k - 1$ and $k_1 \leq k - 2$.

Claim 2. $k_2 \geq 3$.

If $k_2 = 1$, then $r_2 = k-1$ and A_2 is $(k-1)$-regular and $|A_2| \geq 2k^2-6k+6$. On the other hand, $|A_2| = k_1|A_1| < (k-2)(k-1) = k^2 - 3k + 2 < 2k^2 - 6k + 6 \leq |A_2|$, a contradiction.

If $k_2 = 2$, then $r_2 = k-2$. We have $|A_2| \geq n(k-2,6) = 2(k^2-5k+7) = 2k^2 - 10k + 14$ by Lemma 2.5. On the other hand, $|A_2| = \frac{k_1}{k_2}|A_1| < \frac{k-2}{2}(k-1) < 2k^2 - 10k + 14 \leq |A_2|$, a contradiction. Therefore $k_2 \geq 3$.

Claim 3. $k - k_1 - r_1 \geq 3$.

If $k-k_1-r_1 = 2$, $k_2+r_2 = k$, then $|A| \geq n(k-2,6) = 2k^2-10k+14$. On the other hand, $|A| = |A_1|+|A_2| = \frac{k_1+k_2}{k_2}|A_1| < \frac{k-2+k-1}{3}(k-1) = \frac{2k^2-5k+3}{3} < 2k^2 - 10k + 14 \leq |A|$, a contradiction. Therefore, $k - k_1 - r_1 \geq 3$.

Since $|A_1|(k - k_1 - r_1) < \lambda' \leq 2k - 3$, we have $3 \leq k_2 \leq |A_1| \leq \frac{2k-4}{3}$ and $r_2 = k - k_2 \geq k - \frac{2k-4}{3} = \frac{k+4}{3}$. By Lemma 2.6, $|A_2| \geq n(\frac{k+4}{3}, 6) = 2(1 + \frac{k+1}{3} + \frac{(k-1)^2}{9}) = \frac{2k^2+10k+26}{9}$ and $k_1 \leq k - r_1 - 3 \leq k - 3$. On the other hand, $|A_2| = \frac{k_1}{k_2}|A_1| \leq \frac{k-3}{3}\frac{2k-4}{3} = \frac{2k^2-10k+12}{9} < \frac{2k^2+10k+26}{9} \leq |A_2|$, a contradiction.

We complete the proof. □

Remark 2.1. *Note that the graph of Figure.3 is a 3-regular double orbits connected graph with grith $g(G) = 5$. By Theorem 2.1, the graph is non-λ'-optimal. So the girth condition of Theorem 2.2 is sharp.*

A k-regular graph G is called *super-λ* if $\lambda(G) = k$ and every minimum edge-cut of G isolates one vertex. Note that $2k-2 > k$ for $k \geq 3$, then we have:

Corollary 2.1. *A k-regular connected double-orbit graph with girth $k \geq 3, g \geq 6$ is super-λ.*

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