THE INTEGER SEQUENCE $B = B_n(P,Q)$ WITH PARAMETERS P AND Q

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ABSTRACT. In this work, we first prove that every prime number $p \equiv 1 \pmod 4$ can be written of the form $P^2 - 4Q$ with two positive integers P and Q, and then we define the sequence $B_n(P,Q)$ to be $B_0 = 2, B_1 = P$ and $B_n = PB_{n-1} - QB_{n-2}$ for $n \geq 2$ and derive some algebraic identities on it. Also we formulate the limit of cross-ratio for four consecutive numbers B_n, B_{n+1}, B_{n+2} and B_{n+3}.

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1. Preliminaries

Fibonacci, Lucas, Pell and the other special numbers and their generalizations arise in the examination of various areas of science and art. In fact, these numbers are special case of a sequence which is defined as a linear combination as follows:

$$a_{n+k} = c_1 a_{n+k-1} + c_2 a_{n+k-2} + \cdots + c_k a_n, \tag{1.1}$$

where $c_1, c_2, \cdots, c_k$ are real constants.

Fibonacci numbers (sequence A000045 in OEIS) form a sequence defined by $F_0 = 0, F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$. Lucas numbers (sequence A 000032 in OEIS) form a sequence defined by $L_0 = 2, L_1 = 1$ and $L_n = L_{n-1} + L_{n-2}$ for $n \geq 2$. The characteristic equation of them is $x^2 - x - 1 = 0$ and hence the roots of it are

$$\alpha_1 = \frac{1+\sqrt{5}}{2} \text{ and } \beta_1 = \frac{1-\sqrt{5}}{2}. \tag{1.2}$$

So their Binet's formulas are hence

$$F_n = \frac{\alpha_1^n - \beta_1^n}{\alpha_1 - \beta_1} \text{ and } L_n = \alpha_1^n + \beta_1^n$$

for $n \geq 0$.

There are a lot of algebraic relations between Fibonacci and Lucas numbers. For instance, $L_n = F_{n-1} + F_{n+1}, F_{2n} = F_nL_n$, $F_{m+n} = \frac{F_mL_n+L_mF_n}{2}$, $F_{m-n} = \frac{(-1)^n(F_mL_n-L_mF_n)}{2}$ and $L_n^2 - 5F_n^2 = 4(-1)^n$ (see [2, 4, 5, 6, 11, 16, 17]).

Recall that the golden ratio [7] is defined as the ratio that results when a line is divided so that the whole line has the same ratio to the larger segment as the larger segment has to the smaller segment. Expressed algebraically, normalizing the larger part to unit length, it is the positive solution of the equation $\frac{x}{1} = \frac{1}{x-1} \Leftrightarrow x^2 - x - 1 = 0$ which is the characteristic equation of both Fibonacci and Lucas numbers. Johannes Kepler pointed out that the ratio of consecutive Fibonacci numbers converges to the golden ratio as the limit, that is, $\lim\limits_{n\to\infty} \frac{F_n}{F_{n-1}} = \alpha$

The Pell numbers (sequence A000129 in OEIS) form a sequence defined by $P_0 = 0, P_1 = 1$ and $P_n = 2P_{n-1} + P_{n-2}$ for $n \geq 2$. Some identities for Pell numbers can be found in [1, 10, 13]. Pell numbers P_n have a close connection to square triangular numbers, that is,

$$[(P_{k-1} + P_k)P_k]^2 = \frac{(P_{k-1} + P_k)^2[(P_{k-1} + P_k)^2 - (-1)^k]}{2}. \tag{1.3}$$

The left side of (1.3) describes a square number and the right side describes a triangular number (see [3, 9]), so it is a square triangular number (see [12]). Pell–Lucas numbers (sequence A002203 in OEIS) form a sequence defined by $Q_0 = Q_1 = 2$ and $Q_n = 2Q_{n-1} + Q_{n-2}$ for $n \geq 2$. In [10], Melham proved that $P_n^2 + P_{n-1}P_{n+1} = \frac{Q_n^2}{4}$ and $Q_n^2 + Q_{n-1}Q_{n+1} = 16P_n^2$. Martin [8] described that the Pell numbers can be used to form Pythagorean triples, that is, $(2P_nP_{n+1}, P_{n+1}^2 - P_n^2, P_{n+1}^2 + P_n^2)$ is a Pythagorean triple.

2. The Sequences $B = B_n(P, Q)$ with Parameters P and Q.

In this section, our first aim is to define a new integer sequence with two parameters and then we obtain some algebraic identities on it. Before consider our main problem, we first give the following result.

Theorem 2.1. *Every prime number $p \equiv 1 (\mathrm{mod}\ 4)$ can be written of the form $P^2 - 4Q$ for positive integers P and Q.*

Proof. Let p be a prime number such that $p \equiv 1 (\mathrm{mod}\ 4)$, say $p = 1 + 4k$ for an integer $k \geq 1$. Then the quadratic equation $p = P^2 - 4Q$ has a solution for $(P, Q) = (2k + 1, k^2)$. So p can be represented by $P^2 - 4Q$. □

Now let $P = 2k + 1$ and $Q = k^2$. We define the sequence $B = B_n(P, Q)$ as $B_0 = 2$, $B_1 = P$ and

$$B_n = PB_{n-1} - QB_{n-2} = (2k+1)B_{n-1} - k^2 B_{n-2} \tag{2.1}$$

for $n \geq 2$. The characteristic equation of (2.1) is $x^2 - Px + Q = 0$. So its roots are

$$\alpha = \frac{P + \sqrt{D}}{2} \text{ and } \beta = \frac{P - \sqrt{D}}{2},$$

where $D = p$. Hence Binet formula is $B_n = \alpha^n + \beta^n$ for $n \geq 0$.

Now we can give the following theorems.

Theorem 2.2. *Let B_n denote the n^{th} number. Then*

$$\sum_{i=0}^{n} B_i = \frac{B_{n+1} - k^2 B_n + 2k - 1}{2k - k^2}. \tag{2.2}$$

Proof. Note that $B_n = (2k+1)B_{n-1} - k^2 B_{n-2}$. So $B_{n+2} = (2k+1)B_{n+1} - k^2 B_n = 2kB_{n+1} + B_{n+1} - k^2 B_n$ and hence

$$B_{n+2} - B_{n+1} = 2kB_{n+1} - k^2 B_n. \tag{2.3}$$

Applying (2.3), we deduce that

$$\begin{aligned} B_2 - B_1 &= 2kB_1 - k^2 B_0 \\ B_3 - B_2 &= 2kB_2 - k^2 B_1 \\ &\cdots \\ B_{n+1} - B_n &= 2kB_n - k^2 B_{n-1} \\ B_{n+2} - B_{n+1} &= 2kB_{n+1} - k^2 B_n. \end{aligned} \tag{2.4}$$

If we sum both sides of (2.4), then we obtain

$$B_{n+2} - B_1 = (2k - k^2)(B_1 + B_2 + \cdots + B_n) + 2kB_{n+1} - k^2 B_0. \tag{2.5}$$

Since $B_0 = 2$ and $B_1 = 2k+1$, (2.5) becomes $B_{n+2} - (2k+1) = (2k - k^2)(B_1 + B_2 + \cdots + B_n) + 2kB_{n+1} - 2k^2$ and hence

$$B_1 + B_2 + \cdots + B_n = \frac{B_{n+2} - (2k+1) - 2kB_{n+1} + 2k^2}{2k - k^2}. \tag{2.6}$$

Taking $B_{n+2} \to (2k+1)B_{n+1} - k^2 B_n$ and $B_0 = 2$ in (2.6), we conclude that $B_0 + B_1 + B_2 + \cdots + B_n = \frac{B_{n+1} - k^2 B_n + 2k - 1}{2k - k^2}$ as we wanted (Here we note that $k \neq 2$, for $k = 2$, we have $p = 1 + 4 \cdot 2 = 9$ it not a prime). □

Now we want to derive a recurrence relations on B_n numbers. To get this we can give the following theorem.

Theorem 2.3. *Let B_n denote the n^{th} number. Then*

$$B_{2n} = (P^2 - 2Q)B_{2n-2} - Q^2 B_{2n-4}$$
$$B_{2n+1} = (P^2 - 2Q)B_{2n-1} - Q^2 B_{2n-3}$$

for $n \geq 2$.

Proof. Since $B_{2n} = (2k+1)B_{2n-1} - k^2 B_{2n-2}$, we easily get

$$\begin{aligned}
B_{2n} &= (2k+1)B_{2n-1} - k^2 B_{2n-2} \\
&= (2k+1)\left[(2k+1)B_{2n-2} - k^2 B_{2n-3}\right] - k^2 B_{2n-2} \\
&= B_{2n-2}\left[(2k+1)^2 - k^2\right] - k^2(2k+1)\left[(2k+1)B_{2n-4} - k^2 B_{2n-5}\right] \\
&= B_{2n-2}\left[(2k+1)^2 - k^2\right] - k^2(2k+1)^2 B_{2n-4} + k^4(2k+1)B_{2n-5} \\
&= B_{2n-2}\left[(2k+1)^2 - 2k^2\right] + k^2\left[(2k+1)B_{2n-3} - k^2 B_{2n-4}\right] \\
&\quad - k^2(2k+1)^2 B_{2n-4} + k^4(2k+1)B_{2n-5} \\
&= \left[(2k+1)^2 - 2k^2\right] B_{2n-2} - k^4 B_{2n-4} \\
&= (P^2 - 2Q)B_{2n-2} - Q^2 B_{2n-4}.
\end{aligned}$$

The other case is similar. □

Theorem 2.4. *The n^{th} term of B_n is*

$$B_n = \frac{1}{2^{n-1}} \begin{cases} \sum_{i=0}^{\frac{n}{2}} \binom{n}{2i} P^{n-2i} p^i & \text{if } n \text{ is even} \\ \sum_{i=0}^{\frac{n-1}{2}} \binom{n}{2i} P^{n-2i} p^i & \text{if } n \text{ is odd} \end{cases}$$

for $n \geq 1$.

Proof. Let n be even. Then applying Binet's formula, we easily get

$$\begin{aligned}
B_n &= \left(\frac{P+\sqrt{p}}{2}\right)^n + \left(\frac{P-\sqrt{p}}{2}\right)^n \\
&= \frac{1}{2^n}\left[\sum_{i=0}^{n}\binom{n}{i}P^{n-i}(\sqrt{p})^i + \sum_{i=0}^{n}\binom{n}{i}P^{n-i}(-\sqrt{p})^i\right] \\
&= \frac{1}{2^{n-1}}\left[\binom{n}{0}P^n + \binom{n}{2}P^2 p + \cdots + \binom{n}{n}p^{\frac{n}{2}}\right] \\
&= \frac{1}{2^{n-1}}\sum_{i=0}^{\frac{n}{2}}\binom{n}{2i}P^{n-2i}p^i
\end{aligned}$$

as we wanted. □

Example 2.1. *Let $p = 17$. Then $B_n = 9B_{n-1} - 16B_{n-2}$. In this case the first few terms of B_n are*

$$2, 9, 49, 297, 1889, 12249, 80017, 524169, \mathbf{3437249}, \mathbf{22548537}, \cdots.$$

Let $n = 8$. Then

$$B_8 = \frac{1}{2^7}\sum_{i=0}^{4}\binom{8}{2i}9^{8-2i}17^i = 3437249$$

and let $n = 9$, then

$$B_9 = \frac{1}{2^8}\sum_{i=0}^{4}\binom{9}{2i}9^{9-2i}17^i = 22548537.$$

Now we can give the following theorem related to powers of α and β.

Theorem 2.5. *Let B_n denote the n^{th} number. Then*

$$\alpha^n - \beta^n = \frac{1}{\sqrt{p}}\begin{cases} B_{n+1} - QB_{n-1} \\ 2B_{n+1} - PB_n \\ PB_n - 2QB_{n-1} \end{cases}$$

for $n \geq 1$.

Proof. Recall that $B_{n+1} = PB_n - QB_{n-1}$. Hence

$$\begin{aligned} B_{n+1} - QB_{n-1} &= P(\alpha^n + \beta^n) - 2Q(\alpha^{n-1} - \beta^{n-1}) \\ &= P(\alpha^n + \beta^n) - 2(\beta\alpha^n + \alpha\beta^n) \\ &= \alpha^n(P - 2\beta) + \beta^n(P - 2\alpha) \\ &= \sqrt{p}(\alpha^n - \beta^n). \end{aligned}$$

So $\frac{B_{n+1}-QB_{n-1}}{\sqrt{p}} = \alpha^n - \beta^n$. The other cases can be proved similarly. □

From above theorem we can give the following result.

Corollary 2.6. *Let B_n denote the n^{th} number. Then*

$$B_{n+1} - QB_{n-1} = \frac{p}{2^{n-1}}\begin{cases} \sum_{i=0}^{\frac{n-2}{2}} \binom{n}{2i+1} P^{n-2i-1}p^i & \text{if } n \text{ is even} \\ \sum_{i=0}^{\frac{n-1}{2}} \binom{n}{2i+1} P^{n-2i-1}p^i & \text{if } n \text{ is odd.} \end{cases}$$

Now we set the following identities

$$M = \frac{P-2Q+\sqrt{p}}{2},\ N = P-Q-1,\ H = \frac{P+2+\sqrt{p}}{2},$$
$$L = \frac{P-2+\sqrt{p}}{2},\ K = \frac{8Q+\ PQ+4P-2+(3Q+2P)\sqrt{p}}{2}.$$

Then we can give the following theorem.

Theorem 2.7. *Let B_n denote the n^{th} number. Then*

(1)
$$\sum_{i=0}^{n} B_i = \frac{1}{N}[M\alpha^n - \overline{M}\beta^n + P - 2].$$

(2) $B_n + B_{n+1} = H\alpha^n + \overline{H}\beta^n$ *for* $n \geq 0$.

(3) $B_{n+1} + B_{n-1} = K\alpha^{n-2} + \overline{K}\beta^{n-2}$ *for* $n \geq 2$.

(4) $B_n - B_{n-1} = L\alpha^{n-1} + \overline{L}\beta^{n-1}$ *for* $n \geq 1$.

Proof. (1) We proved in Theorem 2.5 that $\frac{B_{n+1}-QB_{n-1}}{\sqrt{p}} = \alpha^n - \beta^n$. So $\alpha^{n+1} - \beta^{n+1} = \frac{B_{n+1}-k^2B_n+2kB_{n+1}-k^2B_n}{\sqrt{p}}$ and hence

$$\begin{aligned}
B_{n+1} - k^2B_n &= \sqrt{p}(\alpha^{n+1} - \beta^{n+1}) - 2kB_{n+1} + k^2B_n \\
&= \alpha^n\left(\alpha\sqrt{p} - 2k\alpha + k^2\right) + \beta^n\left(-\beta\sqrt{p} - 2k\beta + k^2\right) \\
&= \alpha^n\left(\frac{2k+1-2k^2+\sqrt{p}}{2}\right) + \beta^n\left(\frac{2k+1-2k^2-\sqrt{p}}{2}\right) \\
&= \alpha^n\left(\frac{P-2Q+\sqrt{p}}{2}\right) + \beta^n\left(\frac{P-2Q-\sqrt{p}}{2}\right) \\
&= M\alpha^n + \overline{M}\beta^n.
\end{aligned}$$

Applying Theorem 2.2, the result is clear.

(2) Recall that $\alpha^n - \beta^n = \frac{2B_{n+1}-PB_n}{\sqrt{p}}$. So $2(B_n + B_{n+1}) - (2k+3)B_n = (\alpha^n - \beta^n)\sqrt{p}$ and hence

$$\begin{aligned}
B_n + B_{n+1} &= \frac{(2k+3)B_n + (\alpha^n - \beta^n)\sqrt{p}}{2} \\
&= \frac{(2k+3)(\alpha^n + \beta^n) + (\alpha^n - \beta^n)\sqrt{p}}{2} \\
&= \alpha^n\left(\frac{2k+3+\sqrt{p}}{2}\right) + \beta^n\left(\frac{2k+3-\sqrt{p}}{2}\right) \\
&= \alpha^n\left(\frac{P+2+\sqrt{p}}{2}\right) + \beta^n\left(\frac{P+2-\sqrt{p}}{2}\right) \\
&= H\alpha^n + \overline{H}\beta^n.
\end{aligned}$$

(3) Note that $\frac{2B_{n+1}-PB_n}{\sqrt{p}} = \alpha^n - \beta^n$. So we get $(\alpha^n - \beta^n)\sqrt{p} = 2(B_{n+1} + B_{n-1}) - (4k^2+4k+3)B_{n-1} + k^2(2k+1)B_{n-2}$ and hence

$$\begin{aligned}
&B_{n+1} + B_{n-1} \\
&= \frac{(\alpha^n - \beta^n)\sqrt{p} + (4k^2+4k+3)B_{n-1} - k^2(2k+1)B_{n-2}}{2} \\
&= \frac{\alpha^n}{2}\left(\sqrt{p} + \frac{4k^2+4k+3}{\alpha} - \frac{k^2(2k+1)}{\alpha^2}\right) \\
&\quad + \frac{\beta^n}{2}\left(-\sqrt{p} + \frac{4k^2+4k+3}{\beta} - \frac{k^2(2k+1)}{\beta^2}\right) \\
&= \alpha^{n-2}\left(\frac{9k^2+2k^3+8k+2+(3k^2+4k+2)\sqrt{p}}{2}\right) \\
&\quad + \beta^{n-2}\left(\frac{9k^2+2k^3+8k+2-(3k^2+4k+2)\sqrt{p}}{2}\right) \\
&= K\alpha^{n-2} + \overline{K}\beta^{n-2}.
\end{aligned}$$

(4) Since $\frac{2B_{n+1}-PB_n}{\sqrt{p}} = \alpha^n - \beta^n$, we get $2B_n - PB_{n-1} = \sqrt{p}(\alpha^{n-1} - \beta^{n-1})$ and hence $2(B_n - B_{n-1}) + (1-2k)B_{n-1} = \sqrt{p}(\alpha^{n-1} - \beta^{n-1})$. So

$$\begin{aligned}
B_n - B_{n-1} &= \frac{(2k-1)B_{n-1} + \sqrt{p}(\alpha^{n-1} - \beta^{n-1})}{2} \\
&= \frac{(2k-1)(\alpha^{n-1} + \beta^{n-1}) + \sqrt{p}(\alpha^{n-1} - \beta^{n-1})}{2} \\
&= \alpha^{n-1}\left(\frac{2k-1+\sqrt{p}}{2}\right) + \beta^{n-1}\left(\frac{2k-1-\sqrt{p}}{2}\right) \\
&= \alpha^{n-1}\left(\frac{P-2+\sqrt{p}}{2}\right) + \beta^{n-1}\left(\frac{P-2-\sqrt{p}}{2}\right) \\
&= L\alpha^{n-1} + \overline{L}\beta^{n-1}.
\end{aligned}$$

This completes the proof. □

Theorem 2.8. *Let B_n denote the n^{th} number. If $n \geq 2$ is even, then*

$$B_{n+1} - B_n = \frac{1}{2^n}\left[p\sum_{i=0}^{\frac{n-2}{2}}\binom{n}{2i+1}P^{n-2i-1}p^i + (P-2)\sum_{i=0}^{\frac{n}{2}}\binom{n}{2i}P^{n-2i}p^i\right]$$

and if $n \geq 1$ is odd, then

$$B_{n+1} - B_n = \frac{1}{2^n}\left[p\sum_{i=0}^{\frac{n-1}{2}}\binom{n}{2i+1}P^{n-2i-1}p^i + (P-2)\sum_{i=0}^{\frac{n-1}{2}}\binom{n}{2i}P^{n-2i}p^i\right].$$

Proof. We proved in above theorem that $B_n - B_{n-1} = L\alpha^{n-1} + \overline{L}\beta^{n-1}$. So $B_{n+1} - B_n = L\alpha^n + \overline{L}\beta^n$. Note that $L + \overline{L} = P - 2$ and $L - \overline{L} = \sqrt{p}$. Let n be even. Then we deduce that

$$\begin{aligned}
&B_{n+1} - B_n = L\alpha^n + \overline{L}\beta^n \\
&= L\left(\frac{P+\sqrt{p}}{2}\right)^n + \overline{L}\left(\frac{P-\sqrt{p}}{2}\right)^n \\
&= \frac{L}{2^n}\sum_{i=0}^{n}\binom{n}{i}P^{n-i}(\sqrt{p})^i + \frac{\overline{L}}{2^n}\sum_{i=0}^{n}\binom{n}{i}P^{n-i}(-\sqrt{p})^i \\
&= \left(\frac{L+\overline{L}}{2^n}\right)\left[P^n + \binom{n}{2}P^{n-2}(\sqrt{p})^2 + \cdots + (\sqrt{p})^n\right] \\
&\quad + \left(\frac{L-\overline{L}}{2^n}\right)\left[\binom{n}{1}P^{n-1}\sqrt{p} + \cdots + \binom{n}{n-1}P(\sqrt{p})^{n-1}\right] \\
&= \left(\frac{P-2}{2^n}\right)\left[P^n + \binom{n}{2}P^{n-2}(\sqrt{p})^2 + \cdots + (\sqrt{p})^n\right] \\
&\quad + \frac{\sqrt{p}}{2^n}\left[\binom{n}{1}P^{n-1}\sqrt{p} + \cdots + \binom{n}{n-1}P(\sqrt{p})^{n-1}\right] \\
&= \frac{1}{2^n}\left[p\sum_{i=0}^{\frac{n-2}{2}}\binom{n}{2i+1}P^{n-2i-1}p^i + (P-2)\sum_{i=0}^{\frac{n}{2}}\binom{n}{2i}P^{n-2i}p^i\right].
\end{aligned}$$

The second assertion can be proved similarly. □

Now we can also formulate the sum of even and odd B_n numbers by using the powers of α and β as follows.

Theorem 2.9. *Let B_n denote the n^{th} number. Then*

$$\sum_{i=1}^{n} B_{2i} = \begin{cases} K\sum_{i=1}^{\frac{n}{2}}\alpha^{4i-3} + \overline{K}\sum_{i=1}^{\frac{n}{2}}\beta^{4i-3} & \text{if } n \text{ is even} \\ \alpha^{2n} + \beta^{2n} + K\sum_{i=1}^{\frac{n-1}{2}}\alpha^{4i-3} + \overline{K}\sum_{i=1}^{\frac{n-1}{2}}\beta^{4i-3} & \text{if } n \text{ is odd} \end{cases}$$

and

$$\sum_{i=1}^{n} B_{2i-1} = \begin{cases} K\sum_{i=1}^{\frac{n}{2}}\alpha^{4i-4} + \overline{K}\sum_{i=1}^{\frac{n}{2}}\beta^{4i-4} & \text{if } n \text{ is even} \\ \alpha^{2n-1} + \beta^{2n-1} + K\sum_{i=1}^{\frac{n-1}{2}}\alpha^{4i-4} + \overline{K}\sum_{i=1}^{\frac{n-1}{2}}\beta^{4i-4} & \text{if } n \text{ is odd.} \end{cases}$$

Proof. From (3) of Theorem 2.7, we get

$$\begin{aligned}\sum_{i=1}^{n} B_{2i} &= (B_2+B_4)+(B_6+B_8)+\cdots+(B_{2n-2}+B_{2n})\\ &= (K\alpha+\overline{K}\beta)+(K\alpha^5+\overline{K}\beta^5)+\cdots+(K\alpha^{2n-3}+\overline{K}\beta^{2n-3})\\ &= K(\alpha+\alpha^5+\cdots+\alpha^{2n-3})+\overline{K}(\beta+\beta^5+\cdots+\beta^{2n-3})\\ &= K\sum_{i=1}^{\frac{n}{2}}\alpha^{4i-3}+\overline{K}\sum_{i=1}^{\frac{n}{2}}\beta^{4i-3}\end{aligned}$$

and let n be odd, then

$$\begin{aligned}\sum_{i=1}^{n} B_{2i} &= (B_2+B_4)+\cdots+(B_{2n-4}+B_{2n-2})+B_{2n}\\ &= (K\alpha+\overline{K}\beta)+\cdots+(K\alpha^{2n-5}+\overline{K}\beta^{2n-5})+\alpha^{2n}+\beta^{2n}\\ &= K(\alpha+\alpha^5+\cdots+\alpha^{2n-5})+\overline{K}(\beta+\beta^5+\cdots+\beta^{2n-5})+\alpha^{2n}+\beta^{2n}\\ &= \alpha^{2n}+\beta^{2n}+K\sum_{i=1}^{\frac{n-1}{2}}\alpha^{4i-3}+\overline{K}\sum_{i=1}^{\frac{n-1}{2}}\beta^{4i-3}.\end{aligned}$$

The other assertion can be proved similarly. □

Theorem 2.10. *Let B_n denote the n^{th} number. Then*

$$\sum_{n=0}^{\infty} B_n z^n = \frac{2-(2k+1)z}{1-(2k+1)z+k^2z^2}.$$

Proof. Since $z^2 - Pz + Q = 0$, we get

$$\begin{aligned}(1-Pz+Qz^2)B(z) &= (1-Pz+Qz^2)(B_0+B_1z+\cdots+B_nz^n+\cdots)\\ &= B_0+(B_1-PB_0)z+\cdots\\ &\quad+(B_n-PB_{n-1}+QB_{n-2})z^n+\cdots\\ &= 2-Pz.\end{aligned}$$

So we get the desired result since $B_0 = 2, B_1 = P, P = 2k+1, Q = k^2$ and $B_n = PB_{n-1} - QB_{n-2}$. □

For B_n numbers, we set the matrices $M(B_n)$ and $W(B_n)$ to be

$$M(B_n) = \begin{bmatrix} 2k+1 & -k^2 \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad W(B_n) = \begin{bmatrix} B_2 & B_1 \\ B_1 & B_0 \end{bmatrix}.$$

Then we have the following theorem.

Theorem 2.11. *Let B_n denote the n^{th} number. Then*

$$\begin{bmatrix} B_n \\ B_{n-1} \end{bmatrix} = M(B_n)^{n-1} \begin{bmatrix} 2k+1 \\ 2 \end{bmatrix}$$

and

$$\begin{bmatrix} B_{n+1} & B_n \\ B_n & B_{n-1} \end{bmatrix} = M(B_n)^{n-1} W(B_n) \tag{2.7}$$

for $n \geq 1$.

Proof. Note that this relation is true for $n = 1$ since $B_0 = 2, B_1 = 2k+1$. Let us assume that this relation is satisfied for $n-1$, that is,

$$\begin{bmatrix} B_{n-1} \\ B_{n-2} \end{bmatrix} = M(B_n)^{n-2} \begin{bmatrix} 2k+1 \\ 2 \end{bmatrix}.$$

Then we deduce that

$$\begin{bmatrix} B_n \\ B_{n-1} \end{bmatrix} = M(B_n) M(B_n)^{n-2} \begin{bmatrix} 2k+1 \\ 2 \end{bmatrix} = \begin{bmatrix} (2k+1)B_{n-1} - k^2 B_{n-2} \\ B_{n-1} \end{bmatrix}.$$

Hence it is true for every $n \geq 1$ since $B_n = (2k+1)B_{n-1} - k^2 B_{n-2}$. The second assertion can be proved similarly. □

Theorem 2.12. *Let B_n denote the n^{th} number. Then*

(1) $B_{n+1}B_{n-1} - B_n^2 = pQ^{n-1}$ *for $n \geq 1$.*
(2) $B_{n+1}^2 - PB_{n+1}B_n + QB_n^2 = -pQ^n$ *for $n \geq 0$.*

Proof. (1) Note that $\det(W(B_n)) = 4k+1 = p$ and $\det(M(B_n)) = k^2 = Q$. So taking the determinant of both sides of (2.7) yields that $B_{n+1}B_{n-1} - B_n^2 = pQ^{n-1}$.

(2) Recall that $B_{n+1} = (2k+1)B_n - k^2 B_{n-1}$. So

$$\begin{aligned}
& B_{n+1}^2 - PB_{n+1}B_n + QB_n^2 \\
&= [(2k+1)B_n - k^2 B_{n-1}]^2 - (2k+1)[(2k+1)B_n - k^2 B_{n-1}]B_n + k^2 B_n^2 \\
&= (2k+1)^2 B_n^2 - 2k^2(2k+1)B_n B_{n-1} + k^4 B_{n-1}^2 - (2k+1)^2 B_n^2 \\
&\quad + k^2(2k+1)B_{n-1}B_n + k^2 B_n^2 \\
&= B_n B_{n-1}[-k^2(2k+1)] + k^4 B_{n-1}^2 + k^2 B_n^2 \\
&= -k^2 B_{n-1}\left[(2k+1)B_n - k^2 B_{n-1}\right] + k^2 B_n^2 \\
&= -k^2(B_{n+1}B_{n-1} - B_n^2) \\
&= -pQ^n
\end{aligned}$$

as we claimed. □

From above theorem, we can give the following result.

Corollary 2.13. *Let B_n denote the n^{th} number. Then*

(1) $B_{n+1}B_{n-1} - B_n^2 = 4k^{2n-1} + k^{2n-2}$ *for* $n \geq 1$.
(2) $B_{n+1}^2 - PB_{n+1}B_n + QB_n^2 = -4k^{2n+1} - k^{2n}$ *for* ≥ 0.

We can also give the following theorem which can be proved similarly.

Theorem 2.14. *Let B_n denote the n^{th} number. Then*

$$B_{m+n} = B_mB_n - Q^nB_{m-n}$$

for integers m and n such that $m \geq n$.

From above theorem we can give the following result.

Corollary 2.15. *Let B_n denote the n^{th} number. Then $B_{2n} = B_n^2 - 2Q^n$ and $B_{3n} = B_n^3 - 3B_nQ^n$ for $n \geq 0$.*

Proof. We proved in above theorem that $B_{m+n} = B_mB_n - Q^nB_{m-n}$. So we obtain $B_{2n} = B_n^2 - 2Q^n$ since $B_0 = 2$. Similarly we deduce that $B_{3n} = B_{2n+n} = B_{2n}B_n - Q^nB_n = (B_n^2 - 2Q^n)B_n - Q^nB_n = B_n^3 - 3B_nQ^n$. □

By virtue of Corollary 2.15, we can deduce the $(sn)^{tk}$ terms ($s \geq 2$ is an integer) of B_n numbers by terms of B_n and Q^n, for instance we have $B_{4n} = B_n^4 - 4B_n^2Q^n + 2Q^{2n}, B_{5n} = B_n^5 - 5B_n^3Q^n + 5B_nQ^{2n}$, and etc.

For α and β, we define

$$\left(\frac{\alpha,\beta}{p}\right) = \begin{cases} (\frac{D}{p}) & \text{if } p \nmid D \\ 0 & \text{if } p|D \end{cases} \tag{2.8}$$

for primes $p \geq 3$, where $(\frac{\cdot}{p})$ denotes the Legendre symbol. Then the generalized Euler function $\Psi_{\alpha,\beta}(B_n)$ for B_n (when B_n is prime) is defined as

$$\Psi_{\alpha,\beta}(B_n) = B_n - \left(\frac{\alpha,\beta}{B_n}\right). \tag{2.9}$$

Then we can give the following theorem.

Theorem 2.16. *Let B_n denote the n^{th} number. If B_n is prime, then*

$$\Psi_{\alpha,\beta}(B_n) = B_n \tag{2.10}$$

for every prime $p \geq 5$.

Proof. We know that every prime number $p \equiv 1 (\text{mod } 4)$ can be written of the for $P^2 - 4Q$ for positive integers P and Q. Since $D = p = P^2 - 4Q$, we get $(\frac{\alpha,\beta}{p}) = 0$ by (2.8) and hence $\Psi_{\alpha,\beta}(B_n) = B_n$. □

3. Cross–ratio of Four Consecutive B_n Numbers.

Recall that the cross–ratio is also an important quantity in complex analysis and also in the theory of discrete groups. Given four different complex numbers z_1, z_2, z_3 and z_4, the cross–ratio defined as

$$[z_1, z_2; z_3, z_4] = \frac{(z_1 - z_3)(z_2 - z_4)}{(z_2 - z_3)(z_1 - z_4)} \tag{3.1}$$

is invariant under arbitrary Mobius (i.e., linear fractional) transformations. This definition can be extended to the entire Riemann sphere (i.e. $\mathbb{C} \cup \{\infty\}$) by continuity. More generally, the cross–ratio can be defined on any projective line (The Riemann Sphere is just the complex projective line). It is given by the above expression in any affine coordinate chart. Cross–ratios are invariant of projective geometry in the sense that they are preserved by projective transformations. The cross–ratio of four complex numbers is real if and only if the four numbers are either collinear or noncyclic.

In [14], the authors considered the cross–ratio of four consecutive Lucas numbers. They defined the cross–ratio of four consecutive Lucas numbers L_n, L_{n+1}, L_{n+2} and L_{n+3} to be

$$[L_n, L_{n+1}; L_{n+2}, L_{n+3}] = \frac{(L_n - L_{n+1})(L_{n+2} - L_{n+3})}{(L_{n+1} - L_{n+2})(L_{n+3} - L_n)} \tag{3.2}$$

and proved that

$$\lim_{n\to\infty} [L_n, L_{n+1}; L_{n+2}, L_{n+3}] = \frac{-1}{2\alpha_1},$$

and in [15], the authors considered same problem for Fibonacci numbers and using (3.2), they proved that

$$\lim_{n\to\infty} [F_n, F_{n+1}; F_{n+2}, F_{n+3}] = \frac{-1}{2\alpha_1}$$

where α_1 is defined in (1.2).

Similarly we can give the following theorem by using (3.1).

Theorem 3.1. *Let B_n, B_{n+1}, B_{n+2} and B_{n+3} be four consecutive B_n numbers. Then*

$$\lim_{n\to\infty} [B_n, B_{n+1}; B_{n+2}, B_{n+3}] = \frac{\alpha^2 + 2\alpha + 1}{\alpha^2 + \alpha + 1}.$$

Proof. Let B_n, B_{n+1}, B_{n+2} and B_{n+3} be four consecutive B_n numbers. Then we get

$$[B_n, B_{n+1}; B_{n+2}, B_{n+3}] = \frac{(B_n - B_{n+2})(B_{n+1} - B_{n+3})}{(B_{n+1} - B_{n+2})(B_n - B_{n+3})}. \tag{3.3}$$

Since $B_n = (2k+1)B_{n-1} - k^2 B_{n-2}$, we get $B_{n+2} = (2k+1)B_{n+1} - k^2 B_n$ and $B_{n+3} = (3k^2+4k+1)B_{n+1} - (2k^3+k^2)B_n$. Hence

$$\begin{aligned} B_n - B_{n+2} &= -(2k+1)B_{n+1} + (k^2+1)B_n \\ B_{n+1} - B_{n+3} &= (-3k^2-4k)B_{n+1} + (2k^3+k^2)B_n \\ B_{n+1} - B_{n+2} &= -2kB_{n+1} + k^2 B_n \\ B_n - B_{n+3} &= -(3k^2+4k+1)B_{n+1} + (2k^3+k^2+1)B_n. \end{aligned}$$

So (3.3) becomes

$$\begin{aligned} (3.4)\quad & [B_n, B_{n+1}; B_{n+2}, B_{n+3}] \\ &= \frac{[-(2k+1)B_{n+1} + (k^2+1)B_n][(-3k^2-4k)B_{n+1} + (2k^3+k^2)B_n]}{[-2kB_{n+1} + k^2B_n][-(3k^2+4k+1)B_{n+1} + (2k^3+k^2+1)B_n]} \\ &= \frac{[-(2k+1)B_{n+1} + (k^2+1)B_n][(-3k-4)B_{n+1} + (2k^2+k)B_n]}{[-2B_{n+1} + kB_n][-(3k^2+4k+1)B_{n+1} + (2k^3+k^2+1)B_n]}. \end{aligned}$$

Note that

$$(3.5)\qquad B_n = \alpha^n + \beta^n \text{ and } B_{n+1} = \alpha^{n+1} + \beta^{n+1}.$$

Combining (3.4) and (3.5) and taking the limit of both sides of (3.4), we deduce that

$$\lim_{n\to\infty} [B_n, B_{n+1}; B_{n+2}, B_{n+3}] = \frac{\alpha^2 + 2\alpha + 1}{\alpha^2 + \alpha + 1}.$$

This completes the proof. □

By symmetry, can give the following result.

Corollary 3.2. *Let B_n, B_{n+1}, B_{n+2} and B_{n+3} be four consecutive B_n numbers. Then*

$$\lim_{n\to\infty} [B_n, B_{n+1}; B_{n+3}, B_{n+2}] = \frac{\alpha^2 + \alpha + 1}{\alpha^2 + 2\alpha + 1}$$

$$\lim_{n\to\infty} [B_n, B_{n+2}; B_{n+3}, B_{n+1}] = \frac{\alpha^2 + \alpha + 1}{-\alpha}$$

$$\lim_{n\to\infty} [B_n, B_{n+2}; B_{n+1}, B_{n+3}] = \frac{-\alpha}{\alpha^2 + \alpha + 1}$$

$$\lim_{n\to\infty} [B_n, B_{n+3}; B_{n+2}, B_{n+1}] = \frac{\alpha^2 + 2\alpha + 1}{\alpha}$$

$$\lim_{n\to\infty} [B_n, B_{n+3}; B_{n+1}, B_{n+2}] = \frac{\alpha}{\alpha^2 + 2\alpha + 1}.$$

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