MacWilliams Duality in LRTJ-Spaces*

Sapna Jain
Department of Mathematics
University of Delhi
Delhi 110 007
India
E-mail: sapna@vsnl.com

Abstract. In [4], the author introduced a new metric on the space $Mat_{m\times s}(\mathbf{Z}_q)$ which is the module space of all $m \times s$ matrices with entries from the finite ring $\mathbf{Z}_q (q \geq 2)$ generalizing the classical Lee metric [5] and the array RT-metric [8] and named this metric as GLRTP-metric which is further renamed as LRTJ-metric (Lee-Rosenbloom-Tsfasman-Jain Metric) in [1]. In this paper, we introduce a complete weight enumerator for codes over $Mat_{m\times s}(\mathbf{Z}_q)$ endowed with LRTJ-metric and obtain a MacWilliams type identitiy with respect to this new metric for the complete weight enumerator.

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1. Introduction

The choice of a metric for a given parallel channel communication system plays an important role as the channel model should match the metric d to be employed for developing a suitable code, and hence for a communication system to operate reliably. Thus, given a modulation scheme, one metric may be better suited than another. In [4], the author introduced a new metric on $Mat_{m\times s}(\mathbf{Z}_q)$, the module space of all $m \times s$ metrices over the finite ring $\mathbf{Z}_q(q \geq 2)$, generalizing the classical Lee metric [5] and array RT-metric [8] and named this metric as GLRTP-metric which is also known as LRTJ-metric (Lee-Rosenbloom-Tsfasman-Jain Metric) [1]. MacWilliams type identity for RT-weight enumerator and RT-complete weight enumerator have been obtained by various authors ([2], [9], [10]).

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In this paper, we introduce a complete weight enumerator for codes over $Mat_{m\times s}(\mathbf{Z}_q)$ endowed with more general metric viz. LRTJ-metric and obtain a MacWilliams type identity with respect to this new metric for the complete weight enumerator.

2. Definitions and notations

Let $\mathbf{Z}_q$ be the ring of integers modulo q. Let $Mat_{m\times s}(\mathbf{Z}_q)$ be the set of all $m \times s$ matrices with entries from $\mathbf{Z}_q$. Then $Mat_{m\times s}(\mathbf{Z}_q)$ is a module over $\mathbf{Z}_q$. Let V be a $\mathbf{Z}_q$-submodule of the module $Mat_{m\times s}(\mathbf{Z}_q)$. Then V is called an array code (in fact, linear array code). For q prime, $\mathbf{Z}_q$ becomes a field and correspondingly $Mat_{m\times s}(\mathbf{Z}_q)$ and V become the vector space and sub space respectively over the field $\mathbf{Z}_q$. Also, we define the Hamming weight $H(a)$ and Lee weight $L(a)$ of an element $a \in \mathbf{Z}_q$ by

$$H(a) = \begin{cases} 1 & \text{if} \quad a \neq 0 \\ 0 & \text{if} \quad a = 0, \end{cases}$$

$$L(a) = \begin{cases} a & \text{if} \quad 0 \leq a \leq q/2 \\ q-a & \text{if} \quad q/2 < a \leq q-1. \end{cases}$$

We now define the LRTJ-metric [4] as follows:

Definition 2.1. Let $Y \in Mat_{m\times s}(\mathbf{Z}_q)$ with $Y = (y_0, y_1, \cdots, y_{s-1})$. The LRTJ-weight of Y denoted by $\tau(Y)$ is defined as

$$\tau(Y) = \begin{cases} \max\limits_{j=0,1,\cdots,s-1} L(y_j) + \max\limits_{j=0,1,\cdots,s-1} \{j \mid y_j \neq 0\} & \text{if } Y \neq 0 \\ 0 & \text{if } Y = 0. \end{cases}$$

Then $0 \leq \tau(Y) \leq [q/2] + s - 1$. Extending the definition of LRTJ-weight to the class of all $m \times s$ matrices as

$$\tau(A) = \sum_{i=1}^{m} \tau(A_i),$$

where $A = \begin{pmatrix} A_1 \\ A_2 \\ \cdots \\ A_m \end{pmatrix} \in Mat_{m\times s}(\mathbf{Z}_q)$ and A_i denotes the i^{th} row of A. Then LRTJ-weight τ satisfies $0 \leq \tau(A) \leq m([q/2]+s-1)$ for all $A \in Mat_{m\times s}(\mathbf{Z}_q)$ and determines a metric on $Mat_{m\times s}(\mathbf{Z}_q)$ if we set $d(A, A') = \tau(A - A')$ for all $A, A' \in Mat_{m\times s}(\mathbf{Z}_q)$ and is known as LRTJ-metric.

Definition 2.2. Let $V \in Mat_{m \times s}(\mathbf{Z}_q)$ be a linear array code. The LRTJ-weight spectrum of the code V is the set

$$\{w_0, w_1, \cdots, w_{m([q/2]+s-1)}\},$$

where for all $0 \leq r \leq m([q/2] + s - 1), w_r$ is given by

$$w_r = |\{A \in V | \tau(A) = r\}|.$$

The LRTJ-weight enumerator of the code V is defined as

$$W_V(z) = \sum_{r=0}^{m([q/2]+s-1)} w_r z_r = \sum_{A \in V} z^{\tau(A)}. \tag{1}$$

Definition 2.3. Let $Y_1 = (p_0, p_1, \cdots, p_{s-1})$ and $Y_2 = (q_0, q_1, \cdots, q_{s-1})$ be two elements of $Mat_{1 \times s}(\mathbf{Z}_q)$. The inner product of Y_1 and Y_2 is defined by

$$< Y_1, Y_2 >= \sum_{i=0}^{s-1} p_i q_{s-1-i},$$

and this is extended to the inner product of $A = (A_1, A_2, \cdots, A_m)^T$ and $B = (B_1, \cdots, B_m)^T \in Mat_{m \times s}(\mathbf{Z}_q)$ as

$$< A, B >= \sum_{i=1}^{m} < A_i, B_i > .$$

The dual of a linear array code $V \subseteq Mat_{m \times s}(\mathbf{Z}_q)$ is defined as

$$V^{\perp} = \{B \in Mat_{m \times s}(\mathbf{Z}_q) | < A, B >= 0 \text{ for all } A \in V\}.$$

Then $V^{\perp} \in Mat_{m \times s}(\mathbf{Z}_q)$ is also a linear array code.

Now we define the character of a finite Abelien group and canonical additive character of a finite field [6, 7].

Definition 2.4. Let G be a finite Abelian group with respect to addition. Let U be the multiplicative group of complex numbers having absolute value 1 i.e.

$$U = \{z \in \mathbf{C} : \ |z| = 1\}.$$

A character χ of G is a group homomorphism from G into U i.e. $\chi : G \longrightarrow U$ is a map satisfying $\chi(g_1 + g_2) = \chi(g_1)\chi(g_2)$ for all $g_1, g_2 \in G$.

Definition 2.5. Let $F_q = F_{p^n}$ be a finite field having $q = p^n$ elements and U be the group of complex numbers as in Definition 2.4. Consider the additive group F_q^+ of the finite field F_q.

(i) The character $\chi : F_q^+ \to U$ given by

$$\chi(a) = 1 \text{ for all } a \in F_q^+$$

is called the trivial additive character of F_q^+.

(ii) The nontrivial canonical additive character χ of the finite field F_q is a group homomorphism

$$\chi : F_q^+ = F_{p^n}^+ \longrightarrow U \text{ given by}$$

$$\chi(a) = cos2\pi\left(\frac{a + a^p + \cdots a^{p^{n-1}}}{p}\right) + isin2\pi\left(\frac{a + a^p + \cdots a^{p^{n-1}}}{p}\right) \quad (2)$$

for all $a \in F_q^+$.

Observations

1. For the finite field F_q having prime number of elements, the definition of nontrivial canonical additive character χ given in (2) reduces to

$$\begin{aligned} \chi \quad &: \quad F_q^+ \longrightarrow U \text{ is given by} \\ \chi(a) \quad &= \quad cos\frac{2\pi a}{q} + i\ sin\frac{2\pi a}{q} \text{ for all } a \in F_q^+. \end{aligned}$$

2. Over F_2, we have

$$\chi(0) = 1, \quad \chi(1) = -1.$$

3. Over $F_3, \chi(0) = 1, \quad \chi(1) = \frac{-1}{2} + \frac{\sqrt{3}}{2}i, \chi(2) = \frac{-1}{2} - \frac{\sqrt{3}}{2}i.$

3. Motivation for the introduction of complete weight enumerator in LRTJ-spaces

Here, we state the difficulty arising in LRTJ-weight enumerator defined in (1) by way of an example which gives the motivation for the introduction of complete weight enumerator. Similar type of problem for RT spaces was discussed in [9].

Example 3.1. Let V_1 and V_2 be two linear array codes over $Mat_{2\times2}(\mathbf{Z}_2)$ given by

$$V_1 = \left\{\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}\right\} \quad \text{and}$$
$$V_2 = \left\{\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right\}.$$

The LRTJ-weight enumerator of both V_1 and V_2 is $1+z^2$. We find the dual codes of V_1 and V_2.

$$\begin{aligned} V_1^{\perp} &= \left\{\begin{pmatrix} p & q \\ r & s \end{pmatrix} \in Mat_{2\times2}(\mathbf{Z}_2) \mid q+s=0\right\} \\ &= \left\{\begin{pmatrix} p & q \\ r & s \end{pmatrix} \in Mat_{2\times2}(\mathbf{Z}_2) \mid \text{either } q=s=1 \text{ or } q=s=0\right\}. \end{aligned}$$

Similarly,

$$V_2^{\perp} = \left\{\begin{pmatrix} p & q \\ r & s \end{pmatrix} \in Mat_{2\times2}(\mathbf{Z}_2) \mid r=0\right\}.$$

Therefore

$$\begin{aligned} V_1^{\perp} = &\left\{\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix},\right. \\ &\left.\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\right\}, \\ V_2^{\perp} = &\left\{\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},\right. \\ &\left.\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\right\}. \end{aligned}$$

The LRTJ-weight enumerators of $V_1^{\perp}$ and $V_2^{\perp}$ are

$$\begin{aligned} W_{V_1^{\perp}}(z) &= 1+2z+z^2+4z^4, \\ W_{V_2^{\perp}}(z) &= 1+z+3z^2+z^3+2z^4. \end{aligned}$$

Thus we observe that although the LRTJ-weight enumerators of the codes V_1 and V_2 are the same, but the LRTJ-weight enumerators of their duals are different. This problem can be overcome if we define a weight enumerator that preserves the order as well as the absolute value of the entries of an $m \times s$ matrix and gives more information about the code. Motivated by this in Section 4, we introduce complete weight enumerator for codes in LRTJ spaces and obtain MacWilliams type identity for this type of weight enumerator.

4. Complete weight enumerator in LRTJ-spaces

Throughout this section, q is a prime number. We begin with the definition of the complete weight enumerator in LRTJ-spaces.

Definition 4.1. Let $V \subseteq Mat_{m\times s}(\mathbf{Z}_q)$ be a k-dimensional linear array code over $\mathbf{Z}_q$ with $|V| = q^k (k \leq ms) = n$ (say). Let

$$V = \{A^{(1)}, A^{(2)}, \cdots, A^{(n)}\}.$$

Also, for $i \leq i \leq n$, let

$$A^{(i)} = \begin{pmatrix} a_{10}^{(i)} & a_{11}^{(i)} & \cdots & a_{1,s-1}^{(i)} \\ a_{20}^{(i)} & a_{21}^{(i)} & \cdots & a_{2,s-1}^{(i)} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m0}^{(i)} & a_{m1}^{(i)} & \cdots & a_{m,s-1}^{(i)} \end{pmatrix}.$$

Let Y_{ms} and T_{ms} be two ms-tuples of variables given by

$$\begin{aligned} Y_{ms} &= (y_{10}, \cdots, y_{1,s-1}, \cdots, y_{m0}, \cdots, y_{m,s-1}) \quad \text{and} \\ T_{ms} &= (t_{10}, \cdots, t_{1,s-1}, \cdots, t_{m0}, \cdots, t_{m,s-1}). \end{aligned}$$

We define the complete LRTJ-weight enumerator of an array code V by

$$\begin{aligned} W_V(Y_{ms}, T_{ms}) &= \sum_{i=1}^{n} y_{10}^{H(a_{10}^{(i)})} t_{10}^{L(a_{10}^{(i)})} \cdots y_{1,s-1}^{H(a_{1,s-1}^{(i)})} t_{1,s-1}^{L(a_{1,s-1}^{(i)})} \cdots y_{m0}^{H(a_{m0}^{(i)})} t_{m0}^{L(a_{m0}^{(i)})} \\ &\quad \cdots y_{m,s-1}^{H(a_{m,s-1}^{(i)})} t_{m,s-1}^{L(a_{m,s-1}^{(i)})}. \end{aligned}$$

Then the complete LRTJ-weight enumerator $W_V(Y_{ms}, T_{ms})$ of array code V is a polynomial in $2ms$ variables. Further, it is possible to obtain the LRTJ-weight enumerator as a special case of the complete LRTJ-weight enumerator as shown in the following example:

Example 4.1. The complete LRTJ-weight enumerators of the codes $V_1, V_2, V_1^{\perp}$ and $V_2^{\perp}$ of Example 3.1 are

$$W_{V_1}(Y_{22}, T_{22}) = 1 + y_{10}t_{10}y_{20}t_{20},$$

$$\begin{aligned}
W_{V_2}(Y_{22}, T_{22}) &= 1 + y_{21}t_{21},\\
W_{V_1^\perp}(Y_{22}, T_{22}) &= 1 + y_{11}t_{11}y_{21}t_{21} + y_{10}t_{10}y_{11}t_{11}y_{20}t_{20}y_{21}t_{21} +\\
&+ y_{10}t_{10}y_{11}t_{11}y_{21}t_{21} + y_{11}t_{11}y_{20}t_{20}y_{21}t_{21} +\\
&+ y_{10}t_{10} + y_{20}t_{20} + y_{10}t_{10}y_{20}t_{20},\\
W_{V_2^\perp}(Y_{22}, T_{22}) &= 1 + y_{10}t_{10} + y_{11}t_{11} + y_{21}t_{21} + y_{10}t_{10}y_{11}t_{11} +\\
&+ y_{10}t_{10}y_{21}t_{21} + y_{11}t_{11}y_{21}t_{21} + y_{10}t_{10}y_{11}t_{11}y_{21}t_{21}.
\end{aligned}$$

By letting for each $1 \le j \le 2$,

$$y_{j0}^{i_0}t_{j0}^{i_1}y_{j1}^{i_2}t_{j1}^{i_3} = \begin{cases} z^{2i_2+(1-i_2)i_0-1+\max\{i_1,i_3\}} & \text{if either of} \quad i_0, i_2 \neq 0 \\ 1 & \text{if} \quad i_0 = i_2 = 0, \end{cases}$$

in the complete weight enumerators of codes $V_1, V_2, V_1^\perp$ and $V_2^\perp$, we obtain the LRTJ-weight enumerators of these codes discussed in Example 3.1.

In order to state and prove the MacWilliams type identity for the complete LRTJ-weight enumerator of an array code $V \subseteq Mat_{m\times s}(\mathbf{Z}_q)$, we make the following identification:

Definition 4.2. Let q be a prime number. Define

$$\theta_1 : Mat_{1\times s}(\mathbf{Z}_q) \longrightarrow \frac{\mathbf{Z}_q[x]}{< x^s >} \quad \text{as}$$

$$\theta_1((p_0, p_1, \cdots, p_{s-1})) = p_0 + p_1 x + \cdots + p_{s-1}x^{s-1}.$$

Let $P = (P_1, P_2, \cdots, P_m)^T \in Mat_{m\times s}(\mathbf{Z}_q)$ where $P_i = (p_{i0}, p_{i1}, \cdot, p_{i,s-1})$ for all $i \le i \le m$.

Extending θ_1 to $Mat_{m\times s}(\mathbf{Z}_q)$ as

$$\theta : Mat_{m\times s}(\mathbf{Z}_q) \longrightarrow Mat_{m\times 1}\left(\frac{\mathbf{Z}_q[x]}{< x^s >}\right)$$

given by

$$\theta(P) = (p_{00} + p_{01} + \cdots + p_{0,s-1}x^{s-1}, \cdots, p_{m0} + p_{m1}x + \cdots + p_{m,s-1}x^{s-1})^T.$$

Then θ is a Z_q-vector space isomorphism.

The LRTJ-weight of a polynomial $p(x) \in \dfrac{\mathbf{Z}_q[x]}{< x^s >}$ is given by

$$\tau(p(x)) = \deg p(x) + \max_{j=0}^{s-1}\{L(p_j)\},$$

where $p(x) = p_0 + p_1 x + \cdots + p_{s-1}x^{s-1}$.

Further, we define the $l^{th}(0 \leq l \leq s-1)$ coefficient of $p(x)$ as

$$c_l(p(x)) = p_l.$$

Let $P(x) = (P_1(x), \cdots, P_m(x))^T$ and $Q(x) = (Q_1(x), \cdots, Q_m(x))^T$ be two elements in $Mat_{m\times 1}\left(\frac{\mathbf{Z}_q[x]}{< x^s >}\right)$ where for all $1 \leq i \leq m$,

$$\begin{aligned} P_i(x) &= p_{i0} + p_{i1}x + \cdots + p_{i,s-1}x^{s-1} \quad \text{and} \\ Q_i(x) &= q_{i0} + q_{i1}x + \cdots + q_{i,s-1}x^{s-1}. \end{aligned}$$

The inner product of $P(x)$ and $Q(x)$ defined in Section 2 becomes

$$< P(x), Q(x) >= \sum_{i=1}^{m} c_{s-1}(P_i(x)Q_i(x)).$$

Now, we prove the main result for which we require the following lemmas:

Lemma 4.1 [6, 7]. *Let χ be the nontrivial canonical additive character of $\mathbf{Z}_q$ (q prime). Then*

$$\sum_{\alpha \in \mathbf{Z}_q} \chi(\alpha) = 0.$$

□

Lemma 4.2. *Let χ be the nontrivial canonical additive character of $\mathbf{Z}_q$ (q prime). Let $V \subseteq Mat_{m\times s}(\mathbf{Z}_q)$ be an array code. Then*

$$\sum_{P(x)\in V} \chi(< P(x), Q(x) >) = \begin{cases} 0 & \text{if} \quad Q(x) \notin V^{\perp} \\ |V| & \text{if} \quad Q(x) \in V^{\perp}. \end{cases}$$

Proof. If $Q(x) \in V^{\perp}$, then clearly $< P(x), Q(x) >= 0$. This implies that

$$\sum_{P(x)\in V} \chi(< P(x), Q(x) >) = \sum_{P(x)\in V} \chi(0) = |V|.$$

If $Q(x) \notin V^{\perp}$, then we claim that in the summation

$$\sum_{P(x)\in V} \chi(< P(x), Q(x) >),$$

the inner product $< P(x), Q(x) >$ takes every value of $\mathbf{Z}_q$, the same number of times. For this, let $r_j \geq 0$ be the number of elements of V whose inner product with $Q(x)$ is equal to j for all $0 \leq j \leq q-1$. To be more precise, let $P_1^j(x), P_2^j(x), \cdots, P_{r_j}^j(x)$ be all the elements of V such that

$$< P_i^j(x), Q(x) > \; = j \text{ for all } i = 1 \text{ to } r_j \text{ and for all } j = 0 \text{ to } q-1.$$

Choose j such that $1 \leq j \leq q-1$ and fix it. Then

$$P_1^j(x) + P_1^0(x), P_1^j(x) + P_2^0(x), \cdots, P_1^j(x) + P_{r_0}^0(x)$$

are r_0 distinct elements of V such that

$$\begin{aligned} < P_1^j(x) + P_k^0(x), Q(x) > &= < P_1^j(x), Q(x) > + < P_k^0(x), Q(x) > \\ &= j + 0 = j \text{ for all } k = 1 \text{ to } r_0. \end{aligned}$$

This implies that

$$r_0 \leq r_j. \tag{3}$$

Again, let

$$R(x) = \sum_{\substack{l=1 \\ l \neq j}}^{q-1} P_1^l(x) \in V.$$

Then $P_1^j(x) + R(x), P_2^j(x) + R(x), \cdots, P_{r_j}^j(x) + R(x)$ are distinct elements of V such that for every $1 \leq i \leq r_j$, we have

$$\begin{aligned} < P_i^j(x) + R(x), Q(x) > &= < P_i^j(x), Q(x) > + < R(x), Q(x) > \\ &= j + < \sum_{\substack{l=1 \\ l \neq j}}^{q-1} P_1^l(x), Q(x) > \\ &= j + 1 + 2 + \cdots + j - 1 + j + 1 + \cdots + q - 1 \\ &= \text{sum of all the elements of the field } \mathbf{Z}_q = 0. \end{aligned}$$

This implies that

$$r_j \leq r_0. \tag{4}$$

From (3) and (4), we get $r_j = r_0$. But j $(1 \le j \le q-1)$ is arbitrary and hence the claim. Thus

$$\sum_{P(x)\in V} \chi(< P(x), Q(x) >) = \text{multiple of } \sum_{\alpha\in \mathbf{Z}_q} \chi(\alpha) = 0 \text{ (using Lemma 4.1)}$$

□

Lemma 4.3. Let χ be the nontrivial canonical additive character of $\mathbf{Z}_q$ (q prime). Let β be a fixed element of $\mathbf{Z}_q$. Then for $q > 2$, we have

$$\sum_{\alpha\in \mathbf{Z}_q} \chi(\beta\alpha) y^{H(\alpha)} t^{L(\alpha)}$$
$$= \begin{cases} 1 + 2ty\left(\dfrac{t^{(q-1)/2} - 1}{t-1}\right) & \text{if} \quad \beta = 0, \\ 1 + \displaystyle\sum_{1=1}^{(q-1)/2} \Big(\chi(\beta_i) + \chi(\beta(q-i)\Big) yt^i & \text{if} \quad \beta \ne 0. \end{cases}$$

Proof. For $0 \ne \beta \in \mathbf{Z}_q$, we have

$$\begin{aligned} \sum_{\alpha\in \mathbf{Z}_q} \chi(\beta\alpha) y^{H(\alpha)} t^{L(\alpha)} &= \chi(0) + \chi(\beta.1)yt^1 + \chi(\beta.2)yt^2 + \cdots + \\ &\quad + \chi\left(\beta\left(\frac{q-1}{2}\right)\right) yt^{(q-1)/2} + \\ &\quad + \chi\left(\beta\left(\frac{q+1}{2}\right)\right) yt^{(q-1)/2} + \cdots + \chi(\beta(q-1))yt^1 \\ &= \chi(0) + (\chi(\beta.1) + \chi(\beta.(q-1))yt^1 + \cdots + \\ &\quad + \left(\chi\left(\beta.\left(\frac{q-1}{2}\right)\right) + \chi\left(\beta.\left(\frac{q+1}{2}\right)\right)\right) yt^{(q-1)/2} \\ &= 1 + \sum_{i=1}^{(q-1)/2} \Big(\chi(\beta_i) + \chi(\beta(q-i)\Big) yt^i. \end{aligned}$$

Also for $\beta = 0$, we have

$$\begin{aligned} \sum_{\alpha\in \mathbf{Z}_q} \chi(\beta\alpha) y^{H(\alpha)} t^{L(\alpha)} &= \sum_{\alpha\in \mathbf{Z}_q} \chi(0) y^{H(\alpha)} t^{L(\alpha)} \\ &= 1 + yt + yt^2 + \cdots yt^{(q-1)/2} + yt^{(q-1)/2} + \cdots + yt^1 \\ &= 1 + 2ty(1 + t + t^2 + \cdots + t^{(q-3)/2}\Big) \\ &= 1 + 2ty\left(\frac{t^{(q-1)/2} - 1}{t-1}\right). \end{aligned}$$

$\square$

Remark 4.1. The result in Lemma 4.3 is also true for any odd positive integer q and for any character χ of $\mathbf{Z}_q$.

Remark 4.2. If q is any even positive integer and χ is a character of $\mathbf{Z}_q$ then proceeding as in Lemma 4.3, we have

$$\sum_{\alpha\in\mathbf{Z}_q} \chi(\beta\alpha)y^{H(\alpha)}t^{L(\alpha)}$$
$$= \begin{cases} 1+2ty\left(\dfrac{t^{(q-2)/2}-1}{t-1}\right)+yt^{q/2} & \text{if} \quad \beta=0 \\ 1+\displaystyle\sum_{i=1}^{(q-2)/2}\Big(\chi(\beta_i)+\chi(\beta(q-i)\Big)yt^i+\chi\Big(\beta(q/2)\Big)yt^{q/2} & \text{if} \quad \beta\neq 0. \end{cases}$$

On putting $q=2$ and taking χ to be the canonical additive character of $\mathbf{Z}_2$, the above expression reduces to

$$\sum_{\alpha\in\mathbf{Z}_2} \chi(\beta\alpha)y^{H(\alpha)}t^{L(\alpha)} = \begin{cases} 1+yt & \text{if} \quad \beta=0 \\ 1+\chi(1)yt & \text{if} \quad \beta\neq 0. \end{cases}$$
$$= \begin{cases} 1+yt & \text{if} \quad \beta=0 \\ 1-yt & \text{if} \quad \beta\neq 0. \end{cases}$$

Lemma 4.4. *Let χ be the nontrivial canonical additive character of $\mathbf{Z}_q$ (q prime) and i,j be fixed nonnegative integer. Let $p(x)=p_{i0}+p_{i1}x+\cdots+p_{i,s-1}x^{s-1}\in\dfrac{\mathbf{Z}_q[x]}{<x^s>}$. Then for $q=2$,*

$$\sum_{\alpha\in\mathbf{Z}_2}\chi(<p(x),\alpha x^j>)y_{ij}^{H(\alpha)}t_{ij}^{L(\alpha)} = \begin{cases} 1+y_{ij}t_{ij} & \text{if} \quad p_{i,s-1-j}=0 \\ 1-y_{ij}t_{ij} & \text{if} \quad p_{i,s-1-j}\neq 0 \end{cases},$$

and for $q>2$

$$\sum_{\alpha\in\mathbf{Z}_q}(<p(x),\alpha x^j>)y_{ij}^{H(\alpha)}t_{ij}^{L(\alpha)}$$
$$= \begin{cases} 1+2t_{ij}y_{ij}\left(\dfrac{t_{ij}^{(q-1)/2}-1}{t-1}\right) & \text{if} \quad p_{i,s-1-j}=0 \\ 1+\displaystyle\sum_{k=1}^{(q-1)/2}\Big(\chi(p_{i,s-1-j}\times k)+\chi(p_{i,s-1-j}\times(q-k))\Big)y_{ij}t_{ij}^k & \\ & \text{if} \quad p_{i,s-1-j}\neq 0. \end{cases}$$

Proof. Since

$$\begin{aligned} & \sum_{\alpha\in\mathbf{Z}_q} (< p(x), \alpha x^j >) y_{ij}^{H(\alpha)} t_{ij}^{L(\alpha)} \\ = & \sum_{\alpha\in\mathbf{Z}_q} \chi(c_{s-1}(p(x) \times \alpha x^j)) y_{ij}^{H(\alpha)} t_{ij}^{L(\alpha)} \\ = & \sum_{\alpha\in\mathbf{Z}_q} \chi(p_{i,s-1-j} \times \alpha) y_{ij}^{H(\alpha)} t_{ij}^{L(\alpha)}, \end{aligned}$$

the proof now follows from Lemma 4.3 and Remark 4.2. □

Lemma 4.5. *Let V be an array code in $Mat_{m\times 1}\left(\frac{\mathbf{Z}_q[x]}{< x^s >}\right)$. Let f : $Mat_{m\times 1}\left(\frac{\mathbf{Z}_q[x]}{< x^s >}\right) \rightarrow \mathbf{C}[y_{10}, \cdots, y_{m,s-1}, t_{10}, \cdots, t_{m,s-1}]$ be a map where $\mathbf{C}[y_{10}, \cdots, y_{m,s-1}, t_{10}, \cdots, t_{m,s-1}]$ is the polynomial ring in the 2ms commutating variables with coefficients from complex field $\mathbf{C}$. Let χ be the nontrivial canonical additive character $\mathbf{Z}_q$. Then*

$$\sum_{Q(x)\in V^{\perp}} f(Q(x)) = \frac{1}{|V|} \sum_{P(x)\in V} \hat{f}(P(x)),$$

where $\hat{f}$ is the Hadamard transform of f given by

$$\hat{f}(P(x)) = \sum_{Q(x)\in Mat_{m\times 1}\left(\frac{\mathbf{Z}_q[x]}{< x^s >}\right)} \chi(< P(x), Q(x) >) f(Q(x)),$$

and

$$\begin{aligned} P(x) &= (P_1(x), \cdots, P_m(x))^T, \\ Q(x) &= (Q_1(x), \cdots, Q_m(x))^T. \end{aligned}$$

Proof.

$$\begin{aligned} & \sum_{P(x)\in V} \hat{f}(P(x)) \\ = & \sum_{P(x)\in V} \sum_{Q(x)\in Mat_{m\times 1}\left(\frac{\mathbf{Z}_q[x]}{< x^s >}\right)} \chi(< P(x), Q(x) >) f(Q(\end{aligned}$$

$$
\begin{aligned}
&= \sum_{P(x)\in V}\sum_{Q(x)\in V^{\perp}} \chi(< P(x), Q(x) >) f(Q(x)) + \\
&\quad + \sum_{P(x)\in V}\sum_{Q(x)\notin V^{\perp}} \chi(< P(x), Q(x) >) f(Q(x)) \\
&= |V| \sum_{Q(x)\in V^{\perp}} f(Q(x)) \quad \text{(using Lemma 4.2)}
\end{aligned}
$$

$$
\Rightarrow \sum_{Q(x)\in V^{\perp}} f(Q(x)) = \frac{1}{|V|} \sum_{P(x)\in V} \hat{f}(P(x)).
$$

□

Theorem 4.1. Let $q > 2$ be a prime. Let $V \subseteq Mat_{m\times s}(\mathbf{Z}_q)$ be a linear array code equipped with LRTJ-metric. Let $Q(x), P(x) \in Mat_{m\times 1}\left(\frac{\mathbf{Z}_q[x]}{< x^s >}\right)$ be such that $Q(x) = (Q_1(x), \cdots, Q_m(x))^T$ with $Q_i(x) = q_{i0} + q_{i1}x + \cdots + q_{i,s-1}x^{s-1}$ and $P(x) = (P_1(x), \cdots, P_m(x))^T$ with $P_i(x) = p_{i0} + p_{i1}x + \cdots + p_{i,s-1}x^{s-1}$ for all $1 \le i \le m$.

Then

$$
\begin{aligned}
&\sum_{Q(x)\in V^{\perp}} y_{10}^{H(q_{10})} t_{10}^{L(q_{10})} \cdots y_{1,s-1}^{H(q_{1,s-1})} t_{1,s-1}^{L(q_{1,s-1})} \cdots y_{m0}^{H(q_{m0})} t_{m0}^{L(q_{m0})} \cdots \\
&\cdots y_{m,s-1}^{H(q_{m,s-1})} t_{m,s-1}^{L(q_{m,s-1})} \\
= &\frac{1}{|V|}\left(A \times \sum_{P(x)\in V} B\right), \qquad (5)
\end{aligned}
$$

where

$$
A = \prod_{i=1}^{m}\prod_{j=0}^{s-1} A_{ij}, \qquad (6)
$$

$$
B = \prod_{i=1}^{m}\prod_{j=0}^{s-1} \left(\frac{B_{ij}}{A_{ij}}\right)^{H(p_{i,s-1-j})}, \qquad (7)
$$

and

$$
A_{ij} = 1 + 2t_{ij}y_{ij}\left(\frac{t_{ij}^{(q-1)/2} - 1}{t_{ij} - 1}\right), \qquad (8)
$$

$$
B_{ij} = 1 + \sum_{k=1}^{(q-1)/2} \Big(\chi(p_{i,s-1-j} \times k) - \chi(p_{i,s-1-j} \times (q-k))\Big) y_{ij} t_{ik}^{k}. \qquad (9)
$$

Proof. Take $f : Mat_{m\times 1}\left(\dfrac{\mathbf{Z}_q[x]}{<x^s>}\right) \to \mathbf{C}[y_{10},\cdots,y_{m,s-1},t_{10},\cdots,t_{m,s-1}]$ in Lemma 4.5 as

$$\begin{aligned} f(Q(x)) &= y_{10}^{H(q_{10})}t_{10}^{L(q_{10})}\cdots y_{1,s-1}^{H(q_{1,s-1})}t_{1,s-1}^{L(q_{1,s-1})}\cdots y_{m0}^{H(q_{m0})}t_{m0}^{L(q_{m0})}\cdots \\ &\quad \cdots y_{m,s-1}^{H(q_{m,s-1})}t_{m,s-1}^{L(q_{m,s-1})}. \end{aligned}$$

Then

$$\begin{aligned} \text{L.H.S. of (5)} &= \sum_{Q(x)\in V^{\perp}} f(Q(x)) \\ &= \frac{1}{|V|}\sum_{P(x)\in V} \hat{f}(P(x)). \qquad \text{(using Lemma 4.5)} \quad (10) \end{aligned}$$

Now

$$\begin{aligned} \hat{f}(P(x)) &= \sum_{Q(x)\in Mat_{m\times 1}\left(\frac{\mathbf{Z}_q[x]}{<x^s>}\right)} \chi(<P(x),Q(x)>)y_{10}^{H(q_{10})}t_{10}^{L(q_{10})}\cdots \\ &\quad \cdots y_{1,s-1}^{H(q_{1,s-1})}t_{1,s-1}^{L(q_{1,s-1})}\cdots y_{m0}^{H(q_{m0})}t_{m0}^{L(q_{m0})}\cdots y_{m,s-1}^{H(q_{m,s-1})}t_{m,s-1}^{L(q_{m,s-1})} \\ &= \sum_{Q(x)\in Mat_{m\times 1}\left(\frac{\mathbf{Z}_q[x]}{<x^s>}\right)} \prod_{i=1}^{m}\chi(<P_i(x),Q_i(x)>)y_{10}^{H(q_{10})}t_{10}^{L(q_{10})}\cdots \\ &\quad y_{1,s-1}^{H(q_{1,s-1})}t_{1,s-1}^{L(q_{1,s-1})}\cdots y_{m0}^{H(q_{m0})}t_{m0}^{L(q_{m0})}\cdots y_{m,s-1}^{H(q_{m,s-1})}t_{m,s-1}^{L(q_{m,s-1})} \\ &= \sum_{q_{10}\in\mathbf{Z}_q}\chi(<P_1(x),q_{10}>)y_{10}^{H(q_{10})}t_{10}^{L(q_{10})}\times\cdots\times \\ &\quad \times\sum_{q_{1,s-1}\in\mathbf{Z}_q}\chi(<P_1(x),q_{1,s-1}x^{s-1}>)y_{1,s-1}^{H(q_{1,s-1})}t_{1,s-1}^{L(q_{1,s-1})}\times \\ &\quad \times\sum_{q_{20}\in\mathbf{Z}_q}\chi(<P_2(x),q_{20}>)y_{20}^{H(q_{20})}t_{20}^{L(q_{20})}\times\cdots\times \\ &\quad \times\sum_{q_{2,s-1}\in\mathbf{Z}_q}\chi(<P_2(x),q_{2,s-1}x^{s-1}>)y_{2,s-1}^{H(q_{2,s-1})}t_{2,s-1}^{L(q_{2,s-1})}\times\cdots\times \\ &\quad \times\sum_{q_{m0}\in\mathbf{Z}_q}\chi(<P_m(x),q_{m0}>)y_{m0}^{H(q_{m0})}t_{m0}^{L(q_{m0})}\times\cdots\times \\ &\quad \times\sum_{q_{m,s-1}\in\mathbf{Z}_q}\chi(<P_m(x),q_{m,s-1}x^{s-1}>)y_{m,s-1}^{H(q_{m,s-1})}t_{m,s-1}^{L(q_{m,s-1})}. \end{aligned}$$

Applying Lemma 4.4, we get

$$\begin{aligned}
\hat{f}(P(x)) &= \prod_{l=0}^{s-1}\Bigg[\Bigg(1+2t_{1l}y_{1l}\Bigg(\frac{t_{1l}^{(q-1)/2}-1}{t-1}\Bigg)\Bigg)^{1-H(p_{1,s-1-l})}\times \\
&\times\Bigg(1+\sum_{k=1}^{(q-1)/2}\Bigg(\chi(p_{1,s-1-l}\times k) \\
&-\chi(p_{1,s-1-l}\times(q-k))\Bigg)y_{1l}t_{1l}^{k}\Bigg)^{H(p_{1,s-1-l})}\Bigg]\times \\
&\times\cdots\cdots\times\prod_{l=0}^{s-1}\Bigg[\Bigg(1+2t_{ml}y_{ml}\Bigg(\frac{t_{ml}^{(q-1)/2}-1}{t-1}\Bigg)\Bigg)^{1-H(p_{m,s-1-l})}\times \\
&\times\Bigg(1+\sum_{k=1}^{(q-1)/2}\Bigg(\chi(p_{m,s-1-l}\times k) \\
&-\chi(p_{m,s-1-l}\times(q-k))y_{ml}t_{ml}^{k}\Bigg)^{H(p_{m,s-1-l})}\Bigg] \\
&= \prod_{i=0}^{m}\prod_{j=0}^{s-1}(A_{ij})\times\prod_{i=0}^{m}\prod_{j=0}^{s-1}\left(\frac{B_{ij}}{A_{ij}}\right)^{H(p_{i,s-1-j})} \\
&= A\times B
\end{aligned}$$

where A and B are given by (6) and (7) respectively and A_{ij} and B_{ij} are given by (8) and (9) repectively.

Thus

$$\sum_{P(x)\in V}\hat{f}(P(x))=\sum_{P(x)\in V}A\times B=A\times\sum_{P(x)\in V}B. \qquad (11)$$

From (10) and (11) we get

$$\text{L.H.S. of (5)}=\frac{1}{|V|}\Bigg(A\times\sum_{P(x)\in V}B\Bigg).$$

□

Remark 4.3. For $q=2$, proceeding as in Theorem 2.1 and using Lemma 4.4, we get

$$\text{L.H.S. of (5)}=\frac{1}{|V|}\Bigg(A\times\sum_{P(x)\in V}B\Bigg).$$

where

$$A = \prod_{i=0}^{m} \prod_{j=0}^{s-1} (1 + y_{ij}t_{ij}) \tag{12}$$

and

$$B = \prod_{l=1}^{m} \prod_{j=0}^{s-1} \left(\frac{1 - y_{ij}t_{ij}}{1 + y_{ij}t_{ij}} \right)^{H(p_{i,s-1-j})} \tag{13}$$

Example 4.2. Consider the codes V_1 and V_2 of Example 3.1. Here $q = 2, m = s = 2, |V_1| = |V_2| = 2$. We find complete LRTJ-weight enumerator of dual codes $V_1^{\perp}$ and $V_2^{\perp}$ using (5) where A and B are given by (12) and (13) respectively.

$$W_{V_1^{\perp}}(Y_{22}, T_{22}) = \text{ L.H.S. of (5)} = \frac{1}{2}\left(A \times \sum_{P(x)\in V_1} B \right).$$

where (for code V_1)

$$A = \prod_{l=1}^{2} \prod_{j=0}^{1} \left(1 + y_{ij}t_{ij} \right)$$

and

$$B = \prod_{i=1}^{2} \prod_{j=0}^{1} \left(\frac{1 - y_{ij}t_{ij}}{1 + y_{ij}t_{ij}} \right)^{H(p_{i,s-1-j})}.$$

Thus

$$A = (1 + y_{10}t_{10})(1 + y_{11}t_{11})(1 + y_{20}t_{20})(1 + y_{21}t_{21})$$

and

$$B\Big|_{P(x)=\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}} = 1,$$

$$B\Big|_{P(x)=\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}} = \left(\frac{1 - y_{11}t_{11}}{1 + y_{11}t_{11}} \right) \left(\frac{1 - y_{21}t_{21}}{1 + y_{21}t_{21}} \right).$$

Thus

$$WV_1^{\perp}(Y_{22,T_22}) = \frac{1}{2}\Big[(1 + y_{10}t_{10})(1 + y_{11}t_{11})(1 + y_{20}t_{20})(1 + y_{21}t_{21}) \times$$

$$
\begin{aligned}
&\times\left(1+\left(\frac{1-y_{11}t_{11}}{1+y_{11}t_{11}}\right)\left(\frac{1-y_{21}t_{21}}{1+y_{21}t_{21}}\right)\right)\Bigg] \\
= \;& \frac{1}{2}\Bigg[(1+y_{10}t_{10})(1+y_{20}t_{20})\times\Big((1+y_{11}t_{11})(1+y_{21}t_{21}) \\
&+(1-y_{11}t_{11})(1-y_{21}t_{21})\Big)\Bigg] \\
= \;& 1+y_{20}t_{20}+y_{10}t_{10}+y_{10}t_{10}y_{20}t_{20}+y_{11}t_{11}y_{21}t_{21} \\
&+y_{20}t_{20}y_{11}t_{11}y_{21}t_{21}+y_{10}t_{10}y_{11}t_{11}y_{21}t_{21} \\
&+y_{10}t_{10}y_{20}t_{20}y_{11}t_{11}y_{21}t_{21}.
\end{aligned}
$$

Similarly for code V_2, we have

$$WV_2^{\perp}(Y_{22},T_{22}) = \text{ L.H.S. of (5)} = \frac{1}{2}\left(A\times\sum_{(P(x)\in V_2} B\right)$$

where

$$A=\prod_{i=0}^{2}\prod_{j=0}^{1}\left(1+y_{ij}t_{ij}\right)$$

and

$$B=\prod_{i=1}^{2}\prod_{j=0}^{1}\left(\frac{1-y_{ij}t_{ij}}{1+y_{ij}t_{ij}}\right)^{H(p_{i,s-1-j})}.$$

Computing A and B give

$$A=(1+y_{10}t_{10})(1+y_{11}t_{11})(1+y_{20}t_{20})(1+y_{21}t_{21})$$

and

$$
\begin{aligned}
B\Bigg|_{P(x)=\begin{pmatrix}0&0\\0&0\end{pmatrix}} &= 1. \\
B\Bigg|_{P(x)=\begin{pmatrix}0&0\\0&1\end{pmatrix}} &= \left(\frac{1-y_{20}t_{20}}{1+y_{20}t_{20}}\right).
\end{aligned}
$$

Thus

$$
\begin{aligned}
W_{V_2^{\perp}}(Y_{22},T_{22}) = \;& \frac{1}{2}\Bigg[(1+y_{10}t_{10})(1+y_{11}t_{11})(1+y_{20}t_{20})(1+y_{21}t_{21})\times \\
&\times\left(1+\left(\frac{1-y_{20}t_{20}}{1+y_{20}t_{20}}\right)\right)\Bigg]
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{2}\Big[(1+y_{10}t_{10})(1+y_{11}t_{11})(1+y_{21}t_{21}) \times \\
&\quad \times\Big((1+y_{20}t_{20})-(1+y_{20}t_{20})\Big)\Big] \\
&= \frac{1}{2}\Big[2(1+y_{11}t_{11}+y_{10}t_{10}+y_{10}t_{10}y_{11}t_{11})(1+y_{21}t_{21})\Big] \\
&= 1+y_{21}t_{21}+y_{11}t_{11}+y_{11}t_{11}y_{21}t_{21}+y_{10}t_{10} \\
&\quad +y_{10}t_{10}y_{21}t_{21}+y_{10}t_{10}y_{11}t_{11} \\
&\quad +y_{10}t_{10}y_{11}t_{11}y_{21}t_{21}.
\end{aligned}
$$

We note that $W_{V_1^{\perp}}(Y_{22}, T_{22})$ and $W_{V_2^{\perp}}(Y_{22}, T_{22})$ are same as obtained in Example 4.1.

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