Right Triangular Dihedral f–Tilings of the Sphere: $(\alpha, \beta, \frac{\pi}{2})$ and $(\gamma, \gamma, \frac{\pi}{2})$

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Abstract
The classification of all dihedral f-tilings of the Riemannian sphere S^2, whose prototiles are two right triangles with at least one isosceles, is given. The combinatorial structure and the symmetry group of each tiling is also achieved.

Keywords: dihedral f-tilings, combinatorial properties, spherical trigonometry
Mathematics Subject Classification: 52C20, 52B05, 20B35

1. Introduction

Let S^2 be the sphere of radius 1. By a *folding tiling* (*f-tiling*, for short) of the sphere S^2 we mean an edge-to-edge decomposition τ of S^2 by geodesic polygons, such that all the vertices of τ satisfy the *angle-folding relation*, i.e., each vertex of τ is of even valency $2n$, $n \geq 2$, and the sums of alternate angles are equal; that is,

$$\sum_{i=1}^{n} \theta_{2i} = \sum_{i=1}^{n} \theta_{2i-1} = \pi,$$

where the angles θ_i around any vertex of τ are ordered cyclically.

Folding tilings are intrinsically related to the theory of isometric foldings of Riemannian manifolds, introduced by S. A. Robertson [5] in 1977. In fact, the edge-complex associated to a spherical f-tiling is the set of singularities of some spherical isometric folding.

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A spherical f-tiling τ is called *monohedral* if every tile of τ is congruent to one fixed set X; it is called *dihedral* if every tile of τ is congruent to either two fixed sets X and Y.

The classification of f-tilings was initiated by Ana Breda [1], with a complete classification of all spherical monohedral (triangular) f-tilings. Later on, in 2002, Y. Ueno and Y. Agaoka [9] have established the complete classification of all triangular monohedral tilings of the sphere (without any restrictions on angles).

The study of all dihedral spherical f-tilings whose prototiles are an equilateral triangle and an isosceles triangle was presented in [3]. For a list of all dihedral f-tilings of the sphere by equilateral and scalene triangles see [2].

Robert Dawson has also been interested in special classes of spherical tilings, see [6,7,8] for instance.

In this paper we shall discuss dihedral f-tilings by spherical right triangles with at least one isosceles triangle.

2. Main Result

From now on T_1 is a spherical right triangle of internal angles $\frac{\pi}{2}$, α and β, with edge lengths a (opposite to β), b (opposite to α) and c (opposite to $\frac{\pi}{2}$), and T_2 is a spherical isosceles right triangle of internal angles $\frac{\pi}{2}$, γ and γ, with edge lengths d (opposite to γ) and e (opposite to $\frac{\pi}{2}$) (see Figure 1). We will assume throughout the text that T_1 and T_2 are distinct triangles, i.e., $(\alpha, \beta) \neq (\gamma, \gamma)$.

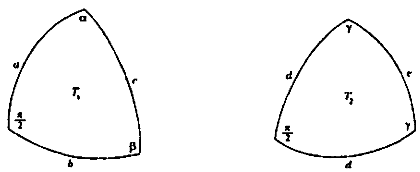


Figure 1. Prototiles: spherical right triangles T_1 and T_2

We shall denote by $\Omega(T_1, T_2)$ the set, up to an isomorphism, of all dihedral f-tilings of S^2 whose prototiles are T_1 and T_2.

Relations between faces, edges, vertices and angles of any dihedral f-tiling of S^2, with prototiles T_1 and T_2, are stated in proposition 1.

Proposition 1 *[3, Proposition 2.1] Let $\tau \in \Omega(T_1, T_2)$. If $N_1 > 0$ and $N_2 > 0$ denote the number of spherical right triangles of τ congruent to T_1 and T_2, respectively, and E and V denote the number of edges and vertices of τ, respectively, then:*

(i) $N_1 + N_2 = 2V - 4 = \frac{2}{3}E \geq 8$;

(ii) $3V = 6 + E$;

(iii) there are, at least, six vertices of valency four;

(iv) the cases $(\alpha + \beta \geq \pi$ *and* $\gamma > \frac{\pi}{2})$ *and* $(\alpha + \beta > \pi$ *and* $\gamma \geq \frac{\pi}{2})$ *cannot occur.*

It follows straightaway that

$$\alpha + \beta > \frac{\pi}{2} \quad \text{and} \quad \gamma > \frac{\pi}{4}. \tag{1}$$

In order to get any dihedral f-tiling $\tau \in \Omega(T_1, T_2)$, we find useful to start by considering one of its *local configurations*, beginning with a common vertex to two tiles of τ in adjacent positions.

In the diagrams of the following sections it is convenient to label the tiles according to the following procedures:

(i) We begin the configuration of a tiling $\tau \in \Omega(T_1, T_2)$ with a right triangle T_1, labelled by 1; then we label with 2 a right triangle T_2, adjacent to T_1;

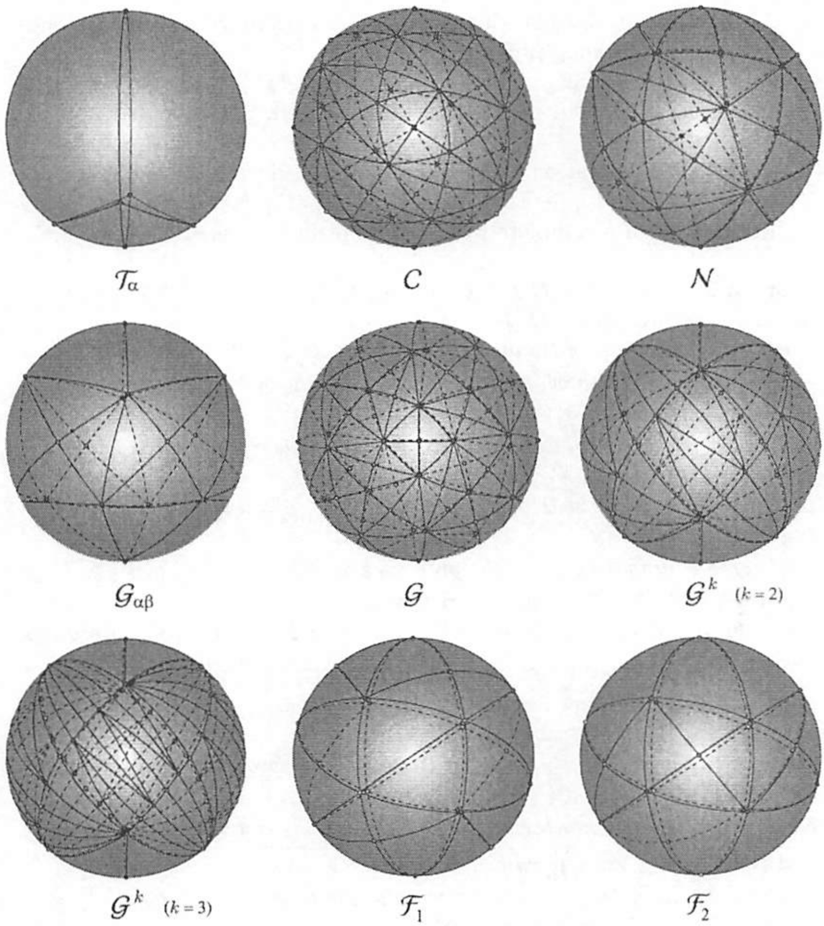
(ii) For $j \geq 3$, the location of tile j can be deduced from the configuration of tiles $(1, 2, 3, \ldots, j-1)$ and from the hypothesis that the configuration is part of a complete f-tiling (except in the cases indicated).

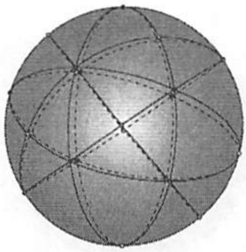
Theorem 2 *(Main Result) Let T_1 and T_2 be two spherical right triangles as described before. Then, $\Omega(T_1, T_2)$ consists of:*

- *a continuous family of tilings $\mathcal{T}_\alpha$, where $\alpha \in (\frac{\pi}{2}, \frac{3\pi}{4})$;*
- *a sporadic f-tiling $\mathcal{C}$ in which the prototiles are the right triangles $(2\beta, \beta, \frac{\pi}{2})$ and $(\pi - 4\beta, \pi - 4\beta, \frac{\pi}{2})$, with $\beta \approx 32.6°$;*
- *a sporadic f-tiling $\mathcal{N}$ in which the prototiles are the right triangles $(\pi - 2\gamma, \pi - 3\gamma, \frac{\pi}{2})$ and $(\gamma, \gamma, \frac{\pi}{2})$, with $\gamma \approx 52.2°$;*
- *a continuous family, with two free parameters, of tilings $\mathcal{G}_{\alpha\beta}$, where $\alpha + \beta \in (\frac{\pi}{2}, \frac{3\pi}{4})$, $\alpha > \beta$;*
- *a sporadic f-tiling $\mathcal{G}$ in which the prototiles are the right triangles $(\frac{\pi}{3}, \beta, \frac{\pi}{2})$ and $(\frac{2\pi}{3} - 2\beta, \frac{2\pi}{3} - 2\beta, \frac{\pi}{2})$, with $\beta \approx 35.9°$;*
- *a family of a discrete parameter $\mathcal{G}^k$, with $k \geq 2$, in which the prototiles are the right triangles $\left(\frac{\pi-\beta}{2}, \beta, \frac{\pi}{2}\right)$ and $\left(\frac{\pi-(2k-1)\beta}{2}, \frac{\pi-(2k-1)\beta}{2}, \frac{\pi}{2}\right)$, where β is completely determined for each k; for instance, if $k = 2$ and $k = 3$, then $\beta \approx 23.1°$ and $\beta \approx 14°$, respectively;*
- *twelve f-tilings $\mathcal{F}_i$, $1 \leq i \leq 12$, in which the prototiles are the right triangles $(\frac{\pi}{2}, \frac{\pi}{4}, \frac{\pi}{3})$ and $(\frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{3})$;*
- *a sporadic f-tiling $\mathcal{D}$ in which the prototiles are the right triangles $(\frac{\pi}{6}, \beta, \frac{\pi}{2})$ and $(\pi - 2\beta, \pi - 2\beta, \frac{\pi}{2})$, with $\beta = \arctan\sqrt{1 + 2\sqrt{3}}$;*
- *a sporadic f-tiling $\mathcal{M}$ in which the prototiles are the right triangles $(\frac{\pi}{3}, \frac{\pi}{2} - \gamma, \frac{\pi}{2})$ and $(\gamma, \gamma, \frac{\pi}{2})$, with $\gamma \approx 48.9°$;*

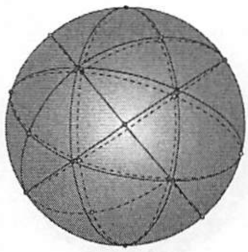
- *a sporadic f-tiling* $\mathcal{H}$ *in which the prototiles are the right triangles* $\left(\frac{\pi}{3}, \frac{2\pi}{3}, \frac{\pi}{2}\right)$ *and* $\left(\frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{2}\right)$;
- *a sporadic f-tiling* $\mathcal{J}$ *in which the prototiles are the right triangles* $\left(\frac{\pi}{3}, \pi - 3\gamma, \frac{\pi}{2}\right)$ *and* $\left(\gamma, \gamma, \frac{\pi}{2}\right)$, *with* $\gamma \approx 48.5°$;
- *for each* $k \geq 4$, *a single f-tiling* $\mathcal{R}^k$ *with prototiles* $\left(\frac{\pi}{k}, \pi - 2\gamma, \frac{\pi}{2}\right)$ *and* $\left(\gamma, \gamma, \frac{\pi}{2}\right)$, *with* $\gamma = \arccos\sqrt{\frac{1}{2}\cos\frac{\pi}{k}}$.

Planar representations, as well as a detailed study of these f-tilings, are included in the following sections. The type of vertices involved in each tiling is also presented. 3D representations of the f-tilings are illustrated in Figure 2.





$\mathcal{F}_3$



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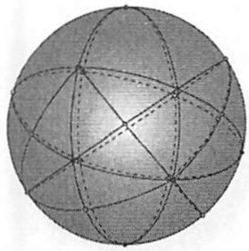
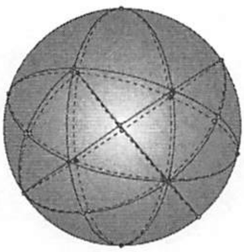
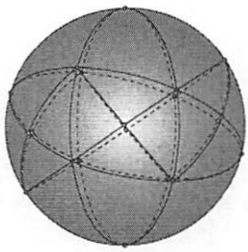


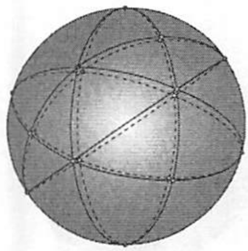
$\mathcal{F}_5$



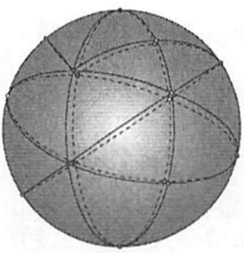
$\mathcal{F}_6$



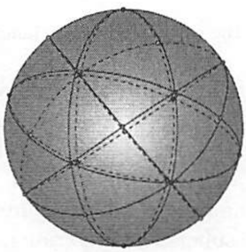
$\mathcal{F}_7$



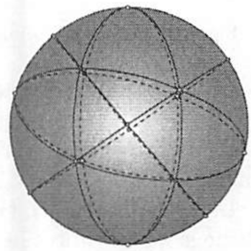
$\mathcal{F}_8$



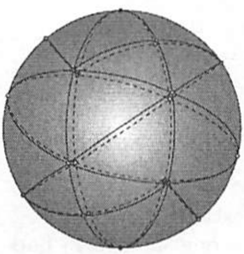
$\mathcal{F}_9$



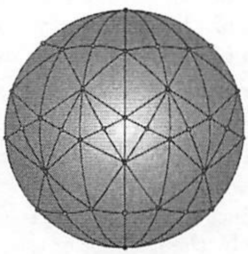
$\mathcal{F}_{10}$



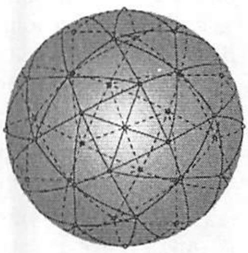
$\mathcal{F}_{11}$



$\mathcal{F}_{12}$



$\mathcal{D}$



$\mathcal{M}$

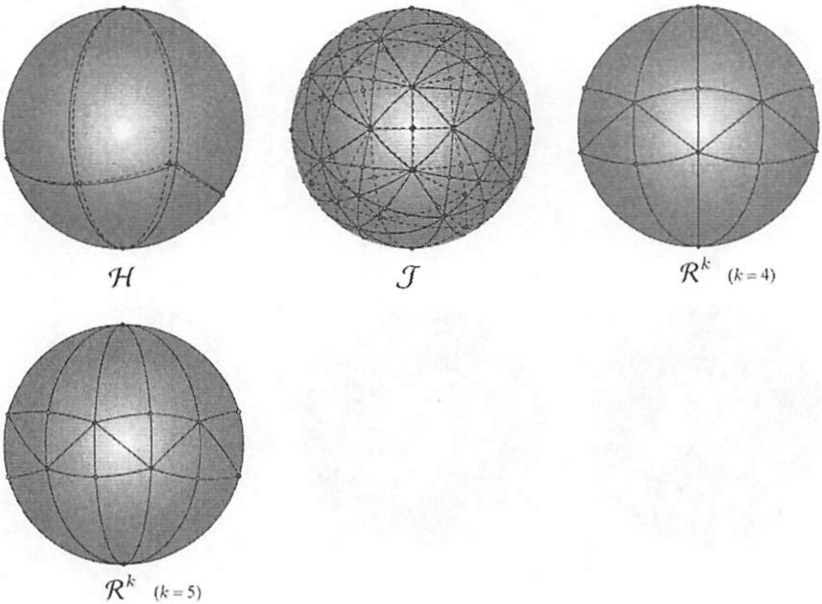


Figure 2. Dihedral f-tilings of the sphere by right triangles $\left(\alpha, \beta, \frac{\pi}{2}\right)$ and $\left(\gamma, \gamma, \frac{\pi}{2}\right)$

Note that the construction of some of these 3D representations, namely $\mathcal{C}$, $\mathcal{N}$, $\mathcal{G}$, $\mathcal{M}$ and $\mathcal{J}$, presented several difficulties inherent to their complexity and absence of symmetries and also great circles of edges contained in these tilings.

In Table 1 is shown the combinatorial structure of the spherical dihedral f-tilings whose prototiles are spherical right triangles $\left(\alpha, \beta, \frac{\pi}{2}\right)$ and $\left(\gamma, \gamma, \frac{\pi}{2}\right)$. Our notation is as follows:

- $\beta_0 \approx 32.6°$ is the solution of (2), with $\alpha = 2\beta$ and $\gamma = \pi - 4\beta$;
- $\gamma_0 \approx 52.2°$ is the solution of (2), with $\alpha = \pi - 2\gamma$ and $\beta = \pi - 3\gamma$;
- $\bar{\beta}_0 \approx 35.9°$ is the solution of (2), with $\alpha = \frac{\pi}{3}$ and $\gamma = \frac{2\pi}{3} - 2\beta$;
- for each $k \geq 3$, $\bar{\bar{\beta}}_0^k$ is the solution of (2), with $\alpha = \frac{\pi-\beta}{2}$ and $\gamma = \frac{\pi-(2k-1)\beta}{2}$;
- $\bar{\bar{\beta}}_0$ is the solution of (3), with $\alpha = \frac{\pi}{6}$ and $\gamma = \pi - 2\beta$;
- $\bar{\gamma}_0 \approx 48.9°$ is the solution of (4), with $\alpha = \frac{\pi}{3}$ and $\beta = \frac{\pi}{2} - \beta$;
- $\bar{\bar{\gamma}}_0$ is the solution of (6), with $\alpha = \frac{\pi}{3}$ and $\beta = \pi - 3\gamma$;
- $|V|$ is the number of distinct classes of congruent vertices;
- N_1 and N_2 are, respectively, the number of triangles congruent to T_1 and T_2, respectively, used in the dihedral f-tilings.
- $G(\tau)$ is the symmetry group of each tiling $\tau \in \Omega\left(T_1, T_2\right)$; C_n is the cyclic group of order n; by D_n we mean the dihedral group of order $2n$; V is the Klein 4-group, isomorphic to $C_2 \times C_2$; S_4 is the symmetry group of

order 24, i.e., the group of orientation preserving symmetries (rotations) of the cube.

f–tiling	α	β	γ	$\lvert V\rvert$	N_1	N_2	$G(\tau)$
$\mathcal{T}_\alpha$	$\left(\frac{\pi}{2}, \frac{3\pi}{4}\right)$	α	$\pi - \alpha$	2	4	4	D_4
$\mathcal{C}$	2β	β_0	$\pi - 4\beta$	5	64	24	C_2
$\mathcal{N}$	$\pi - 2\gamma$	$\pi - 3\gamma$	γ_0	4	16	40	C_1
$\mathcal{G}_{\alpha\beta}$, $\alpha > \beta$	α	$\left(\frac{\pi}{2} - \alpha, \frac{3\pi}{4} - \alpha\right)$	$\left(\frac{\pi}{4}, \frac{\pi}{2}\right)$	3	16	8	C_4
$\mathcal{G}$	$\frac{\pi}{3}$	$\bar{\beta}_0$	$\frac{2\pi}{3} - 2\beta$	4	96	24	D_4
$\mathcal{G}^k$, $k \geq 2$	$\frac{\pi-\beta}{2}$	$\bar{\beta}_0^k$	$\frac{\pi-(2k-1)\beta}{2}$	4	$16(2k - 1)$	8	D_4
$\mathcal{F}_1$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{3}$	5	24	12	V
$\mathcal{F}_2$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{3}$	6	20	14	C_2
$\mathcal{F}_3$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{3}$	7	32	8	V
$\mathcal{F}_4$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{3}$	7	28	10	C_2
$\mathcal{F}_5$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{3}$	7	24	12	V
$\mathcal{F}_6$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{3}$	6	36	6	D_3
$\mathcal{F}_7$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{3}$	4	24	12	D_6
$\mathcal{F}_8$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{3}$	6	16	16	V
$\mathcal{F}_9$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{3}$	5	12	18	D_3
$\mathcal{F}_{10}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{3}$	4	32	8	$C_2 \times D_4$
$\mathcal{F}_{11}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{3}$	3	16	16	$C_2 \times D_4$
$\mathcal{F}_{12}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{3}$	4	24	12	D_6
$\mathcal{D}$	$\frac{\pi}{6}$	$\bar{\bar{\beta}}_0$	$\pi - 2\beta$	4	96	24	D_{12}
$\mathcal{M}$	$\frac{\pi}{3}$	$\frac{\pi}{2} - \gamma$	$\bar{\gamma}_0$	3	48	24	S_4
$\mathcal{H}$	$\frac{\pi}{3}$	$\frac{2\pi}{3}$	$\frac{\pi}{3}$	3	6	4	D_3
$\mathcal{J}$	$\frac{\pi}{3}$	$\pi - 3\gamma$	$\bar{\bar{\gamma}}_0$	4	48	72	D_4
$\mathcal{R}^k$, $k \geq 4$	$\frac{\pi}{k}$	$\pi - 2\gamma$	$\arccos\sqrt{\frac{1}{2}\cos\frac{\pi}{k}}$	3	$4k$	$4k$	D_{2k}

Table 1. Combinatorial structure of the dihedral f–tilings of S^2 by right triangles $\left(\alpha, \beta, \frac{\pi}{2}\right)$ and $\left(\gamma, \gamma, \frac{\pi}{2}\right)$

Note that the angles of each triangular prototile of the monohedral tilings, enumerated by Y. Ueno and Y. Agaoka [9], for instance, are all rationals (except within continuous families). In the dihedral tilings the *relation of adjacency* obtained by two distinct prototiles T_1 and T_2, in adjacent positions, induces a certain condition on the angles of T_1 and T_2 (see e.g. condition (2)). The relation of adjacency does not appear in the case of monohedral tilings, where the determination of the angles is based only on the vertices of the tiling. Such an adjacency relation is behind the appearance of irrational angles in the prototiles of some dihedral f-tilings (Table 1).

3. Case $\alpha = \beta$

In this section, we consider $\alpha = \beta$, with $\alpha \neq \gamma$, $\alpha \neq \frac{\pi}{2}$ and $\gamma \neq \frac{\pi}{2}$ (the cases $\alpha = \frac{\pi}{2}$ and $\gamma = \frac{\pi}{2}$ were studied in [3]). We will assume further, without loss of generality, that $\alpha > \gamma$. By Proposition 1−(iv) and (1), we must have $\gamma \in \left(\frac{\pi}{4}, \frac{\pi}{2}\right)$. We also have $d < a = b$, $0 < d < e < \frac{\pi}{2}$ and $a \neq c$. Moreover, if a vertex v of $\tau \in \Omega(T_1, T_2)$ is surrounded by an angle $\frac{\pi}{2}$, then v is necessarily a vertex of valency four.

Under these assumptions, any element of $\Omega(T_1, T_2)$ has at least two cells such that they are in adjacent positions and in one of the situations illustrated in Figure 3.

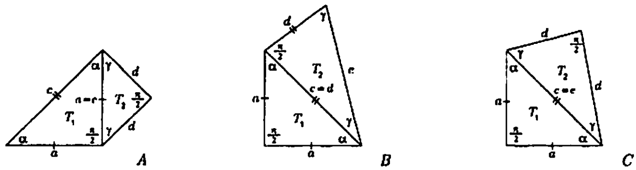


Figure 3. Distinct cases of adjacency when $\alpha = \beta$

Proposition 3 *If $\alpha = \beta$, then $\Omega(T_1, T_2)$ is composed by a continuous family of tilings, denoted by $\mathcal{T}_\alpha$, $\alpha \in \left(\frac{\pi}{2}, \frac{3\pi}{4}\right)$, where $\alpha + \gamma = \pi$ $\left(\alpha \neq \frac{\pi}{2}\right)$. For a planar representation see Figure 6(b). Its 3D representation is given in Figure 2.*

Proof.
1. Suppose firstly that we have two cells in adjacent positions as illustrated in Figure 3 −*A*. With the labelling of Figure 4(a), we must have $\theta_1 = \frac{\pi}{2}$.

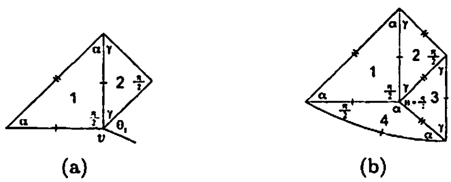
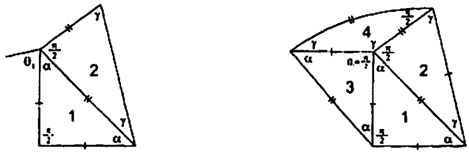


Figure 4. Local configurations

Consequently, $a = e$ and $d = c$ (Figure 4(b)). Using spherical trigonometry, we obtain $\cot\alpha = \cot^2\gamma$ and $\cot\gamma = \cot^2\alpha$. But there is no $\gamma \in \left(\frac{\pi}{4}, \frac{\pi}{2}\right)$ satisfying simultaneously these conditions.

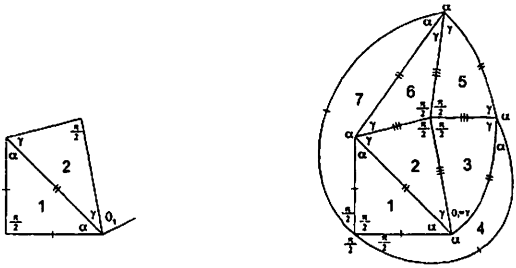
2. Suppose now that we have two cells in adjacent positions as illustrated in Figure 3 −*B*. Analogously to the previous case, we have $\theta_1 = \frac{\pi}{2}$ (Figure 5(a)) and consequently $a = e$ and $d = c$ (Figure 5(b)), which is a contradiction (observe that $\gamma < \frac{\pi}{2} < \alpha$, which leads simultaneously to $c < a$ and $a < c$).
3. Suppose finally that we have two cells in adjacent positions as illustrated in Figure 3 −*C*. As $c = e$, we have $\cot^2\alpha = \cot^2\gamma$, and so $\alpha + \gamma = \pi$. With the labelling of Figure 6(a), we have necessarily $\theta_1 = \gamma$. The



(a) (b)

Figure 5. Local configurations

last configuration is then extended in a unique way to the one given in Figure 6(b). We shall denote such f-tiling by $\mathcal{T}_\alpha$, where $\alpha \in \left(\frac{\pi}{2}, \frac{3\pi}{4}\right)$. Its 3D



(a)

(b) Planar representation of $\mathcal{T}_\alpha$

Figure 6. Local configurations

representation is shown in Figure 2. □

4. Case $\alpha \neq \beta$

Now, we consider $\alpha \neq \beta$, with $\alpha + \beta > \frac{\pi}{2}$, $\gamma > \frac{\pi}{4}$ and $\gamma \neq \frac{\pi}{2}$ (the case $\gamma = \frac{\pi}{2}$ was studied in [3]). Recall that we must have $\alpha + \beta \leq \pi$ or $\gamma < \frac{\pi}{2}$ (Proposition 1–(iv)). We also have $a \neq b$ and $d \neq e$. Under these assumptions, any element of $\Omega\left(T_1, T_2\right)$ has at least two cells such that they are in adjacent positions and in one of the situations illustrated in Figure 7. In the following subsections we will consider separately these distinct cases of adjacency.

4.1. *Case of Adjacency A*

We begin with the following result that states some conditions on the angles and lengths of the sides of T_1 and T_2 in the considered case of

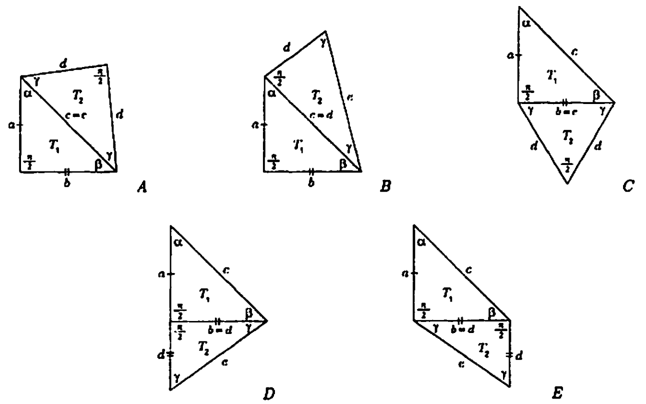


Figure 7. Distinct cases of adjacency when $\alpha \neq \beta$

adjacency.

Lemma 4 *Suppose that there are two cells in adjacent positions as illustrated in Figure 7–A. Then,*

(i)

$$\cot\alpha\cot\beta = \cot^2\gamma; \tag{2}$$

(ii) $a, b, \alpha, \beta \neq \frac{\pi}{2}$*;*

(iii) $a, b, c \neq d$*.*

(iv) the three edges of T_1 *must have different lengths.*

Proof.

(i) It follows immediately observing that $\cos c = \cot\alpha\cot\beta$ and $\cos e = \cot^2\gamma$ (with $c = e$).

(ii) As $\cos a = \frac{\cos\beta}{\sin\alpha}$ and $\cos b = \frac{\cos\alpha}{\sin\beta}$, if $a = \frac{\pi}{2}$, $b = \frac{\pi}{2}$, $\alpha = \frac{\pi}{2}$ or $\beta = \frac{\pi}{2}$, we obtain $\cos c = \cos a\cos b = 0$, i.e., $c = \frac{\pi}{2}$. Using the condition of adjacency $c = e$, we get $\cos e = \cot^2\gamma = 0$, which implies $\gamma = \frac{\pi}{2}$, that is an impossibility.

(iii) If $a = d$ (note that $c = e$), we have $\beta = \gamma$ or $\beta + \gamma = \pi$ (using sine rule). On the other hand, from (i) and using the fact that $\cot\gamma = \frac{\cos\beta}{\sin\alpha}$, we obtain $\sin(2\alpha) = \sin(2\beta)$, and so $\alpha + \beta = \frac{3\pi}{2}$. By Proposition 1–(iv), we must have $\gamma < \frac{\pi}{2}$. Hence, we conclude that $\beta + \gamma = \pi$ (otherwise $\alpha > \pi$). Replacing this equality in (i), we get $\alpha + \gamma = \pi$ and consequently $\alpha = \beta$, which is an impossibility. Analogously we prove that $b \neq d$. $c = d$ implies $d = e$, and so T_2 is equilateral, which is also an impossibility.

(iv) If $a = b$, then $\alpha = \beta$, which is an absurdity. On the other hand, we have $a \neq c$ and $b \neq c$, otherwise we obtain $\beta = \frac{\pi}{2}$ and $\alpha = \frac{\pi}{2}$, respectively. □

Suppose that any element of $\Omega(T_1, T_2)$ has at least two cells congruent, respectively, to T_1 and T_2, such that they are in adjacent positions as illustrated in Figure 8. Taking into account the results of Lemma 4 and with

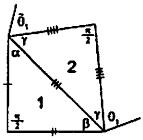


Figure 8. Local configuration

the labelling of this figure, we have

$$\theta_1 \in \left\{\frac{\pi}{2}, \gamma\right\} \quad \text{and} \quad \tilde{\theta}_1 \in \left\{\frac{\pi}{2}, \gamma\right\}.$$

These possibilities for θ_1 and $\tilde{\theta}_1$ will be now analyzed.

Proposition 5 *With the above terminology, if $\Omega(T_1, T_2) \neq \emptyset$, then $\theta_1 = \gamma$ and $\tilde{\theta}_1 = \gamma$.*

Proof. Suppose that we have two cells in adjacent positions as illustrated in Figure 8 and $\theta_1 = \frac{\pi}{2}$ (Figure 9).

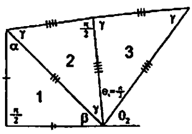
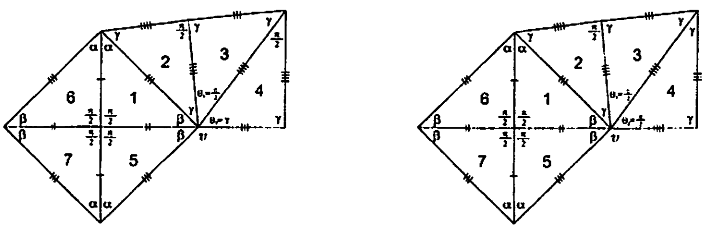


Figure 9. Local configuration

We have $\beta \neq \frac{\pi}{2}$ (Lemma 4(ii)) and so $\beta + \frac{\pi}{2} < \pi$. Consequently, we have $\beta < \frac{\pi}{4} < \alpha$ (recall that $\alpha + \beta > \frac{\pi}{2}$). Now, θ_2 must be γ or $\frac{\pi}{2}$.

1. If $\theta_2 = \gamma$, we obtain the configuration illustrated in Figure 10(a). Vertex v



(a) (b)

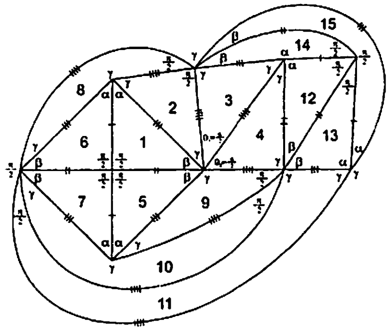
Figure 10. Local configurations

cannot have valency six (see edge lengths), and so we must have $\frac{\pi}{2} + \beta + k\beta =$

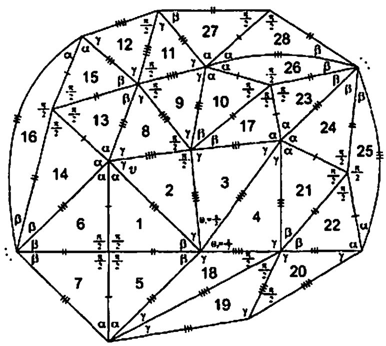
π, $k \geq 1$, or $\frac{\pi}{2}+\beta+\gamma+k\beta=\pi$, $k \geq 0$. But in both cases an incompatibility between sides takes place at this vertex.

2. Suppose now that $\theta_2 = \frac{\pi}{2}$ (Figure 10(b)). At vertex v we must have $\frac{\pi}{2}+\gamma+k\beta=\pi$, with $k \geq 1$, which implies $\beta < \frac{\pi}{4} < \gamma < \alpha$.

If $\alpha+\gamma=\pi$, it follows that $k=1$ and we get the configuration illustrated in Figure 11(a). Although a complete planar representation was possible to



(a)



(b)

Figure 11. Local configurations

draw, we may conclude that such a configuration cannot be realized by a f-tiling since there is no spherical triangles satisfying simultaneously the relations that come from Figure 11(a) and relation (2).

If $\alpha+\gamma<\pi$, the last configuration is extended in a unique way to the one illustrated in Figure 11(b). Note that, at vertex v, we have necessarily $\alpha+\alpha+\gamma=\pi$, implying that $k=1$ ($\frac{\pi}{2}+\gamma+\beta=\pi$). We also have $\bar{k}\beta=\pi$, for some $\bar{k} \geq 5$. Using (2), we conclude that there is no integer $\bar{k}$ satisfying such condition, and so we reach a contradiction. Therefore, $\theta_1 = \gamma$ (Figure 8). Similarly we prove that $\tilde{\theta}_1 = \gamma$. □

By proposition 5, the configuration illustrated in Figure 8 extends to the one that follows.

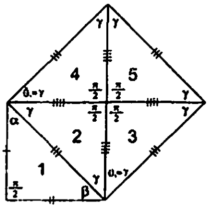


Figure 12. Local configuration

Now, we must obtain all the f-tilings with prototiles T_1 and T_2 that follows from this configuration. Without loss of generality, we may suppose that $\alpha > \beta$.

Proposition 6 *If there are five cells in adjacent positions as illustrated in Figure 12, then $\Omega(T_1, T_2) \neq \emptyset$ iff*

(i) $2\alpha + \gamma = \pi$ *and* $\gamma + 2\beta + \alpha = \pi$ *or*
(ii) $\alpha + 2\gamma = \pi$ *and* $3\gamma + \beta = \pi$, *or*
(iii) $\alpha + \gamma + \beta = \pi$, *with* $\alpha + \beta \in \left(\frac{\pi}{2}, \frac{3\pi}{4}\right)$, $\alpha, \beta \neq \frac{\pi}{2}$, *or*
(iv) $\alpha + \gamma + 2\beta = \pi$ *and* $3\alpha = \pi$, *or*
(v) $\alpha + \gamma + k\beta = \pi$, $k \geq 2$, *and* $2\alpha + \beta = \pi$.

The first case leads to a unique dihedral f-tiling, denoted by C. A planar representation is given in Figure 18. For its 3D representation see Figure 2.

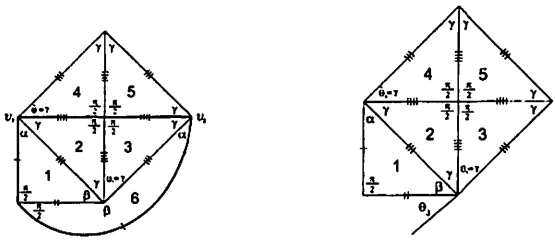
The case (ii) leads also to a unique f-tiling, denoted by $\mathcal{N}$. A planar representation is given in Figure 27. A 3D representation is given in Figure 2.

In the case (iii) there is a family of f-tilings, with two continuous parameters, denoted by $\mathcal{G}_{\alpha\beta}$, with $\alpha + \beta \in \left(\frac{\pi}{2}, \frac{3\pi}{4}\right)$. In Figure 28 is given the corresponding planar representation. A 3D representation is given in Figure 2.

The fourth case leads to a unique f-tiling, denoted by $\mathcal{G}$. A planar representation is given in Figure 30. A 3D representation is given in Figure 2. In the last situation, for each $k \geq 2$ there is an f-tiling, denoted by $\mathcal{G}^k$. A planar representation is given in Figure 34. For their $k = 2$ and $k = 3$ 3D representations see Figure 2.

Proof. Suppose that we have five cells in adjacent positions as illustrated in Figure 12.

If $\beta + \gamma = \pi$, we obtain the configuration illustrated in Figure 13(a). Observing that the vertices v_1 and v_2 at the dark line are in antipodal positions, we conclude that the distance between v_1 and v_2 is $2d = \pi$, i. e., $d = \frac{\pi}{2}$, and so $\gamma = \frac{\pi}{2}$, which is an impossibility.



(a) (b)

Figure 13. Local configurations

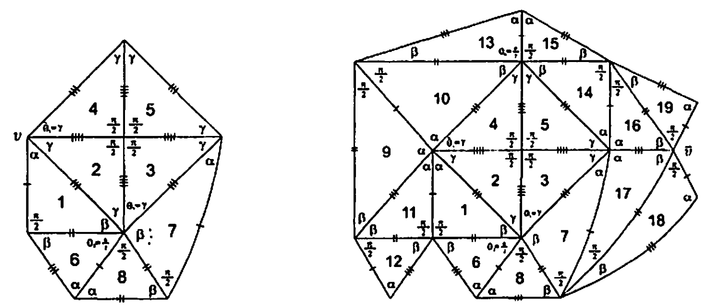
Thus $\beta + \gamma < \pi$ and similarly $\alpha + \gamma < \pi$. Now, we consider separately the cases $\alpha + \beta \geq \pi$ and $\alpha + \beta < \pi$.

1. Suppose firstly that $\alpha + \beta \geq \pi$. If $\alpha > \frac{\pi}{2}$ and $\beta > \frac{\pi}{2}$ (by Lemma 4(ii), $\alpha, \beta \neq \frac{\pi}{2}$), then the local configuration illustrated in Figure 8 cannot be extended. Thus, we shall suppose without loss of generality, that $\alpha > \frac{\pi}{2} > \beta$. Now, one has $\cos e = \cos c = \dfrac{\cos\alpha\cos\beta}{\sin\alpha\sin\beta} < 0$. Hence $e > \frac{\pi}{2}$, and so $\gamma > \frac{\pi}{2}$. It follows an incompatibility at the vertex surrounded by α and γ.

2. Suppose now that $\alpha + \beta < \pi$. With the labelling of Figure 13(b), we have

$$\theta_3 = \frac{\pi}{2} \quad \text{or} \quad \theta_3 = \beta.$$

2.1 If $\theta_3 = \frac{\pi}{2}$, then $\frac{\pi}{2} + \gamma + k\beta = \pi$, for some $k \geq 1$, and hence $\beta < \frac{\pi}{4} < \gamma < \alpha$. We get the configuration illustrated in Figure 14(a). At vertex v we



(a) (b)

Figure 14. Local configurations

have

$$\alpha+\gamma+\alpha=\pi \qquad \text{or} \qquad \alpha+\gamma+\gamma=\pi \qquad \text{or} \qquad \alpha+\gamma+\bar{k}\beta=\pi, \ \ \bar{k}\geq 1.$$

(i) If $\alpha+\gamma+\alpha=\pi$, then we have necessarily $k=1$. And so $\gamma+\beta=\pi$. The last configuration is extended in a unique way to the one illustrated in Figure 14(b).

Note that θ_4 cannot be β in tile 13. In fact, observing Figure 15(a) and taking into account the length sides, we must have $x=\beta$. Therefore, $\frac{\pi}{2}+t\beta=\pi$, for some $t\geq 3$, leading us to conclude that $\beta\leq\frac{\pi}{6}$. On the other hand, $\alpha=\frac{\pi}{4}+\frac{\beta}{2}$ ($2\alpha+\gamma=\pi$ and $\frac{\pi}{2}+\gamma+\beta=\pi$). Now, $\pi=2\alpha+\gamma<3\alpha=\frac{3\pi}{4}+\frac{3\beta}{2}\leq\pi$, which is an impossibility.

At vertex $\bar{v}$, we obtain $\frac{\pi}{2}+\beta+\rho>\pi$, in which ρ must be α or $\frac{\pi}{2}$ attending the length sides, and so we reach a contradiction.

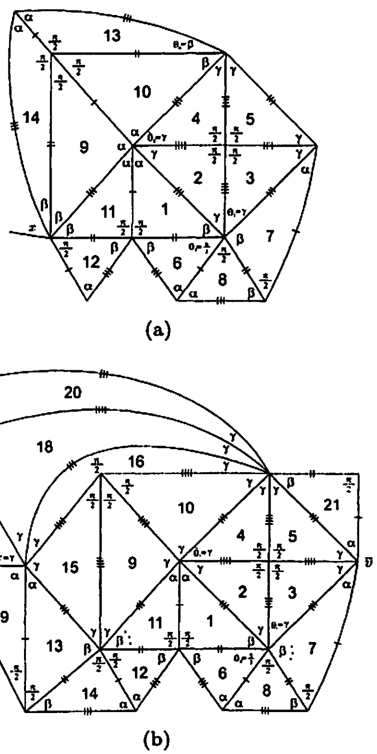


Figure 15. Local configurations

(ii) If $\alpha+\gamma+\gamma=\pi$, then the last configuration is extended in a unique way to the one illustrated in Figure 15(b) (if we consider $x=\beta$ in tile 17 we would obtain $\alpha=\frac{\pi}{2}$, which is a contradiction). At vertex $\bar{v}$, we have $\alpha+\gamma+\rho>\pi$, in which ρ must be α or $\frac{\pi}{2}$, and so we reach a contradiction.
(iii) If $\alpha+\gamma+\bar{k}\beta=\pi$, $\bar{k}\geq 1$, the last configuration is extended in a unique way to the one illustrated in Figure 16(a). At vertex $\bar{v}$, we have $\alpha+\frac{\pi}{2}<\pi$ and $\alpha+\gamma+\rho>\pi$ $\forall\rho\in\{\alpha,\beta,\gamma,\frac{\pi}{2}\}$, which is an impossibility.

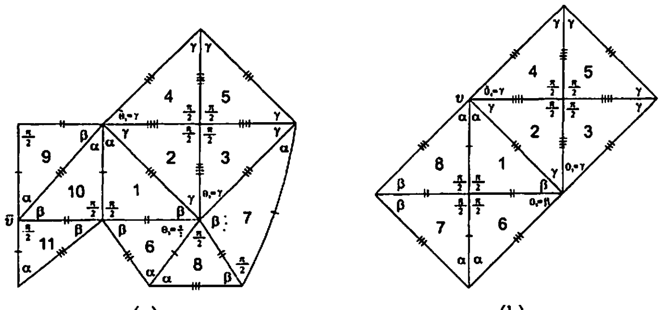


Figure 16. Local configurations

2.2 Suppose now that $\theta_3=\beta$ (Figure 13(b)). As $\beta<\alpha$, we have $\alpha>\frac{\pi}{4}$, which implies $\frac{\pi}{2}+\alpha+\rho>\pi$, $\forall\rho\in\{\alpha,\beta,\gamma,\frac{\pi}{2}\}$, and so tile 7 is completely determined as illustrated in Figure 16(b). Now, at vertex v, we have

$$\alpha+\gamma+\alpha=\pi \qquad \text{or} \qquad \alpha+\gamma+\gamma=\pi \qquad \text{or} \qquad \alpha+\gamma+k\beta=\pi,\quad k\geq 1.$$

(i) If $\alpha+\gamma+\alpha=\pi$, then the last configuration is extended to the one given in Figure 17(a). Now, one has $\frac{\pi}{2}>\alpha$, $\gamma>\frac{\pi}{4}$ and, at vertex $\bar{v}$, we have

$$\begin{gathered}
\gamma+k\beta=\pi,\ k\geq 3,\ \text{ or}\\
\gamma+k_1\beta+\tfrac{\pi}{2}+k_2\beta=\pi,\ k_1\geq 1,\ k_2\geq 0,\ \text{ or}\\
\gamma+k_1\beta+\alpha+k_2\beta=\pi,\ k_1\geq 1,\ k_2\geq 0,\ \text{ or}\\
\gamma+k_1\beta+\gamma+k_2\beta=\pi,\ k_1\geq 1,\ k_2\geq 0,\ \text{ or}\\
\gamma+k_1\beta+\gamma+k_2\beta+\gamma+k_3\beta=\pi,\ k_1\geq 1,\ k_2,\ k_3\geq 0.
\end{gathered}$$

(a) If $\gamma+k\beta=\pi$, $k\geq 3$, then $\beta<\frac{\pi}{4}$, and the last configuration is extended in a unique way to the one illustrated in Figure 17(b). By proposition 5, we have $\theta_4=\gamma$. Now, $\gamma+\gamma+\beta\leq\pi$, and so $\gamma+\gamma+\bar{k}\beta=\pi$ $(1\leq\bar{k}<k)$ or $\gamma+\gamma+\gamma+\bar{\bar{k}}\beta=\pi$ ($\bar{\bar{k}}\geq 1$ and $k\geq 4$, with $\bar{\bar{k}}<k$). In the first case it

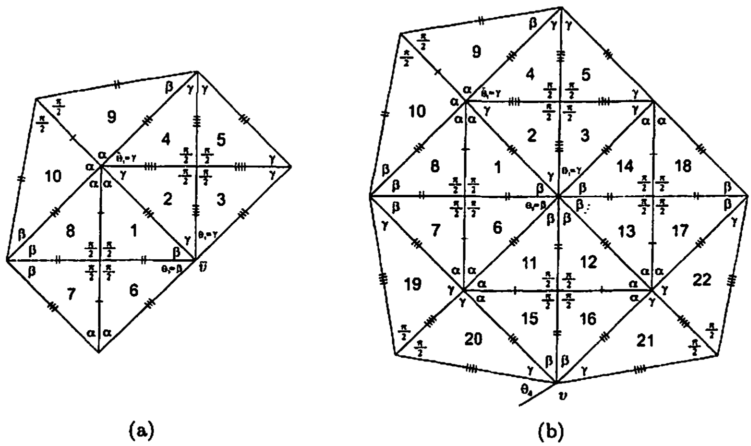


Figure 17. Local configurations

follows that $\gamma = (k - \bar{k})\beta$, while in the second case we obtain $\gamma = \frac{k-\bar{\bar{k}}}{2}\beta$. However, in any case the obtained angles do not satisfy (2).

(b) If $\gamma + k_1\beta + \frac{\pi}{2} + k_2\beta = \pi$, $k_1 \geq 1, k_2 \geq 0$, then we get an impossibility since there is no way to mark the sides around $\tilde{v}$ (Figure 17(b)).

(c) Suppose now that $\gamma + k_1\beta + \alpha + k_2\beta = \pi$, with $k_1 \geq 1$ and $k_2 \geq 0$. We have necessarily $k_1 + k_2 < 3$, and, as $k_1 + k_2 = 1$ implies $\alpha = \beta$, we conclude that $k_1 + k_2 = 2$. Thus, the last configuration extends in a unique way to the one illustrated in Figure 18. Note that we consider $\theta_4 = \alpha$ and $\theta_6 = \beta$ (tile 11 and 21, respectively). If $\theta_4 = \beta$ and/or $\theta_6 = \alpha$, we achieve, by symmetry, the same tiling. The same applies to similar choices $((\gamma, \gamma, \beta, \beta, \alpha, \alpha, \beta, \beta)$ or $(\gamma, \gamma, \beta, \beta, \beta, \beta, \alpha, \alpha))$. Note that we have used the fact that we cannot have $\gamma + \beta + \beta + \gamma + \bar{k}\beta = \pi$ nor $\gamma + \beta + \beta + \gamma + \gamma + \hat{k}\beta = \pi$, $\bar{k} \geq 1$ and $\hat{k} \geq 0$.

Using equation (2), we get $\beta \approx 32.6°$ (all the angles are completely determined). We shall denote this f-tiling by C. A 3D representation of C is given in Figure 2.

(d) Suppose that $\gamma + k_1\beta + \gamma + k_2\beta = \pi$, with $k_1 \geq 1$ and $k_2 \geq 0$. If $k_2 \geq 1$, we obtain the configuration illustrated in Figure 19 (observe that if $x = \beta$ (tile 19), then we have the equalities $2\alpha + \gamma = \pi$, $\alpha + \gamma + 2\beta = \pi$ and $2\gamma + n\beta = \pi$, with $n \geq 2$, which lead to an incompatibility). We get $k\beta = \pi$, for some $k \geq 5$. Now, as $\alpha + \beta > \frac{\pi}{2}$, then $\alpha > \frac{3\pi}{5}$, and so $2\alpha + \gamma > \pi$, which is an impossibility.

Consider now $k_2 = 0$. We separate the cases

$$\gamma + \beta + \gamma = \pi \quad (k_1 = 1) \qquad \text{and} \qquad \gamma + k_1\beta + \gamma = \pi \quad (k_1 \geq 2).$$

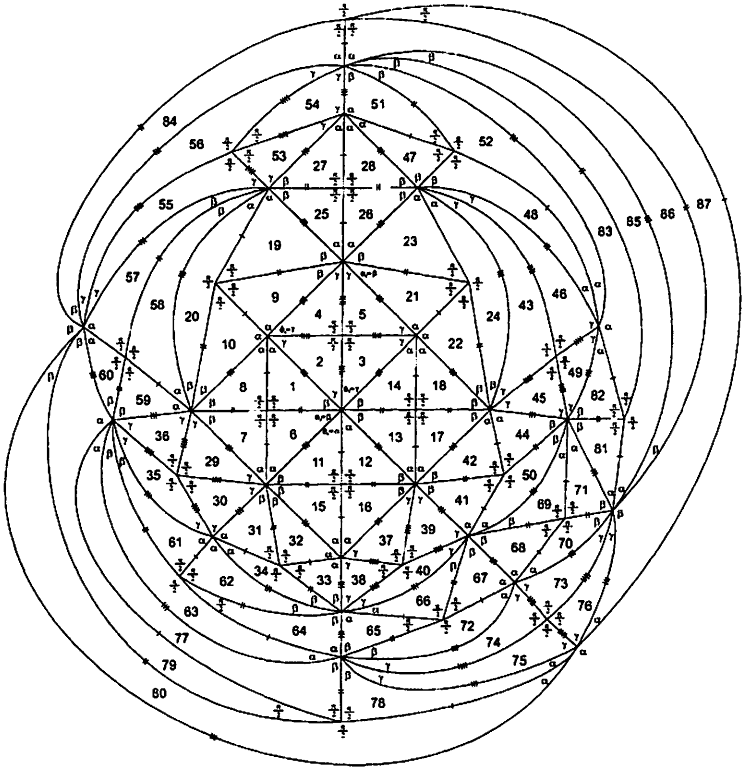


Figure 18. Planar representation of C

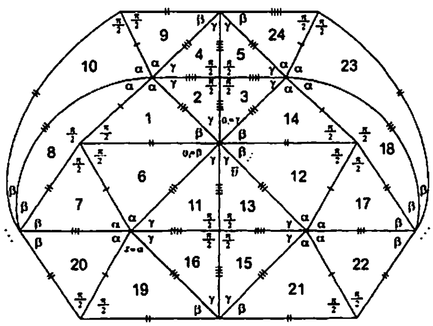
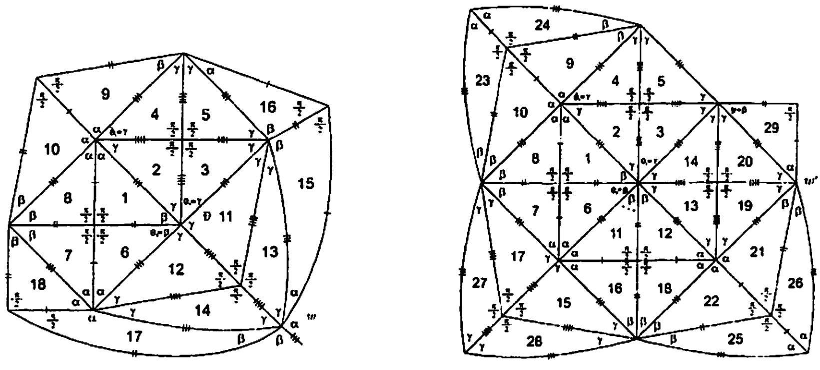


Figure 19. Local configuration

In the first case we have $\gamma > \alpha$ and we obtain the local configuration illustrated in Figure 20(a). At vertex w we obtain $\alpha + \gamma + \beta = \pi$. Hence



(a) (b)

Figure 20. Local configurations

$\alpha = \gamma = \beta$, that is an impossibility.

In the second case we have $\alpha > \gamma$ and we obtain the local configuration illustrated in Figure 20(b). Observe that:

- if $y = \gamma$ (tile 29), then $3\gamma + m\beta = \pi$ $(m \geq 1)$, and so $\gamma \geq 2\beta$; we get an impossibility as $2\alpha + \gamma = \pi$ and $\alpha + \beta > \frac{\pi}{2}$;
- if $y = \alpha$, then $2\gamma + \alpha < \pi$ and $2\gamma + \alpha + \rho > \pi$, $\forall \rho \in \{\alpha, \beta, \gamma\}$, which is also an impossibility.

At vertex w' we obtain $\alpha + \gamma + \beta < \pi$. Therefore, $\alpha + \gamma + 2\beta = \pi$. As seen before, it is incompatible with the equalities $2\alpha + \gamma = \pi$ and $2\gamma + k_1\beta = \pi$ $(k_1 \geq 2)$.

(e) Finally we consider $\gamma + k_1\beta + \gamma + k_2\beta + \gamma + k_3\beta = \pi$, with $k_1 \geq 1$ and $k_2, k_3 \geq 0$ (this condition implies $\alpha > \gamma$). It follows that $k_1 = 1$ and $k_2 = k_3 = 0$. Consequently, $\beta = 6\alpha - 2\pi$ and $\gamma = \pi - 2\alpha$, and from (2) we get $\csc(6\alpha)\left(\cos(4\alpha) + 2\sin^2(3\alpha)\right) = 0$. And so $\alpha = \arccos \frac{\sqrt{5-\sqrt{13}}}{2\sqrt{2}} \approx 65.3°$. Thus, the angles and edge lengths of T_1 and T_2 are completely determined and the configuration illustrated in Figure 17(a) is extended in a unique way to the one given in Figure 21. We obtain $k\beta = \pi$, for some $k \geq 5$. Nevertheless, there is no integer k satisfying $k(6\alpha - 2\pi) = \pi$.

(ii) If $\alpha + \gamma + \gamma = \pi$, then the configuration illustrated in Figure 16(b) is extended to the one given in Figure 22(a). At vertex $\bar{v}$, we have

$$\gamma + k\beta = \pi, \quad k \geq 2, \quad \text{or}$$
$$\gamma + \beta + k\alpha = \pi, \quad k \geq 1, \quad \text{or}$$
$$\gamma + k_1\beta + \tfrac{\pi}{2} + k_2\beta = \pi, \quad k_1 \geq 1, \ k_2 \geq 0, \quad \text{or}$$
$$\gamma + k_1\beta + \alpha + k_2\beta = \pi, \quad k_1 \geq 1, \ k_2 \geq 0, \quad \text{or}$$

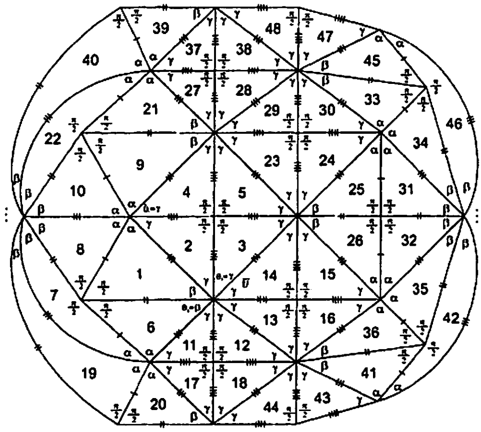
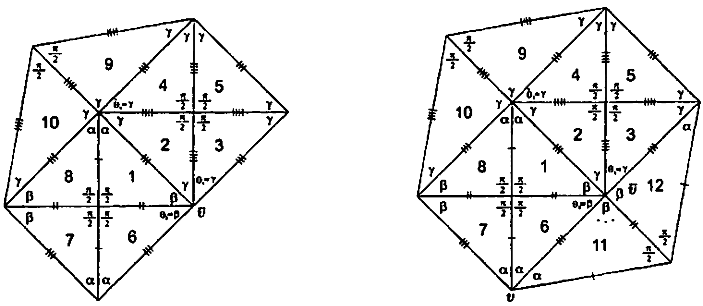


Figure 21. Local configuration



(a) (b)

Figure 22. Local configurations

$$\gamma + k_1\beta + \gamma + k_2\beta = \pi, \ k_1 \geq 1, \ k_2 \geq 0, \ \text{ or}$$
$$\gamma + k_1\beta + \gamma + k_2\beta + \gamma + k_3\beta = \pi, \ k_1 \geq 1, \ k_2, \ k_3 \geq 0.$$

(a) If $\gamma + k\beta = \pi$, $k \geq 2$, then we obtain the configuration illustrated in Figure 22(b). If $k = 2$, $\gamma + \beta + \beta = \pi$, and so $\beta > \gamma > \frac{\pi}{4}$ (recall that $\beta < \alpha$). Now, at vertex v we have no way to satisfy the angle folding relation. On the other hand, if $k \geq 3$, we have $\gamma > \frac{\pi}{4} > \beta$, and so at vertex v we get $\alpha + \alpha + \beta = \pi$. It follows that $\alpha = \frac{2k-1}{4k+1}\pi$, $\beta = \frac{3\pi}{4k+1}$ and $\gamma = \frac{k+1}{4k+1}\pi$. By

(2), we obtain a contradiction.

(b) Consider $\gamma + \beta + k\alpha = \pi$, with $k \geq 1$. As $\beta < \alpha$ and $\gamma > \frac{\pi}{4}$, we must have $k = 1$. It follows that $\beta = \gamma$, and by (2), $\gamma = \alpha$, which is not possible.

(c) Consider $\gamma + k_1\beta + \frac{\pi}{2} + k_2\beta = \pi$, with $k_1 \geq 1$ and $k_2 \geq 0$. We have $\beta < \frac{\pi}{4} < \gamma < \alpha$, and consequently we obtain the configuration illustrated in Figure 23(a). At vertex v we have $\alpha + \frac{\pi}{2} + \rho > \pi$, $\forall \rho \in \{\alpha, \beta, \gamma, \frac{\pi}{2}\}$, and so we reach a contradiction.

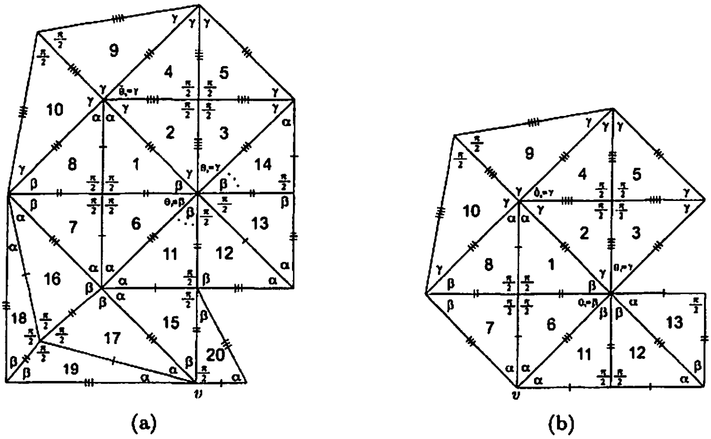


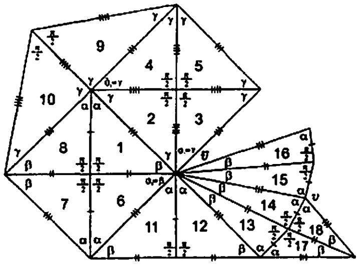
Figure 23. Local configurations

(d) Suppose that $\gamma + k_1\beta + \alpha + k_2\beta = \pi$, with $k_1 \geq 1$, $k_2 \geq 0$ and $k_1 + k_2 \geq 2$ (note that $k_1 + k_2 = 1$ implies $\alpha = \beta = \gamma$, which is not possible).

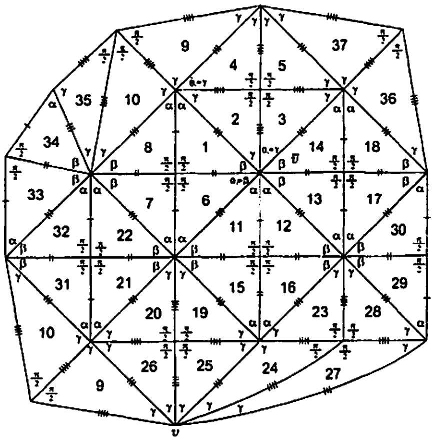
If $k_1 > 1$, we obtain the configuration illustrated in Figure 23(b). As $\beta < \frac{\pi}{4} < \gamma < \alpha$, at vertex v we have necessarily $\alpha + \alpha + \beta = \pi$, and so $\alpha = \pi - 2\gamma$ and $\beta = 4\gamma - \pi$. Using (2), we get $\gamma = \frac{\pi}{3}$, which is an impossibility. Thus, $k_1 = 1$ and at any vertex we cannot have two angles α in the same alternated sum.

Now, if $k_2 > 1$, we obtain the configuration illustrated in Figure 24(a), and consequently a contradiction at vertex v. Therefore, $k_1 = k_2 = 1$ and the angles are completely determined by (2) ($\beta \approx 26.5°$, $\gamma = 2\beta$ and $\alpha = \pi - 4\beta$). The last configuration (Figure 22(a)) is then extended in a unique way to the one illustrated in Figure 24(b). Nevertheless, there is no way to satisfy the angle-folding relation around vertex v (see angles measure).

(e) Consider $\gamma + k_1\beta + \gamma + k_2\beta = \pi$, with $k_1 \geq 1$, $k_2 \geq 0$. It follows immediately that $k_1 + k_2 \geq 2$ ($\alpha > \beta$). Analogously to the previous case, one can show that $k_1 = k_2 = 1$, and so $\beta = \arctan\frac{\sqrt{2}}{2}$ (by (2)). The configuration illustrated in Figure 22(a) is extended to the one given in



(a)



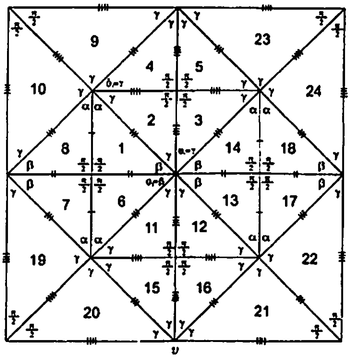
(b)

Figure 24. Local configurations

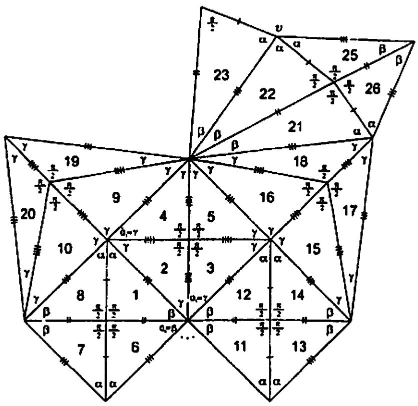
Figure 25(a). Note that $\alpha+\gamma+\beta=\alpha+\frac{\pi}{2}<\pi$ and $\alpha+\gamma+\beta+\rho>\pi$, $\forall\rho\in\{\alpha,\beta,\gamma,\frac{\pi}{2}\}$, and so tiles 19, 21 and 23 are completely determined. At vertex v, we have $\gamma+\gamma+\gamma<\pi$, but $\gamma+\gamma+\gamma+\rho>\pi$, $\forall\rho\in\{\alpha,\beta,\gamma,\frac{\pi}{2}\}$.

(f) Finally, suppose that $\gamma+k_1\beta+\gamma+k_2\beta+\gamma+k_3\beta=\pi$, with $k_1\geq 1$ and $k_2,k_3\geq 0$. Therefore $\alpha>\gamma>\frac{\pi}{4}>\beta$. Analogously to the previous cases, we must have $k_1=1$ and at any vertex we cannot have two angles α in the same alternated sum. Moreover, $k_2,k_3\leq 1$ (as seen before). Now, we consider separately the different arrangements for k_2 and k_3.

The cases $k_3=1$ (with $k_2=0$ or $k_2=1$), and $k_2=1$ and $k_3=0$, give rise to the configurations illustrated in Figure 25(b) and Figure 26, respectively. At vertex v we have $\alpha+\alpha<\pi$ and so $\alpha+\alpha+\beta=\pi$ which implies $\gamma=\frac{\pi}{3}$ (by (2)), which is an impossibility.



(a)



(b)

Figure 25. Local configurations

If $k_2 = k_3 = 0$, then we get the configuration illustrated in Figure 27. Note that, by symmetry, if $\theta_4 = \alpha$ or $\theta_4 = \beta$, we obtain the same f-tiling. We shall denote such f-tiling by $\mathcal{N}$. Its 3D representation is shown in Figure 2. We have $\gamma \approx 52.2°$, $\alpha = \pi - 2\gamma$ and $\beta = \pi - 3\gamma$ (by (2)).

(iii) Consider now $\alpha + \gamma + k\beta = \pi$, $k \geq 1$.

(iii.1) If $k = 1$, then the configuration illustrated in Figure 16(b) is extended to the one given in Figure 28. Note that $\beta \neq \gamma$ and $\alpha \neq \gamma$, otherwise we get, by (2), $\alpha = \beta = \frac{\pi}{3}$, which is not possible (tiles 13 and 21 are then completely determined). We have $\alpha + \beta \in \left(\frac{\pi}{2}, \frac{3\pi}{4}\right)$, with $\alpha > \beta$, $\alpha, \beta \neq \frac{\pi}{2}$, and $\gamma \in \left(\frac{\pi}{4}, \frac{\pi}{2}\right)$. We shall denote such f-tiling by $\mathcal{G}_{\alpha\beta}$. Its 3D representation

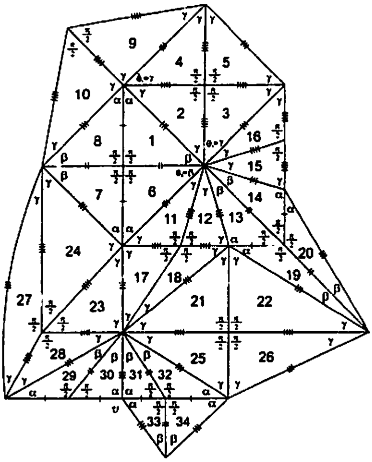


Figure 26. Local configuration

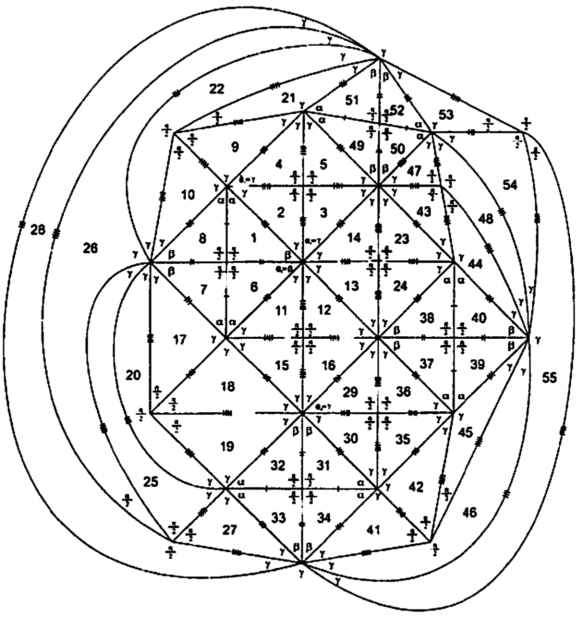


Figure 27. Planar representation of $\mathcal{N}$

is shown in Figure 2.

(iii.2) If $k = 2$, then we get the configuration illustrated in Figure 29 (in this case we obtain $\frac{\pi}{2} > \alpha > \gamma > \frac{\pi}{4} > \beta$). At vertex $\bar{v}$, we have

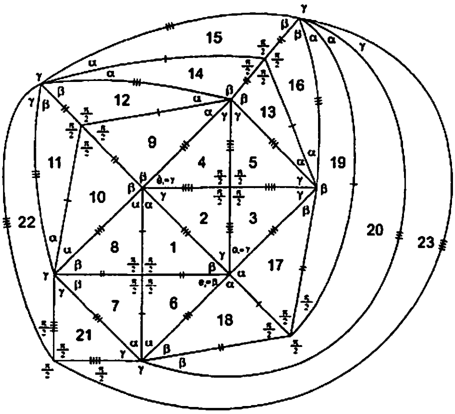


Figure 28. Planar representation of $\mathcal{G}_{\alpha\beta}$, $\alpha+\beta \in \left(\frac{\pi}{2}, \frac{3\pi}{4}\right)$, $\alpha > \beta$

$$\alpha+\alpha+\alpha=\pi, \qquad \alpha+\alpha+\gamma=\pi \quad \text{or} \quad \alpha+\alpha+\beta=\pi.$$

(iii.2.1) If $\alpha+\alpha+\alpha=\pi$, i.e., $\alpha=\frac{\pi}{3}$, then we get the configuration illustrated

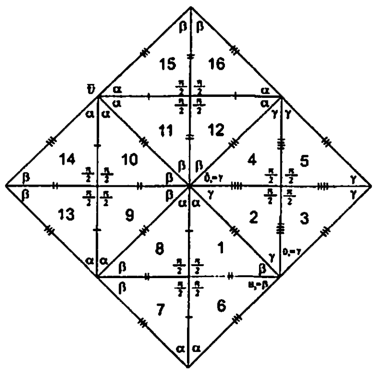


Figure 29. Local configuration

in Figure 30 (note that $\alpha+\beta+\beta+\bar{k}\beta=\pi$ implies $\bar{k}=1$, which is not possible). We shall denote such f-tiling by $\mathcal{G}$. Its 3D representation is shown in Figure 2. We have $\alpha=\frac{\pi}{3}$, $\beta \approx 35.9°$ and $\gamma=\frac{2\pi}{3}-2\beta$ (by (2)).

(iii.2.2) If $\alpha+\alpha+\gamma=\pi$, we get the configuration illustrated in Figure 31(a). At vertex v, we have $\alpha+\gamma+\alpha=\pi$ or $\alpha+\gamma+2\beta=\pi$.

(iii.2.2.1) If $\alpha+\gamma+\alpha=\pi$, at vertex v' (Figure 31(b)) we have $\gamma+\beta+\beta+\alpha=\pi$ or $\gamma+\beta+\beta+\beta+\beta=\pi$ (observe that $\alpha=2\beta$).

(iii.2.2.1.1) If $\gamma+\beta+\beta+\alpha=\pi$, the last configuration is extended to the one given in Figure 32(a). Now, if $\theta_4=\beta$ we obtain another complete

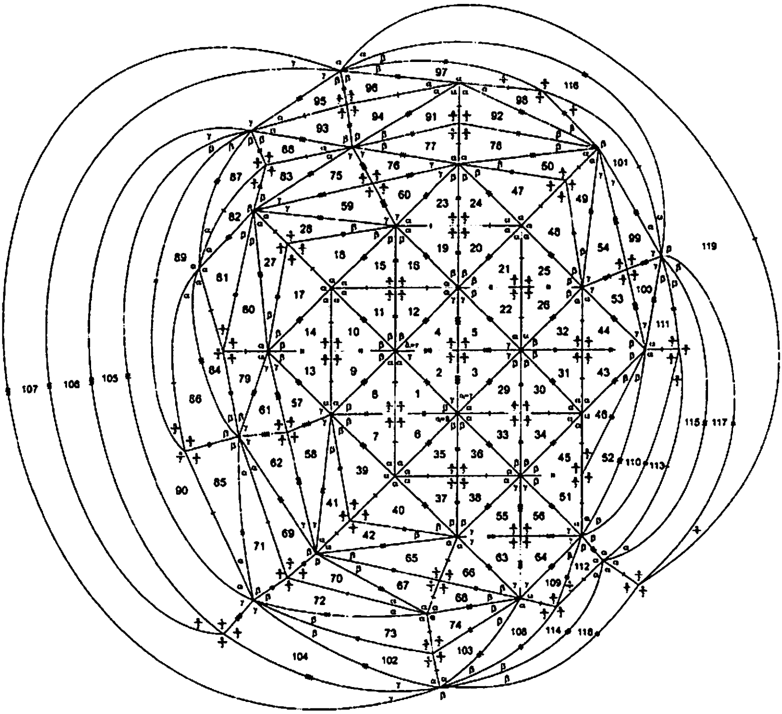
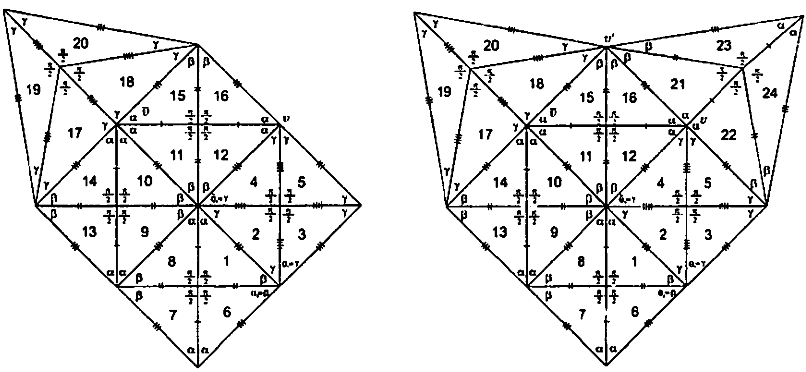


Figure 30. Planar representation of $\mathcal{G}$



(a) (b)

Figure 31. Local configurations

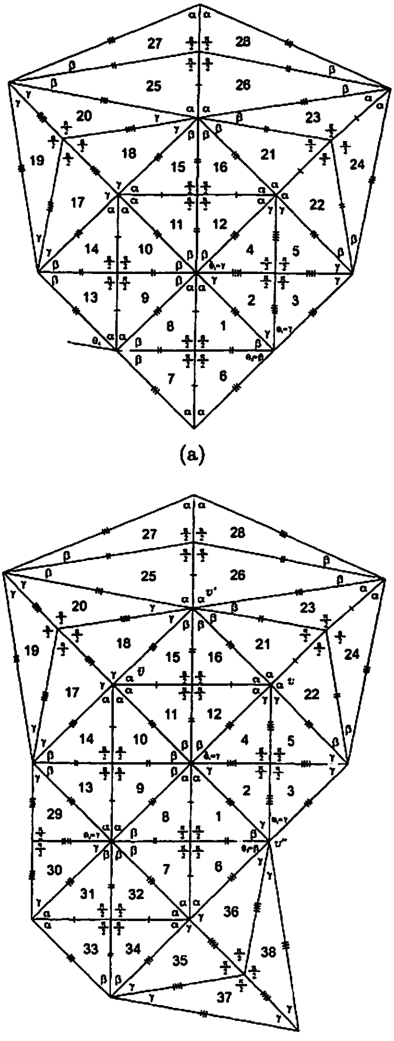


Figure 32. Local configurations

planar representation of the previous f-tiling $\mathcal{C}$. Otherwise, if $\theta_4 = \gamma$, we get the configuration illustrated in Figure 32(b). At vertex v'' we must have $\gamma + \beta + \gamma + \gamma = \pi$ (note that $\gamma + 4\beta = \pi$, and so $\gamma + \beta + \gamma + k\beta \neq \pi$, for all k). Taking into account the relations between angles, we get $\alpha = \frac{4\pi}{11}$,

$\beta = \frac{2\pi}{11}$ and $\gamma = \frac{3\pi}{11}$. But these angles do not satisfy relation (2).

(iii.2.2.1.2) If $\gamma + \beta + \beta + \beta + \beta = \pi$ (Figure 33(a), vertex v'), we reach a contradiction at vertex v''', as $\gamma + \beta + \gamma + \beta < \pi$ and $\gamma + \beta + \gamma + \beta + \rho > \pi$, $\forall \rho \in \{\alpha, \beta, \gamma, \frac{\pi}{2}\}$ (the case $\gamma + \beta + \gamma + \gamma = \pi$ was previously studied and did not give rise to any f-tiling).

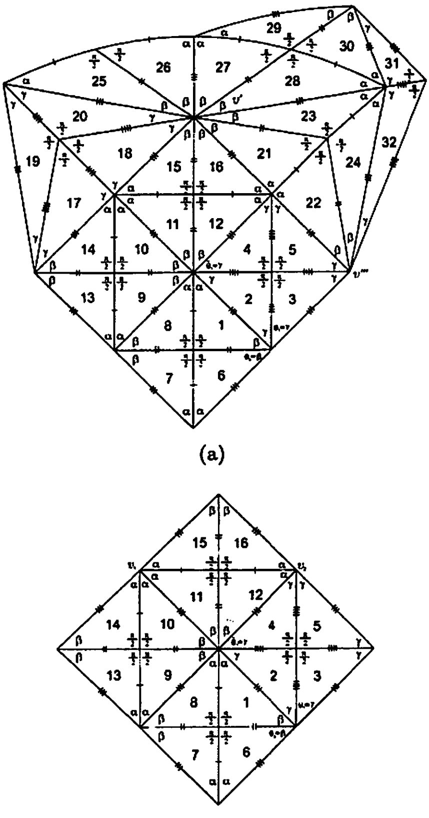


Figure 33. Local configurations

(iii.2.2.2) It is a straightforward exercise to show that, if $\alpha + \gamma + 2\beta = \pi$ (Figure 31(a)), we get either an incongruence or the previous f-tiling $\mathcal{C}$.

(iii.2.3) Finally, if $k = 2$ and $\alpha + \alpha + \beta = \pi$, or $k \geq 3$, the configuration illustrated in Figure 16(b) is extended to the one given in Figure 33(b). At vertex v_1 we have necessarily $\alpha + \alpha + \beta = \pi$, as if $\alpha + \gamma + 3\beta \leq \pi$ and $\alpha + \alpha + \rho = \pi$, with $\rho \in \{\alpha, \gamma\}$, we obtain $2\pi \geq (\alpha + \gamma + 3\beta) + (\alpha + \alpha + \rho) =$

$(\alpha+\beta)+(\alpha+\beta)+(\alpha+\beta)+(\gamma+\rho) > 4\frac{\pi}{2} = 2\pi$, which is impossible. At vertex v_2, we must have $\alpha+\gamma+k\beta = \pi$, since $\alpha+\gamma+\gamma = \pi$ implies $\alpha = \frac{(2k-1)\pi}{4k-1}$, $\beta = \frac{\pi}{4k-1}$ and $\gamma = \frac{k\pi}{4k-1}$, and relation (2) is not satisfied for any k. Thus, the last configuration is extended in a unique way (for any $k \geq 2$) to the one illustrated in Figure 34. We have $\alpha = \frac{\pi-\beta}{2}$ and $\gamma = \frac{\pi-(2k-1)\beta}{2}$, for some

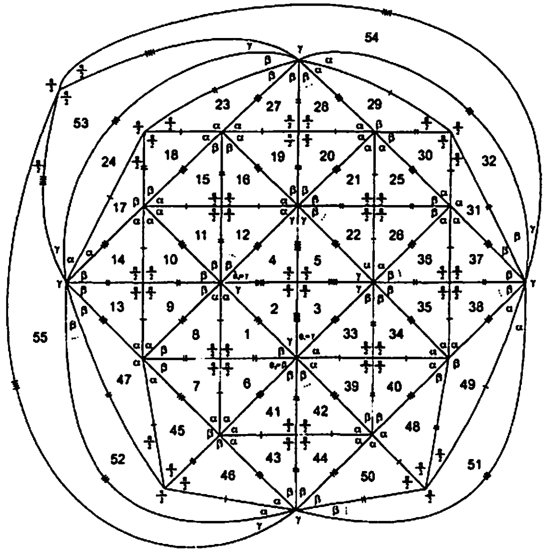


Figure 34. Planar representation of $\mathcal{G}^k$, $k \geq 2$

$k \geq 2$. Using these relations and (2), for each $k \geq 2$, we obtain a unique solution in β. For instance, for $k = 2$ we have $\beta \approx 23.1°$ and for $k = 3$ we have $\beta \approx 14°$. We shall denote such family of dihedral f-tilings by $\mathcal{G}^k$, $k \geq 2$. 3D representations of $\mathcal{G}^2$ and $\mathcal{G}^3$ are illustrated in Figure 2. □

4.2. *Case of Adjacency B*

As in the previous case, we begin stating some conditions on the angles and lengths of the sides of T_1 and T_2.

Lemma 7 *Suppose that there are two cells in adjacent positions as illustrated in Figure 7–B. Then,*

(i)

$$\cot\alpha\cot\beta = \cot\gamma; \qquad (3)$$

(ii) $a, b, \alpha, \beta, \gamma \neq \frac{\pi}{2}$;

(iii) $a, b, c \neq e$.

(iv) the three edges of T_1 must have different lengths.

Proof.

(i) It follows immediately observing that $\cos c = \cot\alpha\cot\beta$ and $\cos d = \cot\gamma$ (and $c = d$).

(ii) Analogous to (ii) of Lemma 4.

(iii) If $a = e$ (note that $c = d$), by the sine rule we obtain $\sin\beta\sin\gamma = 1$, which is an impossible condition since $\beta, \gamma \neq \frac{\pi}{2}$. Analogously we prove that $b \neq e$. Obviously $c \neq e$.

(iv) Analogous to (iv) of Lemma 4. □

Proposition 8 *If there are two cells in adjacent positions as illustrated in Figure 7–B, then $\Omega(T_1, T_2) = \{\mathcal{D}, \mathcal{F}_i, 1 \leq i \leq 12\}$, where $\mathcal{F}_i$ $(1 \leq i \leq 12)$ are non-isomorphic f-tilings obtained in [4, Proposition 1.] and $\mathcal{D}$ is a single tiling whose angles around vertices are positioned as illustrated in Figure 35.*

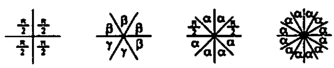


Figure 35. Distinct classes of congruent vertices

Proof. Suppose that any element of $\Omega(T_1, T_2)$ has at least two cells congruent, respectively, to T_1 and T_2, such that they are in adjacent positions as illustrated in Figure 36(a). Taking into account the results of Lemma 7 and

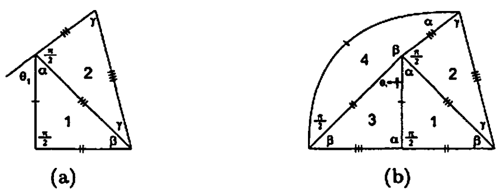


Figure 36. Local configurations

with the labelling of this figure, we have

$$\theta_1 = \frac{\pi}{2} \quad \text{or} \quad \theta_1 = \alpha.$$

1. Suppose firstly that $\theta_1 = \frac{\pi}{2}$. In this case we have necessarily $\alpha + \beta = \pi$ (Figure 36(b)).

If $\alpha > \beta$, then $\alpha > \frac{\pi}{2}$ and the last configuration is extended to the one illustrated in Figure 37(a). At vertex v we have necessarily $\alpha + \gamma = \pi$ as $\alpha + \gamma + \rho > \pi$, $\forall \rho \in \{\alpha, \beta, \gamma, \frac{\pi}{2}\}$. But then $\gamma = \beta$. Using (3), we get $\alpha = \frac{\pi}{4}$, which is a contradiction.

On the other hand, if $\alpha < \beta$, we have $\beta > \frac{\pi}{2}$ and the configuration illustrated in Figure 37(b). At vertex v we cannot have $\beta + \gamma = \pi$, otherwise

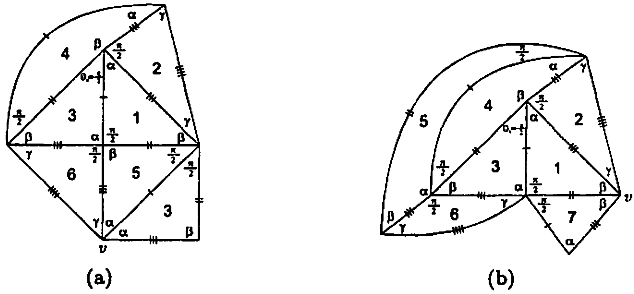


Figure 37. Local configurations

we get $\beta = \frac{\pi}{4}$ (by (3)). Nevertheless, $\beta + \gamma + \rho > \pi$, $\forall \rho \in \{\alpha, \beta, \gamma, \frac{\pi}{2}\}$, and so we reach a contradiction.

2. Suppose now that $\theta_1 = \alpha$. Then, the local configuration extends as illustrated in Figure 38. At vertex v we have

$$\alpha + \frac{\pi}{2} + k\alpha = \pi, \quad k \geq 1, \qquad \text{or} \qquad \alpha + \frac{\pi}{2} + \gamma + k\alpha = \pi, \quad k \geq 0.$$

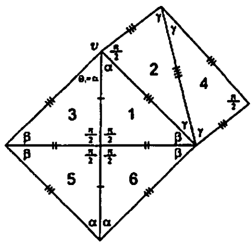


Figure 38. Local configuration

2.1 Suppose that $\alpha + \frac{\pi}{2} + k\alpha = \pi$, with $k \geq 1$. Then $\gamma, \beta > \frac{\pi}{4} \geq \alpha$. The case $k = 1$ corresponds to the analysis developed in [4], where twelve distinct f-tilings, with prototiles $(\frac{\pi}{2}, \frac{\pi}{4}, \frac{\pi}{3})$ and $(\frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{3})$, were achieved, namely $\mathcal{F}_i$, $1 \leq i \leq 12$. 3D representations are illustrated in Figure 2.

Now, if $k > 1$, we have $\alpha \in (0, \frac{\pi}{6}]$ and consequently $\beta > \frac{\pi}{3}$. With the labelling of Figure 39(a), we have

$$\theta_2 = \beta \quad \text{or} \quad \theta_2 = \frac{\pi}{2}.$$

2.1.1 If $\theta_2 = \beta$, we get the configuration illustrated in Figure 39(b). At vertex v_1 we have $\beta + \beta + \alpha = \pi$ or $\beta + \beta + \gamma = \pi$. In the first case, at vertex v_2 we get $\beta + \gamma + \gamma = \pi$ (note that $\beta = \gamma$, by (3), implies $\beta = \frac{\pi}{2}$ or $\alpha = \frac{\pi}{4}$, which is not possible). Applying these conditions in (3), we obtain

$\cot\gamma + \cot 4\gamma \cot 2\gamma = 0$, but this equation has no solution in $\left(\frac{\pi}{4}, \frac{\pi}{2}\right)$. On

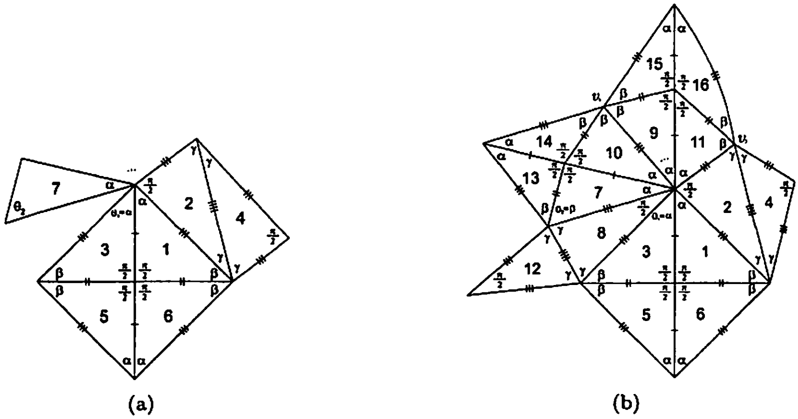


Figure 39. Local configurations

the other hand, if $\beta + \beta + \gamma = \pi$, we must have $k = 2$ (i.e., $\alpha = \frac{\pi}{6}$) and the last configuration is extended in a unique way to the one given in Figure 40. We shall denote such f-tiling by $\mathcal{D}$. A 3D representation of $\mathcal{D}$ is illustrated in Figure 2.

2.1.2 If $\theta_2 = \frac{\pi}{2}$ (Figure 39(a)), we get the configuration illustrated in Figure 41(a). Now, we have

$$\theta_3 = \frac{\pi}{2} \quad \text{or} \quad \theta_3 = \beta.$$

If $\theta_3 = \frac{\pi}{2}$ (Figure 41(b)), observing the vertices v_1', v_2' and v_3', and taking into account that β and $\frac{\pi}{2}$ cannot be adjacent angles, we have necessarily $\beta + \beta + \alpha = \pi$ and $\beta + \gamma + \gamma = \pi$. As we have seen before, these conditions lead to a contradiction.

The analysis of $\theta_3 = \beta$ (Figure 42(a)) is analogous to the one made in $\theta_2 = \beta$, applied to the vertices v_1'' and v_2''.

2.2 Suppose now that $\alpha + \frac{\pi}{2} + \gamma + k\alpha = \pi$, with $k \geq 0$. Then $\beta > \gamma > \frac{\pi}{4} > \alpha$.

If $k = 0$, we get the local configuration illustrated in Figure 42(b). Note that $\theta_2 = \frac{\pi}{2}$, otherwise we get $\theta_3 = \frac{\pi}{2}$, which is not possible. The same applies to the choice of θ_4. If $\beta + \gamma + \bar{k}\alpha = \pi$, for all $\bar{k} \geq 1$, the equation (3) has no solution. On the other hand, if $\beta + \gamma + \beta = \pi$, we obtain $\alpha \approx 35.6°$. This last condition also leads to $\bar{\bar{k}}\alpha = \pi$, for some $\bar{\bar{k}} > 4$, which is not possible. And so, at vertex v_1, we have necessarily $\beta + \gamma + \gamma = \pi$. Nevertheless, there is no way to satisfy the angle-folding relation at vertex v_2.

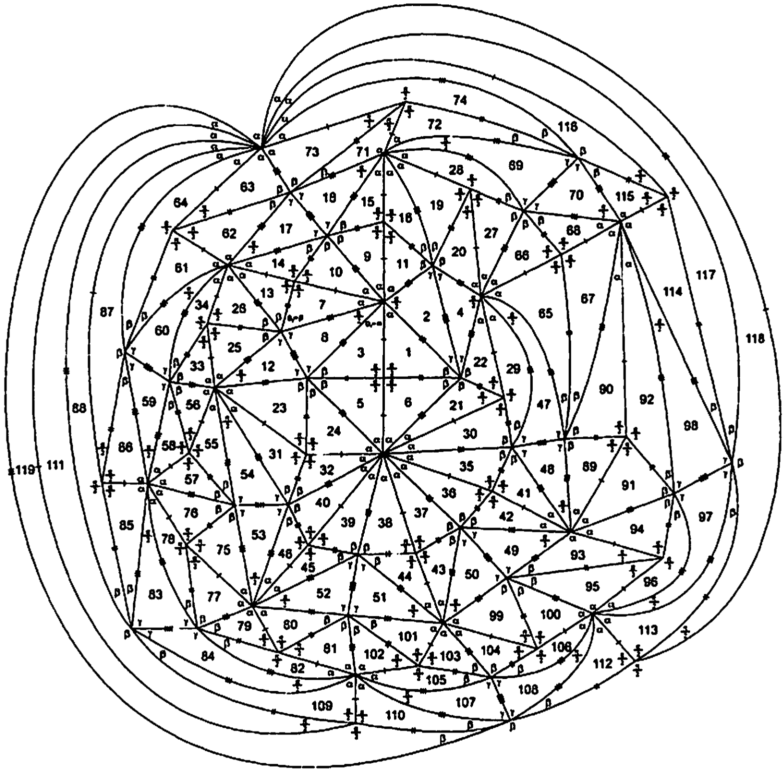


Figure 40. Planar representation of $\mathcal{D}$

If $k \geq 1$ and using analogous argumentation, we get the local configuration illustrated in Figure 43. At vertex v we reach an impossibility. The case $\theta_2 = \alpha$ is similar. □

4.3. *Case of Adjacency C*

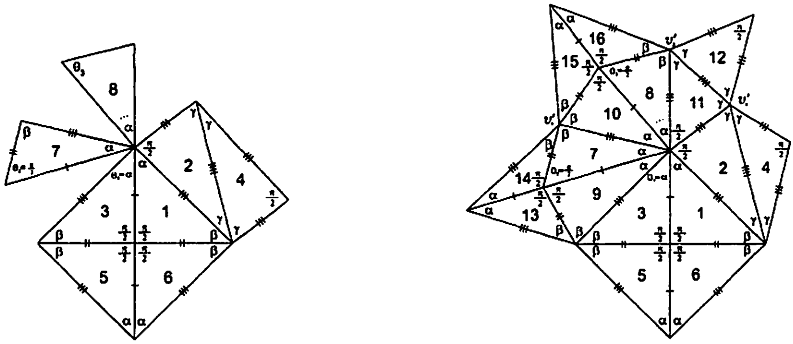
Lemma 9 *Suppose that there are two cells in adjacent positions as illustrated in Figure 7–C. Then,*

(i)

$$\frac{\cos \alpha}{\sin \beta} = \cot^2 \gamma; \tag{4}$$

(ii) $b, \alpha \neq \frac{\pi}{2}$;

(iii) $a, b, d \neq c$ *(observe that if $a = c$, then there are no f-tilings with such prototiles);*



(a) (b)

Figure 41. Local configurations

(iv) if $d = a$, then $a, \beta \neq \frac{\pi}{2}$ and

$$\begin{cases} \cos^2 \beta \sin \beta = \sin^2 \alpha \cos \alpha \\ \cos^2 \gamma \sin \alpha = \sin \gamma \cos \alpha \end{cases} ; \tag{5}$$

(v) the three edges of T_1 must have different lengths.

We omit the proof of this result as it is similar to Lemma 4 and Lemma 7.

Proposition 10 *If there are two cells in adjacent positions as illustrated in Figure 7–C, then $\Omega(T_1, T_2)$ consists of a unique dihedral f-tiling, denoted by $\mathcal{M}$, such that $\alpha = \frac{\pi}{3}$, $\gamma + \beta = \frac{\pi}{2}$ and $\gamma \approx 48.9°$. A planar representation of $\mathcal{M}$ is given in Figure 48 and a 3D representation is given in Figure 2.*

Proof. Suppose that any element of $\Omega(T_1, T_2)$ has at least two cells congruent, respectively, to T_1 and T_2, such that they are in adjacent positions as illustrated in Figure 44(a). Taking into account the results of Lemma 9 and with the labelling of the this figure, we have

$$\theta_1 = \frac{\pi}{2}, \quad \theta_1 = \alpha \quad \text{or} \quad \theta_1 = \gamma.$$

1. Suppose that $\theta_1 = \frac{\pi}{2}$ (Figure 44(b)). The cases $\theta_2 = \alpha$ and $\theta_2 = \gamma$ are illustrated in Figure 45(a) and Figure 45(b), respectively. As $\gamma \neq \frac{\pi}{2}$ and $d \neq c$, in both cases we reach a contradiction at vertex v.

2. If $\theta_1 = \alpha$ (Figure 44(a)), then $\frac{\pi}{2} + \alpha + \gamma + k\alpha = \pi$, for some $k \geq 0$ (note that $\frac{\pi}{2} + \alpha + k\alpha = \pi$, with $k \geq 1$, implies $\gamma = \frac{\pi}{2}$, which is not possible, see length sides). And so $\beta > \gamma > \frac{\pi}{4} > \alpha$. With the labelling of Figure 46(a), we have $\theta_2 = \gamma$ or $\theta_2 = \alpha$. In both cases, taking into account the angles

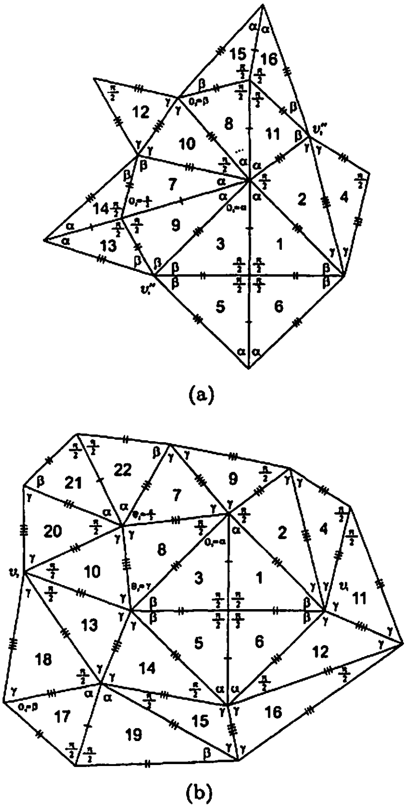


Figure 42. Local configurations

and edge lengths, we get the configuration illustrated in Figure 46(b). We have $\theta_3 = \gamma$, as $\theta_3 = \beta$ implies $\beta + \beta + \rho = \pi$, for some $\rho \in \{\alpha, \beta, \gamma\}$, and in all cases we have no solution satisfying simultaneously the equations (4) and (5). Nevertheless, there is no way to satisfy the angle-folding relation at vertex v.

3. Suppose finally that $\theta_1 = \gamma$ (Figure 47(a)). As $\frac{\pi}{4} < \gamma \neq \frac{\pi}{2}$, we have $\frac{\pi}{2} + \gamma + k\beta = \pi$ or $\frac{\pi}{2} + \gamma + k\alpha = \pi$, for some $k \geq 1$.

3.1 Suppose that $\frac{\pi}{2} + \gamma + k\beta = \pi$, for some $k \geq 1$. Then $\alpha > \gamma > \frac{\pi}{4} > \beta$ and we get the configuration illustrated in Figure 47(b) (recall that $d \neq c$). Now, at vertex v we have $\alpha + \alpha + \alpha = \pi$ or $\alpha + \alpha + \gamma = \pi$.

The first case leads to the planar representation illustrated in Figure 48. Using equation (4) we get $k = 1$, $\gamma \approx 48.9°$ and $\beta = \frac{\pi}{2} - \gamma$. We shall denote

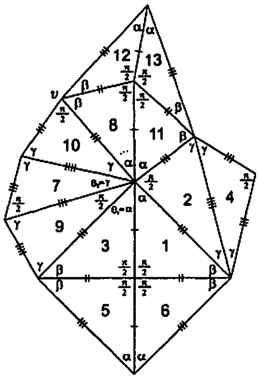
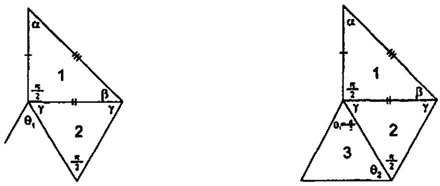
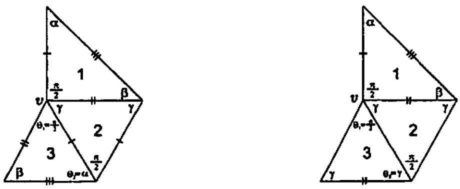


Figure 43. Local configuration



(a) (b)

Figure 44. Local configurations



(a) (b)

Figure 45. Local configurations

such an f-tiling by $\mathcal{M}$. A 3D representation of $\mathcal{M}$ is illustrated in Figure 2.

If $\alpha + \alpha + \gamma = \pi$, then $k = 1$ and $a = d$. But there is no $\alpha \in \left(\frac{\pi}{4}, \frac{3\pi}{8}\right)$ satisfying (4) and (5).

3.2 Suppose now that $\frac{\pi}{2} + \gamma + k\alpha = \pi$, for some $k \geq 1$. We have $\beta > \gamma > \frac{\pi}{4} > \alpha$ and we get the configuration illustrated in Figure 49(a), with $\theta_3 \in \{\gamma, \alpha\}$.

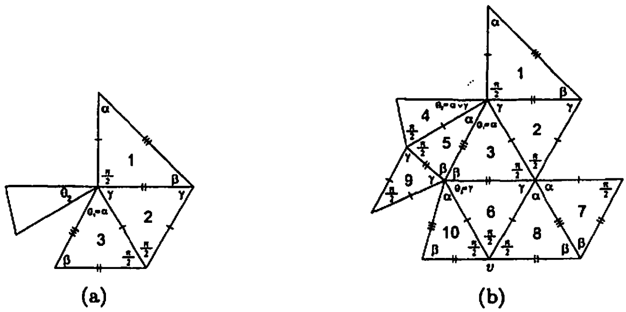


Figure 46. Local configurations

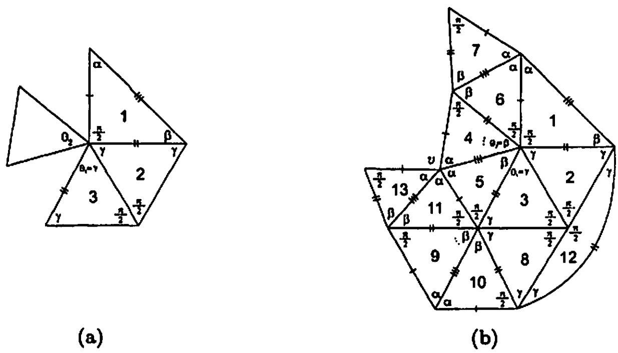


Figure 47. Local configurations

If $\theta_3 = \gamma$, at vertex v we have $\beta + \gamma + \gamma = \pi$ or $\beta + \gamma + \beta = \pi$ (note that, by (4), v cannot have valency four).

If $\beta + \gamma + \gamma = \pi$, we have $a \neq d$, otherwise equations (4) and (5) have no solution. The last configuration is then extended to the one illustrated in Figure 49(b). Nevertheless, at vertex v' there is no way to satisfy the angle-folding relation.

If $\beta+\gamma+\beta = \pi$, then $k = 1$, $a \neq d$ and we get the configuration illustrated in Figure 50(a) (note that tile 17 is uniquely determined as $\alpha < \gamma < 2\alpha$). Again, there is no way to satisfy the angle-folding relation at vertex v''.

If $\theta_3 = \alpha$ (Figure 49(a)), we get $a = d$, $\beta \neq \frac{\pi}{2}$, and the local configuration illustrated in Figure 50(b). At vertex v''' we have $\beta + \beta + \rho = \pi$, for some $\rho \in \{\alpha, \beta, \gamma\}$. But in all cases we have no solution satisfying simultaneously the equations (4) and (5), and so we reach a contradiction. □

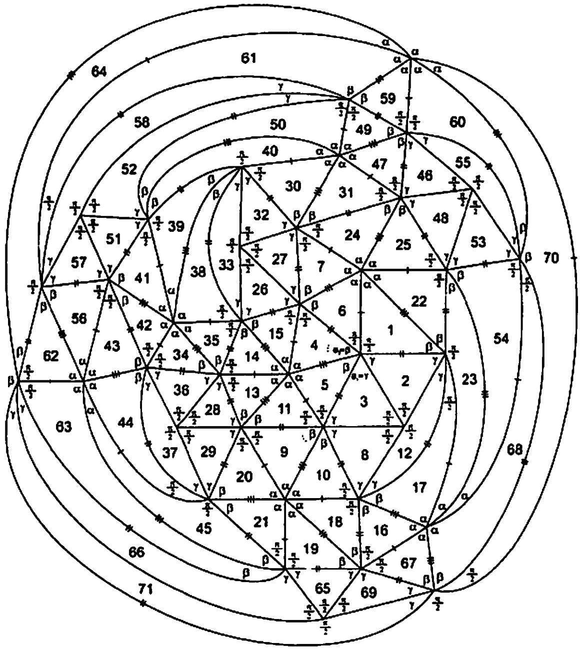
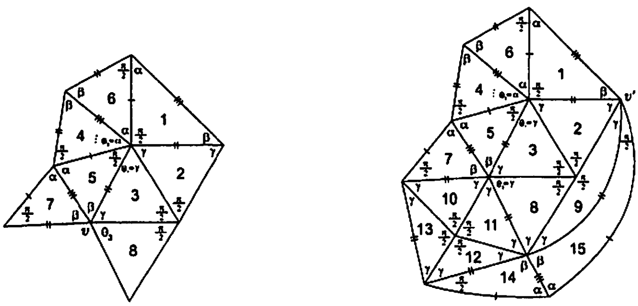


Figure 48. Planar representation of $\mathcal{M}$

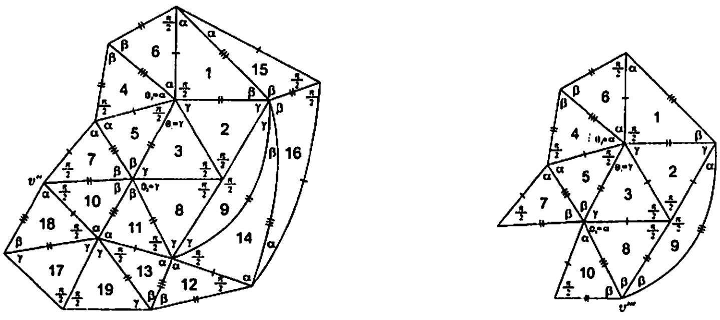


(a) (b)

Figure 49. Local configurations

4.4. *Case of Adjacency D*

Lemma 11 *Suppose that there are two cells in adjacent positions as illustrated in Figure 7–D. Then,*



(a) (b)

Figure 50. Local configurations

(i)

$$\frac{\cos\alpha}{\sin\beta} = \cot\gamma; \tag{6}$$

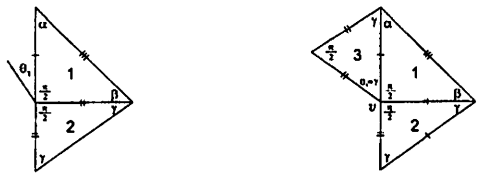
(ii) $b, \alpha \neq \frac{\pi}{2}$ *and* $\alpha, \beta \neq \gamma$;
(iii) $a, b, e \neq c$ *(observe that if* $a = c$*, then there are no f-tilings with such prototiles);*
(iv) if $e = a$*, then* $a, \beta \neq \frac{\pi}{2}$,

$$\begin{cases} \cos\beta\sin^2\beta = \sin\alpha\cos^2\alpha \\ \sin\alpha = \sin\beta\sin\gamma \end{cases} \tag{7}$$

and if $\beta > \frac{\pi}{4}$*, there is at most one angle* β *in each alternating angle sum;*
(v) the three edges of T_1 *must have different lengths.*

Proposition 12 *If there are two cells in adjacent positions as illustrated in Figure 7–D, then* $\Omega(T_1, T_2) \neq \emptyset$ *iff*
(i) $\alpha = \gamma = \frac{\pi}{3}$ *and* $\beta + \gamma = \pi$ *or*
(ii) $\alpha = \frac{\pi}{3}$, $3\gamma + \beta = \pi$, *or*
(iii) $2\gamma + \beta = \pi$ *and* $k\alpha = \pi$*, for some* $k \geq 4$.
The first case leads to a single tiling, denoted by $\mathcal{H}$*. A planar representation is given in Figure 55(b). For its 3D representation see Figure 2.*
The case (ii) leads to a single tiling, denoted by $\mathcal{J}$*, in which* $\gamma \approx 48.5°$*. In Figure 58 is given the corresponding planar representation. A 3D representation is given in Figure 2.*
In the last situation, for each $k \geq 4$*, there is a single tilings, denoted by* $\mathcal{R}^k$*, with* $\gamma = \arccos\sqrt{\frac{1}{2}\cos\frac{\pi}{k}}$*. A planar representation of* $\mathcal{R}^k$ *is given in Figure 60(b). For their* $k = 4$ *and* $k = 5$ *3D representations see Figure 2.*

Proof. Suppose that any element of $\Omega(T_1, T_2)$ has at least two cells congruent, respectively, to T_1 and T_2, such that they are in adjacent positions as illustrated in Figure 51(a). Taking into account the results of Lemma 11



(a) (b)

Figure 51. Local configurations

and with the labelling of the this figure, we have

$$\theta_1 = \gamma, \quad \theta_1 = \alpha \quad \text{or} \quad \theta_1 = \frac{\pi}{2}.$$

1. If $\theta_1 = \gamma$ (Figure 51(b)), we have $a = e$ and, at vertex v, $\frac{\pi}{2} + \gamma + k\alpha = \pi$ or $\frac{\pi}{2} + \gamma + k\beta = \pi$, for some $k \geq 1$. In both cases, there is no way to satisfy the angle-folding relation at vertex v.

2. If $\theta_1 = \alpha$ (Figure 52(a)), at vertex v we have $\frac{\pi}{2} + \alpha + k\alpha = \pi$ or $\frac{\pi}{2} + \alpha + \gamma + \bar{k}\alpha = \pi$, for some $k \geq 1$ and $\bar{k} \geq 0$. As in the previous case, there is no way to satisfy the angle-folding relation at vertex v.

(a) (b)

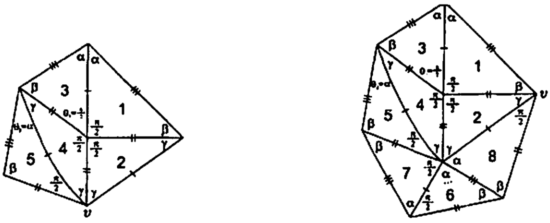
Figure 52. Local configurations

3. If $\theta_1 = \frac{\pi}{2}$ (Figure 52(b)), then we have

$$\theta_2 = \alpha, \quad \theta_2 = \frac{\pi}{2} \quad \text{or} \quad \theta_2 = \gamma.$$

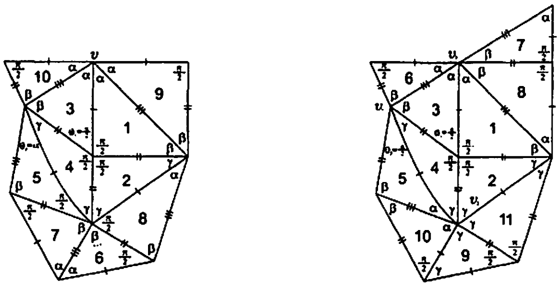
3.1 If $\theta_2 = \alpha$ (Figure 53(a)), $e = a$ and at vertex v we have $\frac{\pi}{2} + \gamma + k\alpha = \pi$ or $\frac{\pi}{2} + \gamma + k\beta = \pi$, for some $k \geq 1$.

If $\frac{\pi}{2} + \gamma + k\alpha = \pi$ ($k \geq 1$), we have $\beta > \gamma > \frac{\pi}{4} > \alpha$ and we get the configuration illustrated in Figure 53(b). At vertex v we reach a contradiction, as $\beta \neq \frac{\pi}{2}$ ($e = a$) and $\frac{\pi}{2} + \beta + \rho > \pi$, for all $\rho \in \{\frac{\pi}{2}, \alpha, \beta, \gamma\}$.



(a) (b)

Figure 53. Local configurations



(a) (b)

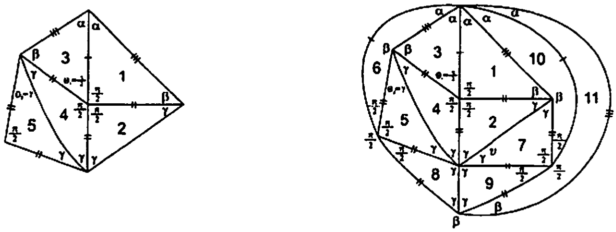
Figure 54. Local configurations

If $\frac{\pi}{2} + \gamma + k\beta = \pi$ $(k \geq 1)$, $\alpha > \gamma > \frac{\pi}{4} > \beta$ and we get the configuration illustrated in Figure 54(a). At vertex v we have $\alpha+\alpha+\alpha = \pi$ or $\alpha+\alpha+\gamma = \pi$. In both cases we have no solution satisfying simultaneously the equations (6) and (7).

3.2 If $\theta_2 = \frac{\pi}{2}$ (Figure 54(b)), then $e = a$ and $\alpha > \gamma > \frac{\pi}{4} > \beta$. Taking into account the analysis of the previous case, we conclude that at vertex v_2 we have $\alpha + \alpha + \beta = \pi$ and at vertex v_3 we have $\alpha + \gamma + \gamma = \pi$; in this last situation note that there is no way to satisfy the angle-folding relation if $\alpha + \gamma + k\beta = \pi$, $k \geq 2$. Nevertheless, the equations (6) and (7) are not satisfied.

3.3 Suppose finally that $\theta_2 = \gamma$ (Figure 55(a)). We consider separately the cases $\beta + \gamma = \pi$ and $\beta + \gamma < \pi$.

3.3.1 If $\beta + \gamma = \pi$, then $e \neq a$ and $\alpha = \gamma$. The last configuration is then extended in a unique way to the one illustrated in Figure 55(b), with $\alpha = \gamma = \frac{\pi}{3}$ and $\beta = \frac{2\pi}{3}$. We denote such f-tiling by $\mathcal{H}$. The corresponding



(a)

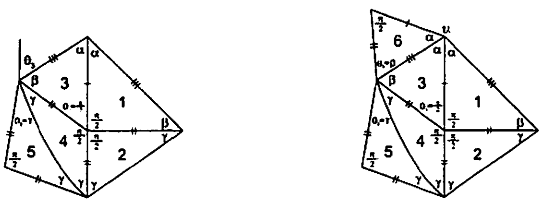
(b) Planar representation of $\mathcal{H}$

Figure 55. Local configurations

3D representation is given in Figure 2.

3.3.2 Suppose now that $\beta+\gamma<\pi$. With the labelling of Figure 56(a), we have $\theta_3=\beta$ or $\theta_3=\alpha$. For each case, we divide the proof in the cases $\alpha>\beta$ and $\alpha<\beta$.

3.3.2.1 If $\theta_3=\beta$ and $\alpha>\beta$, we have $\alpha>\frac{\pi}{4}$ and we get the configuration illustrated in Figure 56(b). At vertex v_1 we have $\alpha+\alpha+\rho=\pi$, for some $\rho\in\{\gamma,\alpha,\beta\}$.



(a) (b)

Figure 56. Local configurations

If $\alpha+\alpha+\gamma=\pi$, then $e=a$ and equations (6) and (7) have no solution.

If $\alpha+\alpha+\alpha=\pi$ (Figure 57), i.e., $\alpha=\frac{\pi}{3}$, then $e\neq a$ and at vertex v_2 we have necessarily $\gamma+\gamma+\gamma+\beta=\pi$ or $\gamma+\gamma+\beta+\beta=\pi$ (note that $\gamma+\gamma+\beta=\pi$ implies $\gamma=\beta=\frac{\pi}{3}$). If $\gamma+\gamma+\gamma+\beta=\pi$, by equation (6), we get $\gamma=48.5°$ and $\beta=34.4°$ ($\alpha=\frac{\pi}{3}$). Taking into account these angles, the last configuration is extended in a unique way to the configuration illustrated in Figure 58. We shall denote such f-tiling by $\mathcal{J}$. Its 3D representation is shown in Figure 2. On the other hand, if $\gamma+\gamma+\beta+\beta=\pi$, we get $\gamma=51.3°$ and $\beta=38.7°$ ($\alpha=\frac{\pi}{3}$). The last configuration is then extended in a unique way to the configuration illustrated in Figure 59(a). Taking into account the angles measure, we reach a contradiction at vertex v.

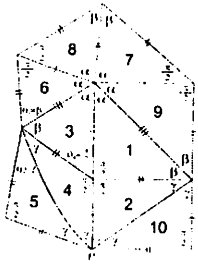


Figure 57. Local configuration

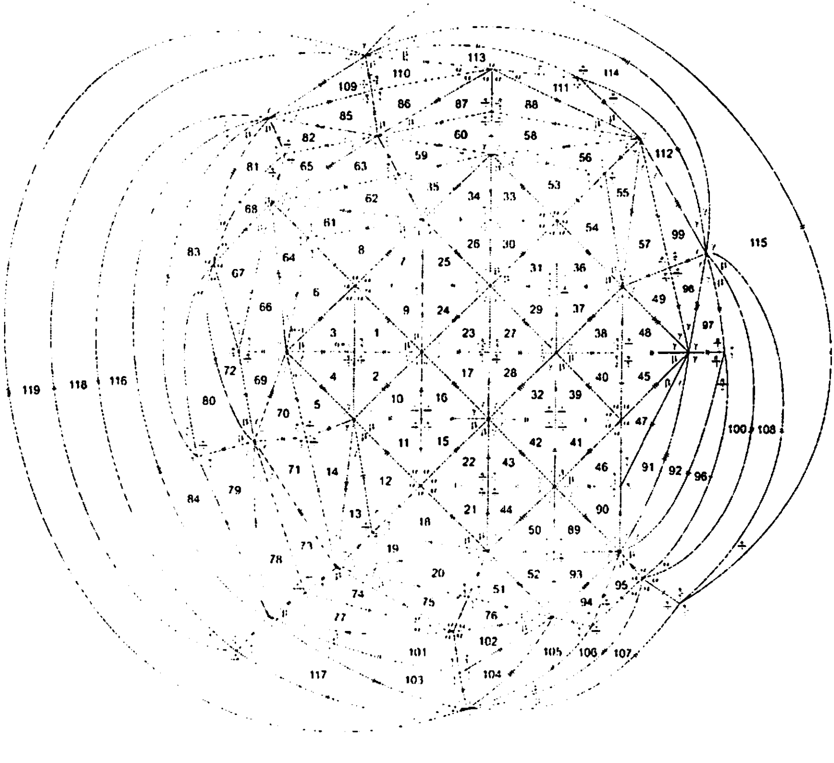
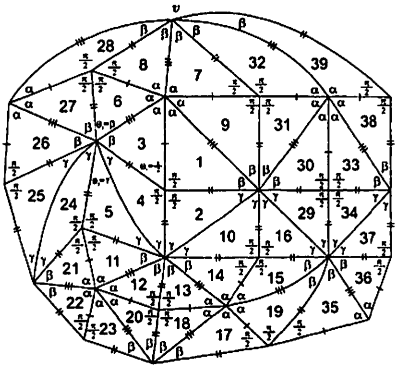


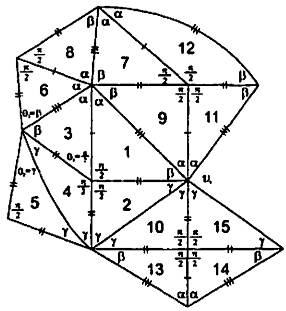
Figure 58. Planar representation of $\mathcal{J}$

If $\alpha + \alpha + \beta = \pi$ (Figure 59(b)), then $e \neq a$. But at vertex v_1 there is no way to satisfy the angle-folding relation.

3.3.2.2 If $\theta_3 = \beta$ and $\alpha < \beta$, we have $\beta > \frac{\pi}{4}$ and we get the configuration illustrated in Figure 60(a). At vertex v_1 we have $\beta + \gamma + \rho = \pi$, for some $\rho \in \{\gamma, \beta\}$, and so $e \neq a$ (otherwise equations (6) and (7) have no solution).

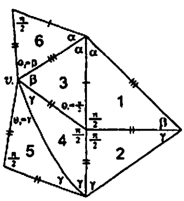


(a)

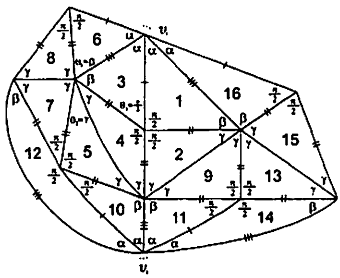


(b)

Figure 59. Local configurations



(a)

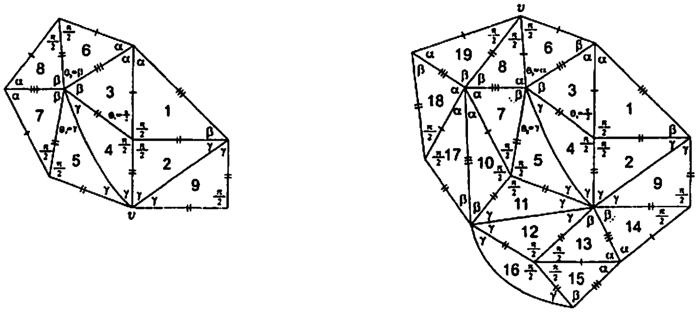


(b) Planar representation of $\mathcal{R}^k$, $k \geq 4$

Figure 60. Local configurations

If $\beta + \gamma + \gamma = \pi$, the last configuration is extended in a unique way to the one illustrated in Figure 60(b). The vertices v_1 and v_2 are in antipodal positions since there exist two distinct geodesics of the same length joining them, and so $a + b + c = \pi$. We have $\alpha = \frac{\pi}{k}$, $k \geq 4$, and $\beta = \pi - 2\gamma$. Using these relations, it is a straightforward exercise to show that $\gamma = \arccos\sqrt{\frac{1}{2}\cos\frac{\pi}{k}}$. We shall denote such family of dihedral f-tilings by $\mathcal{R}^k$, $k \geq 4$. 3D representations of $\mathcal{R}^4$ and $\mathcal{R}^5$ are illustrated in Figure 2.

If $\beta + \gamma + \beta = \pi$, we get the configuration illustrated in Figure 61(a). We reach a contradiction at vertex v, as there is no way to satisfy the angle-folding relation (observe that $\gamma \neq \beta$).



(a) (b)

Figure 61. Local configurations

3.3.2.3 If $\theta_3 = \alpha$ and $\alpha > \beta$ (Figure 56(a)), then, taking into account the edge lengths, we obtain $\alpha + \gamma + k\beta = \pi$, $k \geq 1$. Therefore $e \neq a$ (otherwise, equations (6) and (7) are not satisfied) and we get the configuration illustrated in Figure 61(b). Nevertheless, there is no way to satisfy the angle-folding relation at vertex v (see length sides).

3.3.2.4 Finally, if $\theta_3 = \alpha$ and $\alpha < \beta$ (Figure 62(a)), we have $\beta + \gamma + \gamma = \pi$ or $\beta + \gamma + k\alpha = \pi$, with $k \geq 1$.

If $\beta + \gamma + \gamma = \pi$, then $e = a$ and $\frac{\pi}{2} + \gamma + \alpha = \pi$, which lead to a contradiction.

If $\beta + \gamma + k\alpha = \pi$, $k \geq 1$ (Figure 62(b)), then must $\theta_4 = \gamma$, since $\theta_4 = \frac{\pi}{2}$ and $\theta_4 = \beta$ lead to a contradiction as illustrated in Figure 63(a) and Figure 63(b), respectively, where there is no way to satisfy the angle-folding relation at vertex v. The last configuration is then extended to the one illustrated in Figure 64. However, at vertex v we reach a contradiction. Note that tile 11 is completely determined as $\theta_5 = \alpha$ implies $\theta_6 = \frac{\pi}{2}$. The same applies to tile 12. Note also that $\gamma \neq \frac{\pi}{3}$, otherwise, equation (6) has no solution.

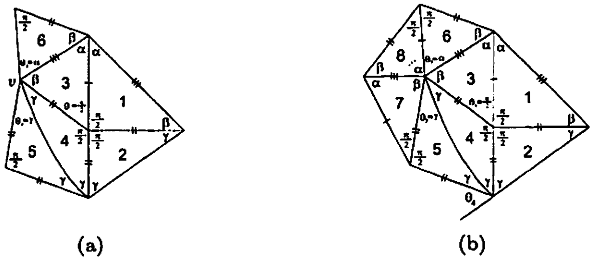


Figure 62. Local configurations

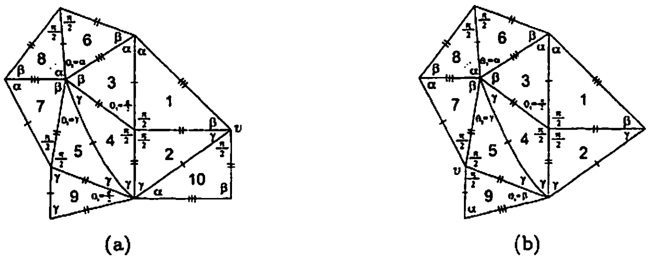


Figure 63. Local configurations

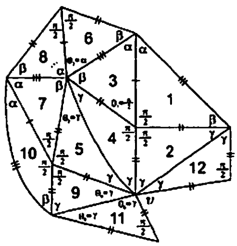


Figure 64. Local configuration

□

4.5. *Case of Adjacency E*

Proposition 13 *If there are two cells in adjacent positions as illustrated in Figure 7–E, then $\Omega(T_1, T_2)$ is the empty set.*

Proof. Suppose that any element of $\Omega(T_1, T_2)$ has at least two cells congruent, respectively, to T_1 and T_2, such that they are in adjacent positions as illustrated in Figure 65(a). Taking into account the results of Lemma 11 and with the labelling of the this figure, we have

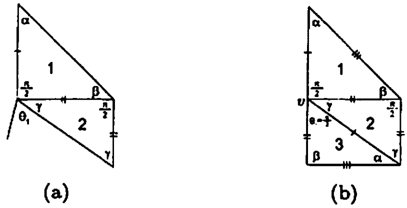


Figure 65. Local configurations

$$\theta_1 = \frac{\pi}{2}, \qquad \theta_1 = \alpha \qquad \text{or} \qquad \theta_1 = \gamma.$$

Note that, by proposition 5 and proposition 6, we have $\theta_1 \neq \beta$.

1. If $\theta_1 = \frac{\pi}{2}$ (Figure 65(b)), we have $a = e$. As $\gamma \neq \frac{\pi}{2}$, we reach a contradiction at vertex v.

2. If $\theta_1 = \alpha$, then by the case of adjacency A, tile 3 is completely determined. On the other hand, as $a = e$, we have $\beta > \gamma > \alpha$ (Figure 66(a)). However, we obtain an incongruence since there is no way to satisfy the angle folding relation at vertices v_1 and v_2 simultaneously.

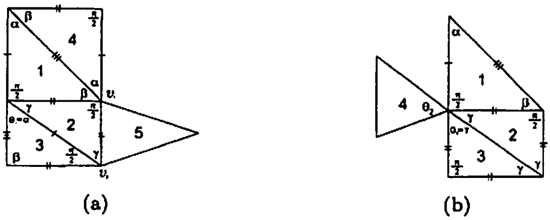


Figure 66. Local configurations

3. Suppose finally that $\theta_1 = \gamma$. With the labelling of Figure 66(b), we have $\theta_2 = \alpha$ or $\theta_2 = \beta$.

3.1 If $\theta_2 = \alpha$, we have $\frac{\pi}{2} + \gamma + k\alpha = \pi$, for some $k \geq 1$. Then $\beta > \gamma > \frac{\pi}{4} > \alpha$ and we get the configuration illustrated in Figure 67(a). Note that we have necessarily $\beta = \frac{\pi}{2}$ (tile 7), and so $e \neq a$. Consequently $\gamma = \frac{\pi}{3}$ and equation (6) has no solution.

3.2 If $\theta_2 = \beta$, then $\frac{\pi}{2} + \gamma + k\beta = \pi$, for some $k \geq 1$ (Figure 67(b)). And so $\alpha > \beta$. As tiles 3 and 6 form the previous case of adjacency, by proposition 12 we have $\Omega(T_1, T_2) = \emptyset$. □

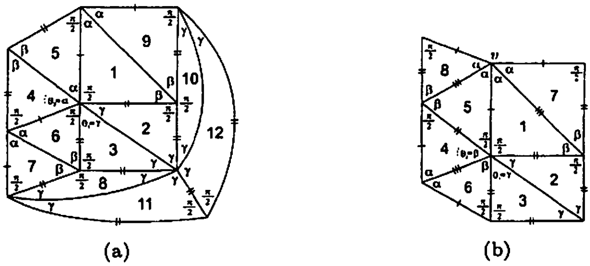


Figure 67. Local configurations

References

[1] A. M. Breda, A Class of Tilings of S^2, *Geometriae Dedicata*, **44** (1992), 241–253.

[2] A. M. Breda, P. S. Ribeiro and A. F. Santos, Dihedral f-Tilings of the Sphere by Equilateral and Scalene Triangles III, *The Electronic Journal of Combinatorics*, **15** (2008), #R147.

[3] A. M. Breda, P. S. Ribeiro and A. F. Santos, A class of spherical dihedral f-tilings, *European Journal of Combinatorics*, **30** (2009), 119–132.

[4] C. P. Avelino and A. F. Santos, Right–Triangular Dihedral f–Tilings of the Sphere: the $\left(\frac{\pi}{2}, \frac{\pi}{4}, \frac{\pi}{3}\right)$, $\left(\frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{3}\right)$ family, submitted.

[5] S. Robertson, Isometric folding of Riemannian manifolds, *Proceedings of the Royal Society of Edinburgh*, **79** (1977), 275–284.

[6] R. J. Dawson, Tilings of the sphere with isosceles triangles, *Discrete and Computational Geometry*, **30** (2003), 467–487.

[7] R. J. Dawson and B. Doyle, Tilings of the sphere with right triangles I: the asymptotically right families, *The Electronic Journal of Combinatorics*, **13** (2006), #R48.

[8] R. J. Dawson and B. Doyle, Tilings of the sphere with right triangles II: the $(1,3,2)$, $(0,2,n)$ family, *The Electronic Journal of Combinatorics*, **13** (2006), #R49.

[9] Y. Ueno and Y. Agaoka, Classification of tilings of the 2-dimensional sphere by congruent triangles, *Hiroshima Mathematical Journal*, **32** (2002), 463–540.