Extremal degree resistance distances in fully loaded unicyclic graphs

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Abstract

Let G be a connected graph, the degree resistance distance of G is defined as $D_R(G) = \sum_{\{u,v\}\subseteq V(G)} [d(u) + d(v)]r(u, v)$, where $d(u)$ (and $d(v)$) is the degree of the vertex u (and v), $r(u, v)$ is the resistance distance between vertices u and v. A fully loaded unicyclic graph is a unicyclic graph with the property that there is no vertex with degree less than 3 in its unique cycle. In this paper, we determine the minimum and maximum degree resistance distance among all fully loaded unicyclic graphs with n vertices, and characterize the extremal graphs.

1 Introduction

All graphs considered here are both connected and simple unless otherwise stated. The distance between vertices u and v of the graph G, denoted by $d(u, v)$, is the length of a shortest path between them. The degree of vertex u is $d(u)$; n, m are the number of vertices and edges of G, respectively. The girth of a graph G is the length of the shortest cycle in G.

The famous Wiener index was introduced by Harold Wiener in 1947, defined as[1]

$$W(G) = \sum_{\{u,v\}\subseteq V(G)} d(u, v) \tag{1}$$

A modified version of the Wiener index is the degree distance, was introduce by A. A. Dobrynin and A. A. Kochetova[2], defined as

$$D(G) = \sum_{\{u,v\}\subseteq V(G)} [d(u) + d(v)]d(u, v) \tag{2}$$

If G is a tree on n vertices, the Wiener index and the degree distance are related as $D(G) = 4W(G) - n(n-1)$ (for details see [3]).

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In [4], a multiplicative variant of the degree distance, namely *modified Schultz index* or *Gutman index* was put forward, defined as

$$S^*(G) = \sum_{\{u,v\} \subseteq V(G)} d(u)d(v)d(u,v) \tag{3}$$

If G is a tree on n vertices, the Wiener index and the modified Schultz index are related as $S^*(G) = 4W(G) - (2n-1)(n-1)$ (for details see [4]). The modified Schultz index of graphs attracted attention recently. In [5] an asymptotic upper bound for $S^*(G)$ was reported. In [6], a relation between the edge Wiener index and modified Schultz index was established, and S. Mukwembi in [7] improved on a bound by Dankelmann, Gutman, Mukwembi and Swart established in [6]. The maximal and minimal modified Schultz index of bicyclic graphs are determined in [8] and [9], respectively.

The concept of resistance distance was introduced by Klein and Randić [10] in 1993, on the basis of electrical network theory. They viewed a graph G as an electrical network N such that each edge of G is assumed to be a unit resistor. The resistance distance between the vertices u and v of a graph G, denoted by $r(u,v)$, is defined to be the effective resistance between nodes u, $v \in N$. Analogous to the definition of the Wiener index, the Kirchhoff index $Kf(G)$ of a graph G is defined as[10, 11]

$$Kf(G) = \sum_{\{u,v\} \subseteq V(G)} r(u,v) \tag{2}$$

If G is a tree, then $r(u,v) = d(u,v)$ for any two vertices u and v, the Kirchhoff and Wiener indices of trees coincide.

The Kirchhoff index is an important molecular structure descriptor[12], it has been well studied in both mathematical and chemical literatures. For a general graph G, I. Lukovits et al. [13] showed that $Kf(G) \geq n-1$ with equality if and only if G is complete graph K_n. J. L. Palacios [14] showed that $Kf(G) \leq \frac{n^3-n}{6}$ with equality if and only if G is a path P_n. For a circulant graph G, ref. [15] showed that $n-1 \leq Kf(G) \leq \frac{n^3-n}{12}$, the first equality holds if and only if G is K_n and the second does if and only if G is C_n. The unicyclic graphs with extremal Kirchhoff index were determined in [16, 17]. H. Deng also studied the Kirchhoff index of fully loaded unicyclic graphs [18] and graphs with cut edges [19]. H. Zhang et al. [20] characterized bicyclic graphs with extremal Kirchhoff indices. B. Zhou [21] characterized the extremal graphs with given matching number, connectivity. H. Wang et al.[22] determined the first three minimal Kirchhoff indices among cacti. R. Li in [22] obtained newer lower bounds for the Kirchhoff index of a connected graph.

The degree Kirchhoff index was put forward in [23], and further studied in [24, 25], defined as

$$R^*(G) = \sum_{\{u,v\}\subseteq V(G)} d(u)d(v)r(u,v) \tag{3}$$

Comparing Eqs. (2) and (3) we can see that the degree Kirchhoff index may be viewed as the resistance distance analogue of the Schultz index. However, there is a much more subtle reason for the introduction of this novel structure descriptor.

The degree resistance distance was introduced by I. Gutman, L. Feng and G. Yu in [26]:

$$D_R(G) = \sum_{\{u,v\}\subseteq V(G)} [d(u)+d(v)]r(u,v) \tag{4}$$

They investigated the degree resistance distance of unicyclic graphs, determined the unicyclic graphs with the minimum and second minimum degree resistance distance. Chen et al. [27] determined unicyclic graphs with the maximum and second maximum degree resistance distance. J. L. Palacios in [28] renamed the degree resistance distance as *additive degree Kirchhoff index*, gave tight upper and lower bounds for the degree resistance distance of a connected undirected graph.

If G is a tree, the degree distance and the degree resistance distance coincide as well, i,e., $D_R(G) = 4W(G) - n(n-1)$.

A graph G is called a unicyclic graph if it contains exactly one cycle. For convenience, we represent a unicyclic graph G with the unique cycle $C_l = v_1v_2\cdots v_lv_1$ as $G = U(C_l; T_1, T_2, \cdots, T_l)$, where T_i is the component of $G - E(C_l)$ and T_i is a tree rooted at v_i; T_i is trivial if it is an isolated vertex. A fully loaded unicyclic graph is a unicyclic graph with the property that there is no vertex with degree less than 3 in its unique cycle. If $G = U(C_l; T_1, T_2, \cdots, T_l)$ is a fully loaded unicyclic graph, then $T_1, T_2, \cdots, T_l$ are all nontrivial. Let $\mathscr{U}(n;l)$ be the set of all fully loaded unicyclic graphs with n vertices and the unique cycle C_l, $\mathscr{U}(n)$ be the set of all fully loaded unicyclic graphs with n vertices; S_n and P_n be the star and the path on n vertices, respectively.

The paper is organized as follows. In Section 2, we introduce some lemmas and two transformations which decrease the degree resistance of graphs and a transformation which increases the degree resistance degree of a graph. In Section 3, we obtain a formula for calculating the degree resistance distance of graphs in $\mathscr{U}(n;l)$, and determine graphs in $\mathscr{U}(n)$ with the maximum and the minimum degree resistance distance by analytic methods.

2 Preliminary Results

For a graph G with $v \in V(G)$, $G - v$ denotes the graph obtained from G by deleting v (and its incident edges); and for an edge uv of the graph G (the complement of G, respectively), $G - uv(G + uv$, respectively) denotes the graph resulting from G by deleting (adding, respectively) the edge uv.

For a vertex $v \in V(G)$, let

$$r(v|G) = \sum_{u \in V(G)} r(u,v), \; S'(v|G) = \sum_{u \in V(G)} d(u)r(u,v)$$

C_n be the cycle on $n \geq 3$ vertices, for any two vertices $v_i, v_j \in V(C_n)$ with $i < j$, by Ohm's law, we have $r(v_i, v_j) = \frac{(j-i)(n+i-j)}{n}$; for any vertex $u \in V(C_n)$, it's suffice to see that $r(u|C_n) = \frac{n^2-1}{6}$, $S'(u|C_n) = \frac{n^2-1}{3}$.

Lemma 2.1([29]). Let T be any n vertices trees, then $(n-1)^2 \leq W(T) \leq \frac{n^3-n}{6}$, the left equality holds if and only if $G \cong S_n$ and the right holds if and only if $G \cong P_n$.

Lemma 2.2([2]). Let x be a cut vertex of a connected graph and a and b be vertices occurring in different components which arise upon deletion of x, then $r(a,b) = r(a,x) + r(x,b)$.

Lemma 2.3([26]). Let G_1 and G_2 be connected graphs with disjoint vertex sets, with n_1, n_2 vertices, and m_1, m_2 edges, respectively; $u_1 \in V(G_1)$, $u_2 \in V(G_2)$. Constructing the graph G by identifying the vertices u_1 with u_2, and denote the so obtained vertex by u. Then $D_R(G) = D_R(G_1) + D_R(G_2) + 2m_2 r(u_1|G_1) + 2m_1 r(u_2|G_2) + (n_2 - 1)S'(u_1|G_1) + (n_1 - 1)S'(u_2|G_2)$.

Let X and Y be two nontrivial connected graphs, such that $u \in V(X)$, $v \in V(Y)$; G be the graph obtained from X and Y by adding the edge uv. We form a graph $G' = \alpha(G,u)$ by identifying vertices u and v, and adding one pendant edge at u. We say that G' is a α-transform of G(see Figure 1).

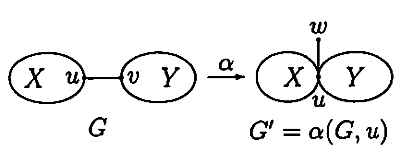


Figure 1. The transformation α

Lemma 2.4. Let $G' = \alpha(G,u)$ be a α-transform of the graph G, then $D_R(G) > D_R(G')$.

Proof. Assume that H is the graph induced by $V(Y) \cup \{u\}$, and $|E(X)| = m_1$, $|E(Y)| = m_2$, $|V(X)| = n_1$, $|V(Y)| = n_2$. By Lemma 2.3, one has $D_R(H) = D_R(Y) + 2r(v|Y) + S'(v|Y) + 2m_2 + n_2 - 1$. Then

$$\begin{aligned} & D_R(G) \\ = & D_R(X) + D_R(H) + 2(m_2+1)r(u|X) + 2m_1 r(u|H) + n_2 S'(u|X) \\ & + (n_1 - 1)S'(u|H) \\ = & D_R(X) + D_R(Y) + 2(m_2+1)r(u|X) + 2(m_1+1)r(v|Y) + n_2 S'(u|X) \\ & + n_1 S'(v|Y) + 2m_1 n_2 + 2m_2 n_1 + n_1 + n_2 \end{aligned}$$

and analogously,

$$\begin{aligned} D_R(G') = & D_R(X) + D_R(Y) + 2(m_2+1)r(u|X) + 2(m_1+1)r(v|Y) \\ & + n_2 S'(u|X) + n_1 S'(v|Y) + 2m_1 + 2m_2 + n_1 + n_2 \end{aligned}$$

So we get $D_R(G) - D_R(G') = 2m_1(n_2 - 1) + 2m_2(n_1 - 1) > 0$.
This proves the result.

Let X, Y and Z be three connected graphs with $u' \in V(Y)$ and $v' \in V(Z)$. Suppose that u and v are two vertices of X. Let G be the graph obtained from X, Y, Z by identifying v with v' and u with u', respectively. Let G' be the graph obtained from X, Y, Z by identifying the vertices u, v', u', and G'' be the graph obtained from X, Y, Z by identifying the vertices v, v', u', as shown in Figure 2. Analogously, we say that G' and G'' are β-transformed from G.

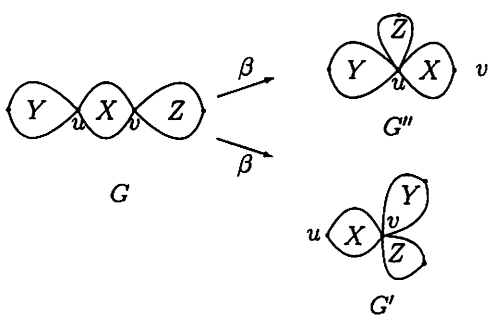


Figure 2. The transformation β

Lemma 2.5. Let G', G'' are graphs β-transformed from G. Then $D_R(G') < D_R(G)$ and $D_R(G'') < D_R(G)$.

Proof. Let $|E(X)| = m_1$, $|E(Y)| = m_2$, $|E(Z)| = m_3$; $|V(X)| = n_1$, $|V(Y)| = n_2$, $|V(Z)| = n_3$. H is the graph induced by $V(X) \cup V(Y)$ in G,

G'. By Lemma 2.3, one has

$$D_R(G) = D_R(Z) + D_R(H) + 2m_3 r(v|H) + 2(m_1+m_2) r(v|Z) \\ + (n_3-1)S'(v|H) + (n_1+n_2-2)S'(v|Z)$$

$$r(v|H) = \sum_{z\in V(H)} r(v,z) = r(v|X) + r(u|Y) + (n_2-1)r(u,v),$$

$$S'(v|H) = \sum_{z\in V(H)} d(z)r(v,z) = S'(v|X) + S'(u|Y) + 2m_2 r(u,v).$$

Thus,

$$D_R(G) = D_R(Z) + D_R(H) + 2m_3[r(v|X) + r(u|Y) + (n_2-1)r(u,v)] + \\ 2(m_1+m_2)r(v|Z) + (n_3-1)[S'(v|X) + S'(u|Y) + 2m_2 r(u,v)] \\ + (n_1+n_2-2)S'(v|Z)$$

and analogously,

$$D_R(G') = D_R(Z) + D_R(H) + 2m_3[r(v|X) + r(u|Y)] + 2(m_1+m_2)r(v|Z) \\ + (n_3-1)[S'(v|X) + S'(u|Y)] + (n_1+n_2-2)S'(v|Z)$$

Thus, $D_R(G) - D_R(G') = [2m_2(n_3-1) + 2m_3(n_2-1)]r(u,v) > 0$.
By the similar argument, one has

$$D_R(G) - D_R(G'') = [2m_2(n_3-1) + 2m_3(n_2-1)]r(u,v) > 0.$$

The proof is completed.

Suppose that G be a graph of order $n \geq 7$ obtained from a connected graph $H \neq P_1$ and a cycle $C_r = u_0u_1\cdots u_r$, $r \geq 4$ by identifying u_0 with a vertex u of the graph H; $G' = G - u_{r-1}u_{r-2} + uu_{r-2}$ (see Figure 3), We say G' is a graph γ-transformed from G.

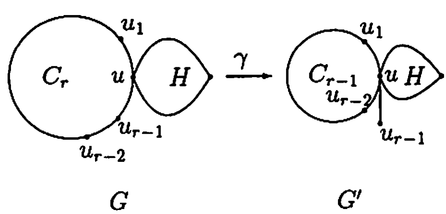


Figure 3. The transformation γ

Lemma 2.6. Let G and G' are two graphs depicted in Figure 3, then $D_R(G) \geq D_R(G')$.

Proof. Assume that $|E(H)| = m_h$, $|V(H)| = n_h$, by Lemma 2.3, one has

$$\begin{aligned}
&D_R(G)\\
&= D_R(C_r) + D_R(H) + 2m_h r(v|C_r) + 2rr(v|H) + (n_h - 1)S'(v|C_r)\\
&\quad + (r-1)S'(v|H)\\
&= \frac{1}{3}(r^3 - r) + D_R(H) + 2m_h\frac{r^2-1}{6} + 2rr(v|H) + (n_h - 1)\frac{r^2-1}{3}\\
&\quad + (r-1)S'(v|H)\\
&= D_R(H) + 2rr(v|H) + (r-1)S'(v|H) + \frac{r^3-1}{3} + \frac{r^2-1}{3}(m_h + n_h - 1)
\end{aligned}$$

and analogously,

$$\begin{aligned}
D_R(G') = D_R(H) + 2rr(v|H) + (r-1)S'(v|H) + \frac{r^3 - r^2 + 7r - 6}{3}\\
+ \frac{r^2 - 2r + 6}{3}m_h + \frac{r^2 - 2r + 3}{3}(n_h - 1)
\end{aligned}$$

Thus, one has

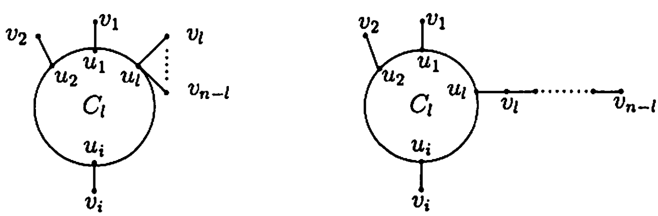
$$\begin{aligned}
D_R(G) - D_R(G') &= \frac{r^2 - 8r + 6}{3} + \frac{2r-7}{3}m_h + \frac{2r-4}{3}(n_h - 1)\\
&= \frac{(r-4)^2 - 10}{3} + \frac{2r-7}{3}m_h + \frac{2r-4}{3}(n_h - 1)\\
&> \frac{-10}{3} + \frac{1}{3}m_h + \frac{4}{3}(n_h - 1) \geq 0
\end{aligned}$$

Lemma 2.7[28]. Let G_0 be a connected graphs with $m_0 > 1$ edges, and $u, v \in V(G_0)$ be two distinct vertices with degree at least 3 in G_0 such that $r(u,v) = l$. Let $P_s = u_1u_2\cdots u_s$ and $P_t = v_1v_2\cdots v_t$ be two paths of order $s \geq 1$ and $t \geq 1$, respectively; $G_{s,t}$ be the graph obtained from G_0, P_s and P_t by adding edges uu_1, vv_1. Suppose that $G_{s-1,t+1} = G_{s,t} - u_ru_{r-1} + v_tu_r$ and $G_{s+1,t-1} = G_{s,t} - v_{t-1}v_t + u_sv_t$. Then either $D_R(G_{s,t}) < D_R(G_{s-1,t+1})$ or $D_R(G_{s,t}) < D_R(G_{s+1,t-1})$.

3 Extremal degree resistance distance in fully loaded unicyclic graphs

3.1 Extremal degree resistance distance of $\mathscr{U}(n;l)$

In this section, we shall determine the graph in $\mathscr{U}(n;l)$ with the maximum and the minimum degree resistance distance.



(a) $U(C_l; K_2, \cdots, K_2, S_{n-2(l-1)})$ (b) $U(C_l; K_2 \cdots, K_2, P_{n-2(l-1)})$

Figure 4.

Theorem 3.1. Let $G \in \mathscr{U}(n; l)$ be an arbitrary fully loaded unicyclic graph with girth l. Then $D_R(U(C_l; K_2, \cdots, K_2, S_{n-2(l-1)})) \leq D_R(G)$ and $D_R(G) \leq D_R(U(C_l; K_2 \cdots, K_2, P_{n-2(l-1)}))$, the first equality holds if and only if $G \cong U(C_l; K_2, \cdots, K_2, S_{n-2(l-1)})$ and the second does if and only if $G \cong U(C_l; K_2 \cdots, K_2, P_{n-2(l-1)})$. $U(C_l; K_2, \cdots, K_2, S_{n-2(l-1)})$ and $U(C_l; K_2, \cdots, K_2, P_{n-2(l-1)})$ are depicted in Figure 4.

Proof. For the left inequality, we suppose that G' be the graph in $\mathscr{U}(n; l)$ has the minimum degree resistance distance. By Lemma 2.4, Lemma 2.5, one has $G' \cong U(C_l; K_2, \cdots, K_2, S_{n-2(l-1)})$.

For the right inequality, suppose that G'' has the maximum degree resistance distance, then all trees $T_i (1 \leq i \leq l)$ attached to the cycle C_l must be paths by Lemma 2.4. Further, from Lemma 2.6, all paths are connected to one vertex of the only cycle C_l, then $G \cong U(C_l; K_2, \cdots, K_2, P_{n-2(l-1)})$.

This completes the proof.

3.2 The minimum degree resistance distance of $\mathscr{U}(n)$

In this section, we determine the graph in $\mathscr{U}(n)$ with the minimum degree resistance distance.

Theorem 3.2. Let $G \in \mathscr{U}(n)$ with $n \geq 6$, then $D_R(G) \geq 3n^2 - \frac{n}{3} - 26$, the equality holds if and only if $G \cong U(C_3; K_2, K_2, S_{n-4})$.

Proof. Firstly, we shall compute $D_R(U(C_l; K_2, \cdots, K_2, S_{n-2(l-1)}))$. As depicted in Figure 4(a), we let G_1 be the graph induced by u_i and v_i for $i = 1, 2, \cdots, l$, G_2 be the graph induced by v_i $(i = l+1, l+2, \cdots, n-l)$

and u_l. By the definition of degree resistance distance, one arrives at

$$\begin{aligned} &D_R(G_1) \\ =& \sum_{1\le i<j\le l}[d(u_i)+d(u_j)]r(u_i,u_j)+\sum_{1\le i<j\le l}[d(v_i)+d(v_j)]r(v_i,v_j) \\ &+\sum_{i=1}^{l}\sum_{j=1}^{l}[d(u_i)+d(v_j)]r(u_i,v_j) \\ =& 6\sum_{1\le i<j\le l}r(u_i,u_j)+2\sum_{1\le i<j\le l}(r(u_i,u_j)+2)+4\sum_{i=1}^{l}\sum_{j=1}^{l}(r(u_i,u_j)+1) \\ =& 6Kf(C_l)+2Kf(C_l)+4\binom{l}{2}+4(2Kf(C_l)+l^2) \\ =& 16Kf(C_l)+6l^2-2l; \end{aligned}$$

and

$$D_R(G_2)=D_R(S_{n-2l+1})=3(n-2l)^2-(n-2l);$$

$$r(u_l|G_2)=n-2l,\quad r(u_l|G_1)=l+\frac{l^2-1}{3};$$

$$S'(u_l|G_2)=n-2l,\quad S'(u_l|G_1)=\frac{2}{3}l^2+l-\frac{2}{3}.$$

By Lemma 2.3, one has

$$\begin{aligned} &D_R(U(C_l;K_2,\cdots,K_2,S_{n-2(l-1)})) \\ =& 16Kf(C_l)+6l^2-2l+3(n-2l)^2-(n-2l)+4l(n-2l) \\ &+2(n-2l)(l+\frac{l^2-1}{3})+(2l-1)(n-2l)+(n-2l)(\frac{2}{3}l^2+l-\frac{2}{3}) \\ =& \frac{1}{3}(-4l^3+4nl^2-9nl+10l+9n^2-10n) \end{aligned}$$

Nextly, we determine the graph in $U(C_l;K_2,\cdots,K_2,S_{n-2(l-1)})$ with the smallest degree resistance distance.

Let $f(l):=D_R(U(C_l;K_2,\cdots,K_2,S_{n-2(l-1)}))=\frac{1}{3}(-4l^3+4nl^2-9nl+10l+9n^2-10n)$, $3\le l\le[\frac{n}{2}]$.

Case 1. If $6\le n\le 7$, then $[\frac{n}{2}]=3$ and $l=3$. $f(l)_{min}=f(3)=3n^2-\frac{n}{3}-26$.

Case 2. If $8\le n\le 10$, then $[\frac{n}{2}]=4,5$ and $l=3,4,5$. Computing directly, one has

(1) $f(3) = \frac{490}{3} < f(4) = \frac{504}{3}$ for $n = 8$.
(2) $f(3) = \frac{642}{3} < f(4) = \frac{675}{3}$ for $n = 9$.
(3) $f(3) = \frac{812}{3} < f(4) = \frac{864}{3} < f(5) = 300$ for $n = 10$.
Case 3. If $n \geq 11$.
The first derivation of $f(l)$ is $f'(l) = -4l^2 + \frac{8n}{3}l - 3n + \frac{10}{3}$, $3 \leq l \leq [\frac{n}{2}]$.
Let $f'(l) = 0$, then $l_{1,2} = \frac{2n \mp \sqrt{4n^2 - 27n + 30}}{6}$.

It's easy to verify that $l_1 < 3$, $l_2 > [\frac{n}{2}]$. Thus, $f'(l) > 0$ in the interval $I := \left[3, 4, \cdots, [\frac{n}{2}]\right]$. Therefore, $f(l)_{min} = f(3) = 3n^2 - \frac{n}{3} - 26$.

Combining above three cases, one proves the theorem.

3.3 The maximum degree resistance distance of $\mathscr{U}(n)$

In this section, we determine the graph in $\mathscr{U}(n)$ with the maximum degree resistance distance.

Theorem 3.3. Let $G \in \mathscr{U}(n)$ with $n \geq 6$, then $D_R(G) \leq \frac{2}{3}n^3 - 28n + 104$, the equality holds if and only if $G \cong U(C_3; K_2, K_2, P_{n-4})$.

Proof. For the graph $U(C_l; K_2, \cdots, K_2, P_{n-2(l-1)})$, as shown in Figure 4(b), we let G_3 be the graph induced by u_i and v_i for $i = 1, 2, \cdots, l$, G_4 be the graph induced by v_i $(i = l, l+1, \cdots, n-l)$.

By Theorem 3.2, one has

$D_R(G_3) = 16Kf(C_l) + 6l^2 - 2l = \frac{4}{3}l^3 + 6l^2 - \frac{10}{3}l$, and

$$\begin{aligned} D_R(G_4) &= D_R(P_{n-2l+1}) \\ &= 4W(P_{n-2l+1}) - (n-2l+1)(n-2l) \\ &= 4 \times \frac{1}{6}\Big((n-2l+1)^3 - (n-2l+1)\Big) - (n-2l+1)(n-2l) \\ &= -\frac{16}{3}l^3 + (8n+4)l^2 - (4n^2+4n+\frac{2}{3})l + \frac{2}{3}n^3 + n^2 + \frac{n}{3}; \end{aligned}$$

$$r(v_l|G_4) = \frac{1}{2}(n-2l)(n-2l+1),\ r(v_l|G_3) = \frac{l^2}{3} + 3l - \frac{7}{3};$$

$$S'(v_l|G_4) = (n-2l)^2,\ \ S'(v_l|G_3) = \frac{2}{3}l^2 + 5l - \frac{8}{3}.$$

By Lemma 2.3, one has

$$D_R(U(C_l;K_2,\cdots,K_2,P_{n-2(l-1)}))$$
$$= \frac{4}{3}l^3+6l^2-\frac{10}{3}l-\frac{16}{3}l^3+(8n+4)l^2-(4n^2+4n+\frac{2}{3})l+\frac{2}{3}n^3+n^2+\frac{n}{3}$$
$$+2l(n-2l)(n-2l+1)+\frac{2}{3}(n-2l)(l^2+9l-7)+\frac{n-2l}{3}(2l^2+10l-8)$$
$$+(2l-1)(n-2l)^2$$
$$=\frac{1}{3}[28l^3-(20n+60)l^2+(39n+32)l+2n^3-21n]$$

In the following, we investigate graph in $U(C_l;K_2,\cdots,K_2,P_{n-2(l-1)})$ with the largest degree resistance distance.

Let $g(l):=D_R(U(C_l;K_2,\cdots,K_2,P_{n-2(l-1)}))=\frac{1}{3}[28l^3-(20n+60)l^2+(39n+32)l+2n^3-21n]$, $3\leq l\leq[\frac{n}{2}]$.

The first derivation of $g(l)$ is $g'(l)=28l^2-\frac{2l}{3}(20n+60)+13n+\frac{32}{3}$.

The roots of $g'(l)=0$ are $l_{1,2}=\dfrac{(10n+30)\mp\sqrt{100n^2-219n+228}}{42}$.

It's easy to verify that $l_1<3$, $l_2>3$ for $n\geq 6$.

Case 1. If $l_2\geq[\frac{n}{2}]$, then $g'(l)<0$ for $3\leq l\leq[\frac{n}{2}]$. $g(l)_{max}=g(3)=\frac{2}{3}n^3-28n+104$.

Case 2. If $l_2<[\frac{n}{2}]$, then $g'(l)<0$ for $3\leq l\leq l_2$ and $g'(l)>0$ for $l_2\leq l\leq[\frac{n}{2}]$.

In this case,

$$\max\{D_R(U(C_l;K_2,\cdots,K_2,P_{n-2(l-1)}))\}=max\{g(3),g([\frac{n}{2}])\}.$$

In the following, we shall compare $g(3)$ with $g([\frac{n}{2}])$. Let

$$h(n)=g(3)-g([\frac{n}{2}]).$$

If n is even and $n\geq 6$, then

$$h(n)=(\frac{2n^3}{3}-28n+104)-(\frac{n^3}{6}+\frac{3n^2}{2}-\frac{5n}{3})$$
$$=\frac{n^3}{2}-\frac{3n^2}{2}-\frac{79n}{3}+104$$
$$=\frac{1}{6}(n-6)(3n^2+9n-104)\geq 0.$$

Thus, $g(3) \geq g(\frac{n}{2})$.

$\max\{D_R(U(C_l; K_2, \cdots, K_2, P_{n-2(l-1)}))\} = D_R(U(C_3; K_2, K_2, P_{n-4}))$.

If n is odd and $n \geq 7$, then

$$\begin{aligned} h(n) &= (\frac{2n^3}{3} - 28n + 104) - (\frac{n^3}{6} + \frac{4n^2}{3} + \frac{11n}{3} - \frac{23}{2}) \\ &= \frac{n^3}{2} - \frac{4n^2}{3} - \frac{95n}{3} + \frac{231}{2} \\ &= \frac{1}{6}(n-7)(3n^2 + 13n - 99) \geq 0. \end{aligned}$$

So, $g(3) \geq g(\frac{n-1}{2})$.

$\max\{D_R(U(C_l; K_2, \cdots, K_2, P_{n-2(l-1)}))\} = D_R(U(C_3; K_2, K_2, P_{n-4}))$.

Combining the two cases above, one arrives at the desired result.

Acknowledgements: The authors would like to express their sincere gratitude to the referees for a very careful reading of the paper and for all their insightful comments and valuable suggestions, which led to a number of improvements in this paper. Projects supported by National Natural Science Foundation of China (11401192, 11326057), Natural Science Foundation of Hunan Province(2015JJ3031), Scientific Research Fund of Hunan Provincial Education Department (14A206).

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