An improved Fan-Type degree condition for k-linked graphs*

Jiuying Dong[a,b], Xueliang Li[c]
[a] School of Statistics, Jiangxi University of Finance and Economics, Nanchang 330013, China
[b] Research Center of Applied Statistics, Jiangxi University of Finance and Economics, Nanchang 330013, China
[c] Center for Combinatorics and LPMC-TJKLC Nankai University, Tianjin 300071, China
jiuyingdong@126.com, lxl@nankai.edu.cn

Abstract

Let G be a graph of order at least $2k$ and $s_1, s_2, \cdots, s_k, t_1, t_2, \cdots, t_k$ be any $2k$ distinct vertices of G, if there exist k disjoint paths $P_1, P_2, \cdots, P_k$ such that P_i is an $s_i - t_i$ path for $1 \leq i \leq k$, we call that G is k-linked. K. Kawarabayashi et al. showed that if $n \geq 4k-1 (k \geq 2)$ with $\sigma_2(G) \geq n+2k-3$, then G is k-linked. Li et al. showed that if G is a graph of order $n \geq 232k$ with $\sigma_2^*(G) \geq n+2k-3$, then G is k-linked. For sufficiently large n, it implied the result of K. Kawarabayashi et al. The main purpose of this paper is to lower the down bound of n in the result of Li et al.. We show that if G is a graph of order $n \geq 111k+9$ with $\sigma_2^*(G) \geq n+2k-3$, then G is k-linked. Thus, we improve the order bound to $111k+9$, and when $n \geq 111k+9$, it implies the result of K. Kawarabayashi et al.

Keywords: connected graph; k-linked graph; Fan-type degree condition

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1 Introduction

Let $G=(V,E)$ be a finite and undirected connected graph. Let A and B be two subsets of V, and let both $G[A]$ and $G[B]$ be the induced subgraph of A and B in G respectively. Let $N_G(A)$ denote the neighbor of A in G, simply denoted by $N(A)$. Let $N_B(A)$ denote the neighbor of A in B. We denote $e(G)=|E(G)|$. Let G be a graph of order at least $2k$ and $s_1, s_2, \cdots, s_k, t_1, t_2, \cdots, t_k$ be any $2k$ distinct vertices of G, if there exist k disjoint paths $P_1, P_2, \cdots, P_k$ such that P_i is an $s_i - t_i$ path for $1 \leq i \leq k$, we call that G is *k-linked.*

Finding the minimum positive integer $f(k)$ such that every $f(k)$-connected graph is k-linked is an interesting problem. Bollobás and Thomason [1] showed that if G is a $22k$-connected graph, then G is k-linked. It is the first linear upper

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bound for $f(k)$. Later, Thomas and Wollan [8] improved this bound to $f(k) \leq 10k$. It is currently the best result on determining the minimum value of $f(k)$. The result of Thomas and Wollan [8] is the following.

Theorem 1 *If G is a $2k$-connected graph with at least $5kn$ edges, then G is k-linked.*

Hence from the result, we know that if G is a $10k$-connected graph, then G is k-linked. There are also many other results on k-linked graph. Chen et al. [3] showed the following result.

Theorem 2 *If G is a $2k$-connected graph that contains a k-linked subgraph H, then G is k-linked.*

Y. Manoussakis [6] got the following result.

Theorem 3 *Let G be a graph and v be a vertex in G with $d(v) \geq 2k-1$. If $G-v$ is k-linked, then G is k-linked.*

Let $\sigma_2(G)$ denote the minimum degree sum of a pair of nonadjacent vertices. Ore [7] proved that every n-vertex graph G with $\sigma_2(G) \geq n$ is Hamiltonian. K. Kawarabayashi et al.[4] used the parameter $\sigma_2(G)$ to study the sufficient condition of G being k-linked. They showed the following result.

Theorem 4 *If $k \geq 2$ and*

$$\sigma_2(G) \geq \begin{cases} n+2k-3, & n \geq 4k-1 \\ \frac{2(n+5k)}{3} - 3, & 3k \leq n \leq 4k-2 \\ 2n-3, & 2k \leq n \leq 3k-1 \end{cases}$$

then G is k-linked.

And they gave out examples to present that these bounds are the best possible for all n and k.

Let $\sigma_2^*(G)$ denote the minimum degree sum of a pair of nonadjacent vertices at distance 2. Fan [2] showed that if G is a 2-connected graph of order $n(n \geq 3)$ and x, y are any two vertices of G at distance 2 with $max\{d(x), d(y)\} \geq \frac{n}{2}$, then G is Hamiltonian. Li et al.[5] used the parameter $\sigma_2^*(G)$ to study the sufficient condition of G being k-linked. They showed the following result.

Theorem 5 *If G is a graph of order $n \geq 232k$ with $\sigma_2^*(G) \geq n+2k-3$, then G is k-linked.*

In this paper, we continue to use the parameter $\sigma_2^*(G)$ to study the sufficient condition of G being k-linked, and try to lower the order bound as small as possible. Our main result is as below.

Theorem 6 *If G is a graph of order $n \geq 111k+9$ with $\sigma_2^*(G) \geq n+2k-3$, then G is k-linked.*

The result of Theorem 6 lower the order bound to $n \geq 111k+9$. And when $n \geq 111k+9$, Theorem 6 implies Theorem 4.

2 Proof of Theorem 6

Proof. Let G be a graph of order $n \geq 111k + 9$ with $\sigma_2^*(G) \geq n + 2k - 3$. Let S be a minimum cut of G, A and B be any two components of $G \setminus S$, without loss of generosity, assume that $|A| \leq |B|$. Let $N_A(S)$ be the neighbor of S in A and $N_B(S)$ be the neighbor of S in B. As S is a minimum cut, for every vertex $s \in S$, there exists some vertex $a_s \in N_A(S)$ such that $a_s s \in E(G)$, and some vertex $b_s \in N_B(S)$ such that $b_s s \in E(G)$. That is, the distance of a_s and b_s is 2. Hence $d(a_s) + d(b_s) \geq n + 2k - 3$. Since

$$n + 2k - 3 \leq d(a_s) + d(b_s) \leq |A| + |S| + |B| + |S| - 2 \leq n + |S| - 2, \quad (1)$$

we can get $|S| \geq 2k - 1$. So G is $(2k - 1)$-connected and the minimum degree of G is at least $2k - 1$. By Theorem 1, if $|S| \geq 10k$, then G is k-linked, so in the following, we assume that $2k - 1 \leq |S| \leq 10k - 1$.

We will divide two cases to show our theorem.

Case 1. $|S| = 2k - 1$

From (1) and $|S| = 2k - 1$, we can get $|A| + |S| + |B| = n$, $d(a_s) = |A| - 1 + |S|$ and $d(b_s) = |B| - 1 + |S|$. Hence both $G[N_A(S)]$ and $G[N_B(S)]$ are complete graphs. If $N_A(S) \subset A$, then $|N_A(S)| \geq |S|$, otherwise S is not a minimum cut. If $|B| \leq| S|$, then

$$n \leq 2|B| + |S| \leq 2|S| + |S| = 3|S| \leq 30k - 3,$$

a contradiction to $n \geq 111k + 9$. So $|B| \geq| S|$ and $|N_B(S)| \geq |S|$.

We can get that for any vertex $b'_s \in N_B(S)$, there exists some vertex $s' \in S$ such that $s'b'_s \in E(G)$ and there exists some vertex $a'_s \in N_A(S)$ such that $s'a'_s \in E(G)$, so the distance of a'_s and b'_s is 2. Thus from (1) and $|S| = 2k - 1$, we can get $|A|+|S|+|B| = n$, $d(a'_s) = |A|-1+|S|$ and $d(b'_s) = |B|-1+|S|$. For every vertex of $N_A(S)$, we can similarly get the above result. Thus, we can conclude that the following facts: for any vertex $b'_s \in N_B(S)$ the vertex b'_s is adjacent to every vertex of S and B; for any vertex $b \in B \setminus N_B(S)$ (if $N_B(S) \subset B$, then $|N_B(S)) \geq |S|$), the vertex b is adjacent to every vertex of $N_B(S)$; for any vertex $s \in S$ the vertex s is adjacent to every vertex of $N_B(S)$ and $N_A(S)$; for any vertex $a'_s \in N_A(S)$ the vertex a'_s is adjacent to every vertex of S and A; for any vertex $a \in A \setminus N_A(S)$ (if $N_A(S) \subset A$, then $|N_A(S)) \geq |S|$), the vertex a is adjacent to every vertex of $N_A(S)$.

In the following, we let $S = \{s_1, s_2, \cdots, s_{|S|}\}$, then $G[N_B(S) \cup \{s_1\}]$ is a complete graph of order at least $2k$, so $G[N_B(S) \cup \{s_1\}]$ is k-linked. Let $G_1 = G[N_B(S) \cup \{s_1\}]$ and $V_1 = V(G_1)$. Since s_2 is adjacent to every vertex of $N_B(S)$, and $|N_B(S)| \geq |S| \geq 2k - 1$, by Theorem 3, we know that $G[V_1 \cup \{s_2\}]$ is k-linked. Similarly we can get $G[V_1 \cup \{s_2\} \cup \{s_3\}]$ is k-linked, and so on, finally we can get that $G[V_1 \cup S]$ is k-linked. Since every vertex of $N_A(S)$ is adjacent to every vertex of S, by Theorem 3, we can get that $G[V_1 \cup S \cup N_A(S)]$ is k-linked. If $N_A(S) \subset A$, then every vertex of $A \setminus N_A(S)$ is adjacent to every vertex of $N_A(S)$, by Theorem 3, we have that $G[V_1 \cup S \cup A]$ is k-linked. If $N_B(S) \subset B$, then every vertex of $B \setminus N_B(S)$ is adjacent to every vertex of $N_B(S)$, by Theorem 3, we have that $G[B \cup S \cup A]$ is k-linked, that is G is k-linked.

Case 2. $2k \le |S| \le 10k-1$

If $G[A]$ is a complete subgraph of G, we will show that G is k-linked.

First, we claim that $|A| \le 2k$. In fact, if $|A| \ge 2k$, then $G[A]$ is k-linked, by Theorem 2, we know that G is k-linked. So we assume that $|A| \le 2k$. We know that

$$d(G[B]) \ge |N_B(S)|(n + 2k - 3 - |A| + 1 - |S| - | S|).$$

If $|N_B(S)| \ge 11k$, then

$$\begin{aligned} d(G[B]) - 10k|B| &\ge | N_B(S)|(n + 2k - 3 - |A| + 1 - |S| - | S|) - 10k|B| \\ &\ge k(n + 22k - 22 - |A| - 12|S|) \\ &\ge k[(n + 22k - 22 - 2k + 1 - 12(10k - 1)] \\ &= k(n - 100k - 9) \\ &\ge 0 (n \ge 100k + 9). \end{aligned}$$

So, when $n \ge 100k + 9$, we have $d(G[B]) \ge 10k|B|$. Hence we can get $e(G[B]) \ge 5k|B|$. Therefore by Theorem 1, we can get that $G[B]$ is k-linked. And again by Theorem 2, we can get G is k-linked. In the following we assume that $|N_B(S)| \le 11k - 1$. Let $N_B(N_B(S))$ be the neighbor of $N_B(S)$ in B. Then for every vertex v of $N_B(N_B(S))$, there must be some vertex v' of S such that the distance $dist(v, v') = 2$. We claim that $|N_B(N_B(S))| \ge 11k$. In fact, if $|N_B(N_B(S))| \le 11k - 1$, then for any $w \in N_B(S)$, we have

$$\begin{aligned} d(w) &\le |S| + |N_B(N_B(S))| + |N_B(S)| - 1 \\ &\le 10k - 1 + 11k - 1 + 11k - 1 \\ &= 32k - 3. \end{aligned}$$

Let $w' \in A$ such that the distance $dist(w, w') = 2$. Then

$$\begin{aligned} d(w) + d(w') &\le 32k - 3 + |A| - 1 + |S| \\ &\le 32k - 3 + 2k - 1 + 10k - 1 \\ &= 44k - 5, \end{aligned}$$

a contradiction to

$$d(w) + d(w') \ge n + 2k - 3 \ge 111k + 9 + 2k - 3 = 113k + 6.$$

Hence $|N_B(N_B(S))| \ge 11k$. We know that

$$d(G[B]) \ge |N_B(N_B(S))|(n + 2k - 3 - |A| -| S| + 1 - |N_B(S)|).$$

Thus we have

$$\begin{aligned} d(G[B]) - 10k|B| &\ge 11k(n + 2k - 3 - |A| -| S| + 1 - |N_B(S)|) - 10k|B| \\ &\ge k(n + 22k - 22 - |A| -| S| - 11|N_B(S)|) \\ &\ge k(n + 22k - 22 - 2k + 1 - 10k + 1 - 11(11k - 1)) \\ &\ge k(n - 111k - 9) \\ &\ge 0 (n \ge 111k + 9). \end{aligned}$$

That is $d(G[B]) \geq 10k|B|$. Hence we can get $e(G[B]) \geq 5k|B|$. By Theorem 1, we know that $G[B]$ is k-linked. And again by Theorem 2, we also can get G is k-linked.

If $G[A]$ is not a complete subgraph of G, we are still able to show that G is k-linked.

In fact, since $G[A]$ is not a complete subgraph of G, $G[A]$ has non-adjacent vertices and $\sigma_2^*(G[A]) \geq n+2k-3-2|S|$. Since $n \geq 48k-1$ and $|S| \leq 10k-1$, we have $n \geq 16k + 3|S| + 2$, that is $n - 16k - 3|S| - 2 \geq 0$. So we get

$$n + 2k - 1 - 2|S| - \frac{n - |S|}{2} - 10k \geq 0.$$

Since $|A| + |B| + |S| \leq n$ and $|A| \leq |B|$, we get $|A| \leq \frac{n-|S|}{2}$. Hence,

$$n + 2k - 1 - 2|S| -| A| - 10k \geq 0.$$

Therefore

$$\sigma_2^*(G[A]) \geq n + 2k - 3 - 2|S| \geq |A| + 10k - 2.$$

Similar to the previous proof, we can obtain that $G[A]$ is $10k$-connected, so $\delta(G[A]) \geq 10k$. Hence $e(G[A]) \geq \frac{1}{2}10k|A| = 5k|A|$. By Theorem 1, we know that $G[A]$ is k-linked. And again by Theorem 2, we also get G is k-linked.

Combining Cases 1 and 2, we complete the proof of Theorem 6. ∎

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