Sharp lower bounds on signed domination numbers of digraphs*

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Abstract: Let D be a finite and simple digraph with vertex set $V(D)$ and let $f : V(D) \to \{-1, 1\}$ be a two-valued function. If $\sum_{x \in N_D^-[v]} f(x) \geq 1$ for each $v \in V(D)$, where $N_D^-[v]$ consists of v and all vertices of D from which arcs go into v, then f is a *signed dominating function* on D. The sum $\sum_{v \in V(D)} f(v)$ is called the *weight* of f. The *signed domination number*, denoted by $\gamma_S(D)$, of D is the minimum weight of a signed dominating function on D.

In this work, we present different lower bounds on $\gamma_S(D)$ for general digraphs, show that these bounds are sharp, and give an improvement of a known lower bound obtained by Karami in 2009 [H. Karami, S.M. Sheikholeslami, A. Khodkar, Lower bounds on the signed domination numbers of directed graphs, Discrete Math. 309 (2009), 2567-2570]. Some of our results are extensions of well-known properties of the signed domination number of graphs.

Keywords: Digraph; Signed dominating function; Signed domination number

1 Introduction and terminology

The dominating theory has received considerable attention in recent years for its strong applicability. The classical domination has been developed to the signed vertex-domination [1, 2, 3, 4, 11, 13] and signed edge-domination [5, 8, 9, 10]. Moreover, the graphs in which these concepts are considered have been extended to digraphs [6, 7, 12]. In 1995, Dunbar et al. [3] first introduced the concept of signed domination in graphs. Afterwards,

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Zelinka [12] transferred it to digraphs. In 2009, Karami et al. [6] presented some lower bounds for signed domination number of digraphs in terms of the order, the maximum degree and the chromatic number of a digraph. In this work, we further study the lower bounds on signed domination number of digraphs, present some new lower bounds on this parameter, and show that these bounds are sharp. Moreover, we give an improvement of a known lower bound obtained by Karami et al. in [6]. Some of our results are extensions of well-known properties of the signed domination number of graphs in [4, 6, 13].

Let G be be a finite and simple graph with vertex set $V(G)$, and let $N_G[v]$ be the closed neighborhood of the vertex v consisting of v and all vertices of G adjacent to v. A *signed dominating function* on G, proposed in [3], is a function $f : V(G) \to \{-1, 1\}$ such that $\sum_{x \in N_G[v]} f(x) \geq 1$ for all $v \in V(G)$. The sum $\sum_{x \in V(G)} f(x)$ is the *weight* of f. The minimum of weights, taken over all signed dominating functions on G, is called the *signed domination number* of G, denoted by $\gamma_S(G)$.

Let D be a finite simple digraph with vertex set $V(D)$ and arc set $A(D)$. Let $V_0(D)$ be the set of vertices of odd indegree in D and denote $|V_0(D)| = n_0(D)$ (if the digraph D is clear from the context, we write V_0 and n_0 instead of $V_0(D)$ and $n_0(D)$, respectively). If D_1 is a subdigraph of D, then $e(D_1)$ denotes the number of arcs in D_1. If $X \subseteq V(D)$ and $x \in V(D)$, then we use $e(X, x)$ to denote the number arcs from X to x. If X and Y are two disjoint vertex sets of a digraph D, then $e(X, Y)$ means the number of arcs from X to Y. We use $\Delta^-(D)$ ($\Delta^+(D)$) to denote the maximum indegree (outdegree) of D and $\delta^-(D)$ ($\delta^+(D)$) to denote the minimum indegree (outdegree) of D. Let $N_D^-[v]$ be the set consisting of v and all vertices of D from which arcs go into v. When no ambiguity arises, we drop the index D of the notation. For a real-valued function $f : V(D) \to R$ the *weight* of f is $w(f) = \sum_{v \in V(D)} f(v)$, and for $S \subseteq V$, we define $f(S) = \sum_{v \in S} f(v)$, so $w(f) = f(V(D))$. For convenience, we denote $f(N_D^-[v])$ by $f[v]$.

A *signed dominating function* (abbreviated SDF) on a digraph D, proposed by Zelinka [12], is a two-valued function $f : V(D) \to \{-1, 1\}$ such that $f[v] \geq 1$ for every $v \in V(D)$. The minimum of weights $w(f)$, taken over all signed dominating functions f on D, is called the *signed domination number* of D, denoted by $\gamma_S(D)$. A *$\gamma_S(D)$-function* is an SDF on D of weight $\gamma_S(D)$. According to this definition, one can easily obtain $\gamma_S(C_n) = n$ for any directed cycle C_n of order n.

The *associated digraph* $D(G)$ of a graph G is the digraph obtained when each edge e of G is replaced by two oppositely oriented arcs with the same ends as e. Since $N_{D(G)}^-[v] = N_G[v]$ for each vertex $v \in V(G) = V(D(G))$, the following useful observation is valid.

Observation 1.1. If $D(G)$ is the associated digraph of a graph G, then $\gamma_S(D(G)) = \gamma_S(G)$.

2 Main results

Throughout this paper, we define $P = \{v \in V(D) \mid f(v) = 1\}$, $M = \{v \in V(D) \mid f(v) = -1\}$, where f is a $\gamma_S(D)$-function on D, and let $|P| = p$, $|M| = m$. Clearly, $|V(D)| = p + m$ and $\gamma_S(D) = p - m = |V(D)| - 2m = 2p - |V(D)|$.

Theorem 2.1. *If D is a digraph of order n, then*

$$\gamma_S(D) \geq 2\left\lceil \frac{\Delta^-(D)}{2} \right\rceil + 2 - n.$$

Furthermore, this lower bound is sharp.

Proof. Let $v \in V(D)$ be a vertex of maximum indegree $d^-(v) = \Delta^-(D)$, and let f be a $\gamma_S(D)$-function. Assume first that $f(v) = 1$. Since $f[v] \geq 1$, we deduce that $e(P,v) \geq e(M,v)$, and then,

$$p \geq e(P,v) + 1 \geq \frac{e(P,v) + e(M,v)}{2} + 1 = \frac{\Delta^-(D)}{2} + 1.$$

Hence, we have $p \geq \lceil \frac{\Delta^-(D)}{2} \rceil + 1$. This yields $\gamma_S(D) = 2p - n \geq 2\lceil \frac{\Delta^-(D)}{2} \rceil + 2 - n$.

Assume now that $f(v) = -1$. As $f[v] \geq 1$, we know $e(P,v) \geq e(M,v) + 2$. It follows that

$$p \geq e(P,v) \geq \frac{e(P,v) + e(M,v) + 2}{2} = \frac{\Delta^-(D)}{2} + 1,$$

and then, $p \geq \lceil \frac{\Delta^-(D)}{2} \rceil + 1$. This implies again that $\gamma_S(D) \geq 2\lceil \frac{\Delta^-(D)}{2} \rceil + 2 - n$.

In order to show the sharpness of this bound, we consider the digraph D with $V(D) = \{v, v_1, v_2, ..., v_{2r}\}$ and $A(D) = \{v_j v \mid j = 1, 2, ..., 2r\} \cup \{v_{2i}v_{2i-1} \mid i = 1, 2, .., r\} \cup \{v_{2i}v_{2i+1} \mid i = 1, 2, .., r\}$, where $v_{2r+1} = v_1$. It is easy to see that $n = 2r + 1$, $d^-(v) = 2r$, $d^-(v_{2i}) = 0$, $d^-(v_{2i-1}) = 2$ for $i = 1, 2, ..., r$. Since $d^-(v_{2i}) = 0$, then for any SDF f on D, we have $f(v_{2i}) = 1$ for $i = 1, 2, ..., r$ and since $f[v] \geq 1$, at most r of the rest $r + 1$ vertices can be assigned to -1 . Define a signed dominating function f on D as follows

$$f(v) = 1, f(v_{2i}) = 1, f(v_{2i-1}) = -1, \text{ for } i = 1, 2, ..., r.$$

Obviously, f is a $\gamma_S(D)$-function on D. So $\gamma_S(D) = 1 = 2\lceil \frac{\Delta^-(D)}{2} \rceil + 2 - n$ and the result follows. □

Corollary 2.1. *Let D be a digraph of order n. If D contains a vertex of indegree k, then $\gamma_S(D) \geq k+2-n$.*

By using Observation 1.1 and Corollary 2.1, one can easily derive the following result which was obtained by Haasa et al. in 2004.

Corollary 2.2. (Haasa et al. [4]) *If G is a graph of order n and contains a vertex of degree k, then $\gamma_S(G) \geq k+2-n$.*

The following theorem provides a lower bound on the signed domination number of digraphs, which implies some well-known results on the signed domination number of graphs as well as digraphs.

Theorem 2.2. *Let D be a digraph of order n. Then*

$$\gamma_S(D) \geq \frac{n(\delta^+(D)+2-\Delta^+(D))+2n_0}{\delta^+(D)+2+\Delta^+(D)}.$$

Furthermore, this lower bound is sharp.

Proof. Let f be a $\gamma_S(D)$-function. Then for any vertex $v \in V_0$, we have $f[v] \geq 2$. It follows that

$$\sum_{v\in V(D)} f[v] = \sum_{v\in V_0} f[v] + \sum_{v\in V(D)\setminus V_0} f[v] \geq 2n_0 + n - n_0 = n + n_0.$$

In addition,

$$\begin{array}{rcl} \sum_{v\in V(D)} f[v] & = & \sum_{v\in V(D)}(d^+(v)+1)f(v) \\ & = & \sum_{v\in P}(d^+(v)+1) - \sum_{v\in M}(d^+(v)+1) \\ & \leq & p(\Delta^+(D)+1) - m(\delta^+(D)+1) \\ & = & n(\Delta^+(D)+1) - m(\delta^+(D)+2+\Delta^+(D)). \end{array}$$

The above two inequality chains imply

$$m \leq \frac{n\Delta^+(D) - n_0}{\delta^+(D)+2+\Delta^+(D)},$$

and hence, we obtain the desired bound as follows

$$\gamma_S(D) = n - 2m \geq \frac{n(\delta^+(D)+2-\Delta^+(D))+2n_0}{\delta^+(D)+2+\Delta^+(D)}.$$

Clearly, any directed cycle of order n achieves this bound. So this lower bound is sharp. □

Corollary 2.3. *Let D be a digraph of order n such that $d^+(v) = r$ for all $v \in V(D)$. Then*

$$\gamma_S(D) \geq \frac{n+n_0}{r+1} \geq \frac{n}{r+1}.$$

Corollary 2.4. *Let D be an r-regular digraph of order n. If r is odd, then*

$$\gamma_S(D) \geq \frac{2n}{r+1}.$$

Combining Observation 1.1 with Theorem 2.2 one can easily obtain the following lower bound on $\gamma_S(G)$ for any undirected graph G.

Corollary 2.5. (Zhang et al. [13]) If G is a graph of order n, maximum degree $\Delta(G)$, minimum degree $\delta(G)$ and $n_0(G)$ is the number of vertices of odd degree in G, then

$$\gamma_S(G) \geq \frac{n(\delta(G)+2-\Delta(G))+2n_0(G)}{\delta(G)+2+\Delta(G)} \geq \frac{\delta(G)+2-\Delta(G)}{\delta(G)+2+\Delta(G)}n.$$

Now we give another type of lower bound on $\gamma_S(D)$ in terms of $|V(D)|$, $|A(D)|$, $\delta^+(D)$, $\Delta^+(D)$ and $n_0(D)$.

Theorem 2.3. *Let D be a digraph of order n. Then*

$$\gamma_S(D) \geq \max\left\{\frac{n(1-\Delta^+(D))+|A(D)|+n_0}{1+\Delta^+(D)}, \frac{n(1+\delta^+(D))-|A(D)|+n_0}{1+\delta^+(D)}\right\}.$$

Furthermore, this lower bound is sharp.

Proof. Let f be a $\gamma_S(D)$-function. Then following the proof of Theorem 2.2, we obtain

$$\begin{aligned}
n+n_0 &\leq \textstyle\sum_{v\in V(D)} f[v] = \sum_{v\in P}(d^+(v)+1) - \sum_{v\in M}(d^+(v)+1)\\
&= p-m+\textstyle\sum_{v\in P} d^+(v) - \sum_{v\in M} d^+(v)\\
&= 2p-n+2\textstyle\sum_{v\in P} d^+(v) - \sum_{v\in V(D)} d^+(v)\\
&= 2p-n+\textstyle\sum_{v\in V(D)} d^+(v) - 2\sum_{v\in M} d^+(v)\\
&= 2p-n+2\textstyle\sum_{v\in P} d^+(v) - |A(D)|\\
&= 2p-n+|A(D)|-2\textstyle\sum_{v\in M} d^+(v).
\end{aligned}$$

This leads to

$$n+n_0 \leq 2p-n+2p\Delta^+(D)-|A(D)|$$

as well as

$$n+n_0 \leq 2p-n+|A(D)|-2(n-p)\delta^+(D),$$

and thus

$$2p \geq \frac{2n+|A(D)|+n_0}{1+\Delta^+(D)} \tag{1}$$

and

$$2p \geq \frac{2n-|A(D)|+n_0+2n\delta^+(D)}{1+\delta^+(D)}. \tag{2}$$

Using (1) and (2), we deduce that

$$\gamma_S(D) = 2p - n \geq \frac{n(1-\Delta^+(D)) + |A(D)| + n_0}{1+\Delta^+(D)}$$

and

$$\gamma_S(D) = 2p - n \geq \frac{n(1+\delta^+(D)) - |A(D)| + n_0}{1+\delta^+(D)},$$

and these imply the desired bound. Obviously, this bound is sharp for any directed cycle of order n. □

Theorem 2.3 also implies Corollaries 2.3 and 2.4. The following theorem provides a new type of lower bound for the signed domination number of digraphs, which implies the result in [6] immediately and improves it in some cases.

Theorem 2.4. *Let D be digraph of order n with outdegree sequence $d_1^+ \geq d_2^+ \geq \ldots \geq d_n^+$ and let λ be the smallest positive integer for which $\sum_{k=1}^{\lambda} d_k^+ - \sum_{k=\lambda+1}^{n} d_k^+ \geq 2(n-\lambda)+n_0$. Then $\gamma_S(D) \geq 2\lambda - n$. Furthermore, this lower bound is sharp.*

Proof. Let f be a $\gamma_S(D)$-function. Following the proof of Theorem 2.2, we obtain

$$\begin{aligned} n + n_0 &\leq p - m + \textstyle\sum_{v\in P} d^+(v) - \sum_{v\in M} d^+(v) \\ &\leq 2p - n + \textstyle\sum_{k=1}^{p} d_k^+ - \sum_{k=p+1}^{n} d_k^+. \end{aligned}$$

Thus $\sum_{k=1}^{p} d_k^+ - \sum_{k=p+1}^{n} d_k^+ \geq 2(n-p) + n_0$. By the assumption on λ, we deduce that $p \geq \lambda$, which implies $\gamma_S(D) = 2p - n \geq 2\lambda - n$.

To see the sharpness of this bound, we consider the digraph D with $V(D) = \{v_1, v_2, v_3, v\}$ and $A(D) = \{v_1v, v_2v, v_3v\}$. It is easy to check that for this digraph, $n = 4$, $n_0 = 1$, $\lambda = 3$ and $\gamma_S(D) = 2$. So the proof of Theorem 2.4 is complete. □

Corollary 2.6. (Karami et al. [6]) *Let D be a digraph of order n with outdegree sequence $d_1^+ \geq d_2^+ \geq \ldots \geq d_n^+$ and let μ be the smallest positive integer for which $\sum_{k=1}^{\mu} d_k^+ - \sum_{k=\mu+1}^{n} d_k^+ \geq 2(n-\mu)$. Then $\gamma_S(D) \geq 2\mu - n$.*

Note that Corollary 2.6 is sharp for $n_0 = 0$. However, for $n_0 \neq 0$, the lower bound in Theorem 2.4 is better than Corollary 2.6. See the following example.

Example. Let D be an r-regular digraph with order n such that r is odd and $n \geq 2r+2$. Then we can see that $n_0 = n$, the parameter λ in Theorem 2.4 equals $\lceil \frac{3n+nr}{2r+2} \rceil$ and the parameter μ in Corollary 2.6 is equal to $\lceil \frac{2n+nr}{2r+2} \rceil$. Since $n \geq 2r+2$, we have $\lambda = \lceil \frac{3n+nr}{2r+2} \rceil \geq \lceil \frac{2n+nr}{2r+2} \rceil + 1 = \mu + 1$, so $2\lambda - n \geq 2\mu - n + 2$.

The next two theorems provide two sharp lower bounds for the signed domination number of a digraph D depending on its order and a parameter

λ, which is determined on the basis of the outdegree sequence as well as indegree sequence of D.

Theorem 2.5. *Let D be a digraph of order n with indegree sequence $d^-_{i_1} \geq d^-_{i_2} \geq \ldots \geq d^-_{i_n}$ and outdegree sequence $d^+_{j_1} \leq d^+_{j_2} \leq \ldots \leq d^+_{j_n}$. Then $\gamma_S(D) \geq n - 2\lambda$, where $\lambda \geq 0$ is the largest integer such that*

$$\sum_{k=1}^{\lambda} \left(d^+_{j_k} - \left\lfloor \frac{d^-_{i_k}}{2} \right\rfloor + 1 \right) \leq \sum_{k=1}^{n-\lambda} \left\lfloor \frac{d^-_{i_k}}{2} \right\rfloor.$$

Furthermore, this lower bound is sharp.

Proof. Let f be a $\gamma_S(D)$-function. The condition $f[v] \geq 1$ implies that $e(P,v) \geq e(M,v)$ for $v \in P$ and $e(P,v) \geq e(M,v) + 2$ for $v \in M$. Thus we obtain $d^-(v) = e(P,v) + e(M,v) \geq 2e(M,v)$ and so $e(M,v) \leq \lfloor \frac{d^-(v)}{2} \rfloor$ for each vertex $v \in P$. It follows that

$$e(M,P) = \sum_{v \in P} e(M,v) \leq \sum_{v \in P} \left\lfloor \frac{d^-(v)}{2} \right\rfloor \leq \sum_{k=1}^{n-m} \left\lfloor \frac{d^-_{i_k}}{2} \right\rfloor \leq (n-m) \left\lfloor \frac{\Delta^-(D)}{2} \right\rfloor. \tag{3}$$

In addition, $d^-(v) = e(P,v) + e(M,v) \geq 2e(M,v) + 2$ and thus $e(M,v) \leq \lfloor \frac{d^-(v)}{2} \rfloor - 1$ for each vertex $v \in M$. So we have

$$e(D\langle M \rangle) = \sum_{v \in M} e(M,v) \leq \sum_{v \in M} \left(\left\lfloor \frac{d^-(v)}{2} \right\rfloor - 1 \right) \leq \sum_{k=1}^{m} \left(\left\lfloor \frac{d^-_{i_k}}{2} \right\rfloor - 1 \right),$$

which implies that

$$\begin{aligned} e(M,P) &= \textstyle\sum_{v \in M} d^+(v) - e(D\langle M \rangle) \\ &\geq \textstyle\sum_{k=1}^{m} d^+_{j_k} - \sum_{k=1}^{m} \left(\left\lfloor \frac{d^-_{i_k}}{2} \right\rfloor - 1 \right) \\ &\geq m\delta^+(D) - m(\lfloor \frac{\Delta^-(D)}{2} \rfloor - 1). \end{aligned} \tag{4}$$

Inequalities (3) and (4) lead to

$$\sum_{k=1}^{m} \left(d^+_{j_k} - \left\lfloor \frac{d^-_{i_k}}{2} \right\rfloor + 1 \right) \leq \sum_{k=1}^{n-m} \left\lfloor \frac{d^-_{i_k}}{2} \right\rfloor \tag{5}$$

and

$$m\delta^+(D) - m(\lfloor \frac{\Delta^-(D)}{2} \rfloor - 1) \leq (n-m) \left\lfloor \frac{\Delta^-(D)}{2} \right\rfloor. \tag{6}$$

By the assumption on λ, we know from (5) that $m \leq \lambda$, and then, $\gamma_S(D) = n - 2m \geq n - 2\lambda$. Clearly, this bound is sharp for any directed cycle of order n. □

From (6), we have $m \leq \frac{n\lfloor \frac{\Delta^-(D)}{2} \rfloor}{\delta^+(D)+1}$, so the following result holds.

Corollary 2.7. *If D is a digraph of order n, then*

$$\gamma_S(D) \geq \frac{1+\delta^+(D) - 2\lfloor \frac{\Delta^-(D)}{2} \rfloor}{1+\delta^+(D)} n.$$

Counting the arcs from P to M, we obtain the next theorem analogously to the proof of Theorem 2.5.

Theorem 2.6. *Let D be a digraph of order n with indegree sequence $d^-_{i_1} \leq d^-_{i_2} \leq ... \leq d^-_{i_n}$ and outdegree sequence $d^+_{j_1} \geq d^+_{j_2} \geq ... \geq d^+_{j_n}$. Then $\gamma_S(D) \geq n - 2\lambda$, where $\lambda \geq 0$ is the largest integer such that*

$$\sum_{k=1}^{\lambda} \left(\left\lceil \frac{d^-_{i_k}}{2} \right\rceil + 1 \right) \leq \sum_{k=1}^{n-\lambda} \left(d^+_{j_k} - \left\lceil \frac{d^-_{i_k}}{2} \right\rceil \right).$$

Furthermore, this lower bound is sharp.

Corollary 2.8. *If D is a digraph of order n, then*

$$\gamma_S(D) \geq \frac{1-\Delta^+(D) + 2\lceil \frac{\delta^-(D)}{2} \rceil}{1+\Delta^+(D)} n.$$

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References

[1] W. Chen, E. Song, Lower bounds on several versions of signed domination number, Discrete Math. 308 (2008) 1837-1846.

[2] D. Delić, C. Wang, Upper signed k-domination in a general graph, Information Processing Lett. 110 (2010) 662-665.

[3] J.E. Dunbar, S.T. Hedetniemi, M.A. Henning, P.J. Slater, Signed domination in graphs, in: Graph Theory, Combinatorics, and Applications, vol. 1, John Wiley and Sons, Inc., New York, 1995, pp. 311-322.

[4] R. Haasa, T.B. Wexler, Signed domination numbers of a graph and its complement, Discrete Math. 283 (2004) 87-92.

[5] H. Karami, S.M. Sheikholeslami, A. Khodkar, Some notes on signed edge domination in graphs, Graphs Comb. 24 (2008) 29-35.

[6] H. Karami, S.M. Sheikholeslami, A. Khodkar, Lower bounds on the signed domination numbers of directed graphs, Discrete Math. 309 (2009) 2567-2570.

[7] W. Meng, S. Li, Q. Guo, Y. Guo, Signed cycle domination numbers of digraphs, Ars Combin. 115 (2014) 425-433.

[8] X. Pi, H. Liu, On the characterization of trees with signed edge domination numbers 1, 2, 3, or 4, Discrete Math. 309 (2009) 1779-1782.

[9] C. Wang, The signed star domination numbers of the Cartesian product, Discrete Applied Math. 155 (2007) 1497-1505.

[10] B. Xu, Two classes of edge domination in graphs, Discrete Applied Math. 154 (2006) 1541-1546.

[11] B. Zelinka, Signed total domination number of a graph, Czechoslovak Math. J. 51 (126) (2001) 225-229.

[12] B. Zelinka, Signed domination numbers of directed graphs, Czechoslovak Mathematical Journal 55 (2005) 479-482.

[13] Z. Zhang, B. Xu, Y. Li, L. Liu, A note on the lower bounds of signed domination number of a graph, Discrete Math. 195 (1999) 295-298.