On $h(x)$-Lucas quaternion polynomials

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Abstract

In this paper, we introduce $h(x)$-Lucas quaternion polynomials that generalize k-Lucas quaternion numbers that generalize Lucas quaternion numbers. Also we derive the Binet formula and generating function of $h(x)$-Lucas quaternion polynomial sequence.

Keywords and Phrases: Lucas numbers; Lucas polynomials; Lucas quaternions; Quaternion algebra; Generating Function.

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1 Introduction

Investigation of normed division algebras is a topic of a great interest today. It is well known that the quaternions **H** are the noncommutative normed division algebra over the real numbers $\mathbb{R}$. Due to the noncommutativity, one cannot directly extend various results on real and complex numbers to quaternions. The book by Conway and Smith [3] gives a great deal of useful background on quaternions, much of it based on Coxeter's paper [4].

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Quaternions made further appearance ever since in associative algebras, analysis, topology, and physics. Nowadays quaternions play an important role in computer science, quantum physics, signal and color image processing, and so on (e.g. [1]).

The investigation of special number sequences over **H** and **O** which are not analogs of ones over $\mathbb{R}$ and $\mathbb{C}$ has attracted some recent attention (see, e.g.,[2, 7, 8, 10, 11, 12]). While majority of papers in the area are devoted to some Fibonacci-type special number sequences over $\mathbb{R}$ and $\mathbb{C}$, only few of them deal with Fibonacci-type special number sequences over **H** and **O** (see, e.g., [2, 7, 8, 10, 11, 12]), notwithstanding the fact that there are a lot of papers on various types of Lucas number sequences over $\mathbb{R}$ and $\mathbb{C}$ (see, for example, [6, 13, 19, 20] and the references therein).

In this note, we introduce $h(x)$-Lucas quaternion polynomials that generalize k-Lucas quaternion numbers that generalize Lucas quaternion numbers. Also we derive the Binet formula and generating function of $h(x)$-Lucas quaternion polynomial sequence.

The rest of the paper is structured as follows. Some preliminaries which are required are given in Section 2. In Section 3 we introduce $h(x)$-Lucas quaternion polynomials that generalize k-Lucas quaternion numbers that generalize Lucas quaternion numbers and derive the Binet formula and generating function of $h(x)$-Lucas quaternion polynomial sequence. Section 4 ends with our conclusion.

2 Some Preliminaries

We start this section by introducing some definitions and notations that will greatly help us in the statement of the results.

Definition 1 ([5, 15, 21]) *The classic Lucas* $\{L_n\}_{n\in\mathbb{N}}$ *sequence is de-*

fined by

$$L_0 = 2,\ L_1 = 1 \text{ and } L_n = L_{n-1} + L_{n-2} \quad \text{for } n \geq 2, \tag{1}$$

The books written by Hoggat [9], Koshy [15] and Vajda [21] collects and classifies many results dealing with these number sequences , most of them are obtained quite recently.

In [17], the authors introduced the $h(x)$-Lucas polynomials.

Definition 2 ([17]) *Let $h(x)$ be a polynomial with real coefficients. The $h(x)$-Lucas polynomials $\{L_{h,n}(x)\}_{n=0}^{\infty}$ are defined by the recurrence relation*

$$L_{h,n+1}(x) = h(x)L_{h,n}(x) + L_{h,n-1}(x),\ n \geq 1, \tag{2}$$

with initial conditions $L_{h,0}(x) = 2,\ L_{h,1}(x) = h(x)$.

For k any real number and $h(x) = k$, it is obtained the k-Lucas numbers $L_{k,n}$. For $k = 1$ it is obtained the usual Lucas numbers L_n.

The quaternion, which is a type of hypercomplex numbers, was formally introduced by Hamilton in 1843.

Definition 3 *The real quaternion is defined by*

$$q = q_r + q_i i + q_j j + q_k k$$

where q_r, q_i, q_j and q_k are real numbers and i, j and k are complex operators obeying the following rules

$$i^2 = j^2 = k^2 = -1, ij = -ji = k, jk = -kj = i, ki = -ik = j.$$

The quaternions in Clifford algebra $\mathbf{C}$ are a normed division algebra with four dimensions over the real numbers larger than the complex numbers. The field $\mathbf{H} \cong \mathbf{C}^2$ of quaternions

$$\alpha = \sum_{s=0}^{3} \alpha_s e_s,\ \alpha_s \in \mathbb{R}, s = 0, 1, 2, 3. \tag{3}$$

is an four-dimensional non-commutative $\mathbb{R}$-field generated by four base elements $e_0 \cong 1, e_1 \cong i, e_2 \cong j$ and $e_3 \cong k$. The multiplication rules for the basis of $\mathbf{H}$ are listed in the following table

$\times$	1	e_1	e_2	e_3
1	1	e_1	e_2	e_3
e_1	e_1	-1	e_3	$-e_2$
e_2	e_2	$-e_3$	-1	e_1
e_3	e_3	e_2	$-e_1$	-1

(4)

Table 1. The multiplication table for the basis of **H**.

A quaternion $\alpha = \sum_{s=0}^{3} \alpha_s e_s \in \mathbf{H}$ is pieced into two parts with scalar piece $S_\alpha = \alpha_0$ and vectorial piece $\overrightarrow{V_\alpha} = \sum_{s=1}^{3} \alpha_s e_s$. We also write $\alpha = S_\alpha + \overrightarrow{V_\alpha}$. The conjugate of $\alpha = S_\alpha + \overrightarrow{V_\alpha}$ is then defined as

$$\overline{\alpha} = S_\alpha - \overrightarrow{V_\alpha} = \alpha_0 e_0 - \sum_{s=1}^{3} \alpha_s e_s.$$

We call a real quaternion pure if its scalar part vanishes. Let α and β be two quaternions such that $\alpha = S_\alpha + \overrightarrow{V_\alpha}$ and $\beta = S_\beta + \overrightarrow{V_\beta}$, where $S_\alpha = \alpha_0, S_\beta = \beta_0$, $\overrightarrow{V_\alpha} = \sum_{s=1}^{3} \alpha_s e_s$ and $\overrightarrow{V_\beta} = \sum_{s=1}^{3} \beta_s e_s$. Summation of α and β is defined as

$$\alpha + \beta = (S_\alpha + S_\beta) + (\overrightarrow{V_\alpha} + \overrightarrow{V_\beta}) = \sum_{s=0}^{3} (\alpha_s + \beta_s) e_s.$$

Multiplication of the quaternion α with a scalar $\lambda \in \mathbb{R}$ is defined as

$$\lambda\alpha = \lambda S_\alpha + \lambda\overrightarrow{V_\alpha} = \sum_{s=0}^{3} (\lambda\alpha_s)\, e_s.$$

In addition, quaternionic multiplication of α and β is defined as

$$\alpha\beta = S_\alpha S_\beta + S_\alpha V_\beta + V_\alpha S_\beta - V_\alpha \cdot V_\beta + V_\alpha \times V_\beta,$$

where $V_\alpha \cdot V_\beta = \alpha_1\beta_1 + \alpha_2\beta_2 + \alpha_3\beta_3$ and $V_\alpha \times V_\beta = (\alpha_2\beta_3 - \alpha_3\beta_2)i - (\alpha_1\beta_3 - \alpha_3\beta_1)j + (\alpha_1\beta_2 - \alpha_2\beta_1)k$.

In recent years there has been a flurry of activity for doing research with Lucas quaternion. Horadam defined the nth Lucas quaternions as follows:

Definition 4 ([10]) *The Lucas quaternion numbers that are given for the n^{th} classic Lucas L_n number are defined by the following recurrence relations:*

$$T_n = L_n + iL_{n+1} + jL_{n+2} + kL_{n+3} \tag{5}$$

where $n = 0, \pm 1, \pm 2, \ldots$.

The basic properties of Lucas quaternion numbers can be found in [10]. In [7], the author investigate the Lucas quaternions and give the generating functions and Binet's formulas for these quaternions.

3 The $h(x)$-Lucas quaternion polynomials

In this section, we introduce $h(x)$-Lucas quaternion polynomials that generalize k-Lucas quaternion numbers. Also we derive the Binet formula and and generating function of $h(x)$-Lucas quaternion polynomial sequence.

Let e_i, $i = 0, 1, 2, 3$ be a basis of $\mathbf{H}$ which satisfy the non-commutative multiplication rules are listed in Table 1 in (4). Let $h(x)$ be a polynomial with real coefficients.

We now introduce $h(x)$-Lucas quaternion polynomials that generalize k-Lucas quaternion numbers and derive the Binet formula and and generating function of $h(x)$-Lucas quaternion polynomial sequence.

Definition 5 *The $h(x)$-Lucas quaternion polynomials $\{T_{h,n}(x)\}_{n=0}^{\infty}$ are defined by the recurrence relation*

$$T_{h,n}(x) = \sum_{s=0}^{3} L_{h,n+s}(x)e_s \tag{6}$$

where $L_{h,n}(x)$ is the n^{th} $h(x)$-Lucas polynomial.

For k any real number and $h(x) = k$, it is obtained the k-Lucas numbers $L_{k,n}$ from the $h(x)$-Lucas polynomials $L_{h,n}(x)$, and thus for $h(x) = k$ we obtain k-Lucas quaternion numbers $T_{k,n}$ from the $h(x)$-Lucas quaternion polynomials $T_{h,n}(x)$.

Generating functions are known as the most surprising, useful, and clever tools in mathematics. The ordinary generating function (OGF) of the sequence $\{a_n\}_{n=0}^{\infty}$ is defined by $g(x) = \sum\limits_{n=0}^{\infty} a_n x^n$ [18] and the exponential generating function (EGF) of a sequence $\{a_n\}_{n=0}^{\infty}$ is defined by $g(x) = \sum\limits_{n=0}^{\infty} a_n \frac{x^n}{n!}$ [14].

Throughout this paper, we shall be concerned with the ordinary generating functions. The generating function $g_T(t)$ of the sequence $\{T_{h,n}(x)\}_{n=0}^{\infty}$ is defined by

$$g_T(t) = \sum_{n=0}^{\infty} T_{h,n}(x)t^n. \tag{7}$$

We consider $g_T(t)$ as a formal power series which is needed not take care of the convergence. For general material on generating functions the interested reader may refer to the books [16, 22] and the references therein.

For convenience, we use the following notations: $L_{h,n} = L_{h,n}(x)$ and $T_{h,n} = T_{h,n}(x)$.

Equipped with the definitions and properties above, we can present the fundamental theorems of this paper as follows.

Theorem 6 *The generating function for the $h(x)$-Lucas quaternion polynomials $T_{h,n}(x)$ is*

$$g_T(t) = \frac{T_{h,0} + (T_{h,1} - h(x)T_{h,0})\, t}{1 - h(x)t - t^2} \tag{8}$$

and

$$g_T(t) = \frac{1}{1 - h(x)t - t^2} \sum_{s=0}^{3} (L_{h,s} + L_{h,s-1}t)\, e_s. \tag{9}$$

Proof. The generating function $g_T(t)$ of the $h(x)$-Lucas quaternion polynomials is

$$g_T(t) = T_{h,0} + T_{h,1}t + T_{h,2}t^2 + ... + T_{h,n}t^n + ... \quad (10)$$

The orders of $T_{h,n-1}$ and $T_{h,n-2}$, respectively, are 1 and 2 less than the order of $T_{h,n}$. Thus, we obtain

$$\begin{aligned} h(x)g_T(t)t &= h(x)T_{h,0}t + h(x)T_{h,1}t^2 + h(x)T_{h,2}t^3 + ... \quad (11) \\ &\quad + h(x)T_{h,n-1}t^n + ... \end{aligned}$$

and

$$g_T(t)t^2 = T_{h,0}t^2 + T_{h,1}t^3 + T_{h,2}t^4 + ... + T_{h,n-2}t^n + ... \quad (12)$$

From Definition 5, $(10), (11)$ and (12), we have

$$\left(1 - h(x)t - t^2\right) g_T(t) = T_{h,0} + \left(T_{h,1} - h(x)T_{h,0}\right) t \quad (13)$$

and we thus obtain (8). From Definition 5, it follows that

$$T_{h,1} - h(x)T_{h,0} = \sum_{s=0}^{3} L_{h,s-1}e_s. \quad (14)$$

Combining (13) and (14), then (9) is evident. ■

Similarly, we have the following result.

Theorem 7 *Suppose that $h(x)$ is an odd polynomial. Then for $g_T(t) = \sum_{n=0}^{\infty} T_{h,n}(-x)(-t)^n$ we have*

$$g_T(t) = \frac{T_{h,0}(-x) - \left(T_{h,1}(-x) + h(x)T_{h,0}(-x)\right) t}{1 - h(x)t - t^2} \quad (15)$$

and

$$g_T(t) = \frac{1}{1 - h(x)t - t^2} \sum_{s=0}^{3} (-1)^s \left(L_{h,s}(x) + L_{h,s-1}(x)t\right) e_s. \quad (16)$$

Proof. We now consider

$$g_T(t) = \sum_{n=0}^{\infty} T_{h,n}(-x)(-t)^n.$$

$g_T(t)$ is a formal power series. Therefore, we need not take care of the convergence of the series. Thus, we write

$$g_T(t) = T_{h,0}(-x) - T_{h,1}(-x)t + T_{h,2}(-x)t^2 - \ldots + T_{h,n}(-x)(-t)^n + \ldots. \quad (17)$$

The orders of $T_{h,n-1}(-x)$ and $T_{h,n-2}(-x)$, respectively, are 1 and 2 less than the order of $T_{h,n}(-x)$. Thus, since $h(-x) = -h(x)$ we obtain

$$\begin{aligned} -h(x)g_T(t)t &= -h(x)T_{h,0}(-x)t + h(x)T_{h,1}(-x)t^2 \\ &\quad -h(x)T_{h,2}(-x)t^3 + \ldots - (-1)^{n-1}h(x)T_{h,n-1}(-x)t^n + \ldots \end{aligned} \quad (18)$$

and

$$\begin{aligned} -g_T(t)t^2 &= -T_{h,0}(-x)t^2 + T_{h,1}(-x)t^3 - T_{h,2}(-x)t^4 \\ &\quad - \ldots + (-1)^{n-1}T_{h,n-2}(-x)(t)^n + \ldots. \end{aligned} \quad (19)$$

From (17), (18) and (19) we get (15). On the other hand, by using Definition (5), we can compute

$$T_{h,1}(-x) + h(x)T_{h,0}(-x) = \sum_{s=0}^{3} \left(L_{h,s+1}(-x) + h(x)L_{h,s}(-x)\right) e_s. \quad (20)$$

Combining Definition (2) and (20) gives the following equality

$$T_{h,1}(-x) + h(x)T_{h,0}(-x) = \sum_{s=0}^{3} L_{h,s-1}(-x)e_s. \quad (21)$$

Since $L_{h,n}(-x) = (-1)^n L_{h,n}(x)$ [17, Theorem 3.3] from (15) and (21) we have (16). ■

Binet's formulas are well known in the theory of the Fibonacci and Lucas numbers. These formulas can also be carried out for the $h(x)$-Lucas quaternion polynomials.

The characteristic equation associated with the recurrence relation (2) is $v^2 = h(x)v + 1$. The roots of this equation are

$$r_1(x) = \frac{h(x) + \sqrt{h(x)^2 + 4}}{2}, \ r_2(x) = \frac{h(x) - \sqrt{h(x)^2 + 4}}{2}.$$

The following basic identities is needed for our purpose in proving.

$$\left.\begin{array}{c} r_1(x) + r_2(x) = h(x), \\ r_1(x) - r_2(x) = \sqrt{h(x)^2 + 4}, \\ r_1(x).r_2(x) = -1. \end{array}\right\} \tag{22}$$

For convenience of representation, we adopt the following notations: $r_1 = r_1(x)$ and $r_2 = r_2(x)$.

The following lemma is directly useful for stating our next main results.

Lemma 8 *For the generating function $g_T(t)$ in (7) of the $h(x)$-Lucas quaternion polynomials $T_{h,n}(x)$, we have*

$$g_T(t) = \frac{1}{r_1 - r_2}\left[\frac{T_{h,1} - r_2 T_{h,0}}{1 - r_1 t} - \frac{T_{h,1} - r_1 T_{h,0}}{1 - r_2 t}\right].$$

Proof. The proof can be obtained easily from (22). ∎

We obtain following Binet's formula for $T_{h,n}(x)$.

Theorem 9 *For $n \geq 0$, Binet's formula for the $h(x)$-Lucas quaternion polynomials $T_{h,n}(x)$ is as follows*

$$T_{h,n}(x) = \alpha^* r_1^n + \beta^* r_2^n \tag{23}$$

where $\alpha^* = \sum_{s=0}^{3} \alpha^s e_s$ and $\beta^* = \sum_{s=0}^{3} \beta^s e_s$.

Proof. From Lemma 8, we obtain

$$\begin{aligned} g_T(t) &= \frac{1}{r_1 - r_2}\left[\frac{T_{h,1} - r_2 T_{h,0}}{1 - r_1 t} - \frac{T_{h,1} - r_1 T_{h,0}}{1 - r_2 t}\right] \\ &= \frac{1}{r_1 - r_2}\left[(T_{h,1} - r_2 T_{h,0}) \sum_{n=0}^{\infty} r_1^n t^n \right. \\ &\quad \left. - (T_{h,1} - r_1 T_{h,0}) \sum_{n=0}^{\infty} r_2^n t^n \right] \end{aligned} \tag{24}$$

By taking (6) into (24), we can get

$$g_T(t) = \frac{1}{r_1 - r_2}\left[\sum_{s=0}^{3}(L_{h,s+1} - r_2 L_{h,s})\, e_s \sum_{n=0}^{\infty} r_1^n t^n \right. \tag{25}$$
$$\left. - \sum_{s=0}^{3}(L_{h,s+1} - r_1 L_{h,s})\, e_s \sum_{n=0}^{\infty} r_2^n t^n \right].$$

Since $L_{h,s+1} - r_2 L_{h,s} = r_1^s(r_1 - r_2)$ and $L_{h,s+1} - r_1 L_{h,s} = -r_2^s(r_1 - r_2)$ and from (25) we have

$$g_T(t) = \frac{1}{r_1 - r_2}\left[\sum_{s=0}^{3} r_1^s(r_1 - r_2) e_s \sum_{n=0}^{\infty} r_1^n t^n \right. \tag{26}$$
$$\left. + \sum_{s=0}^{3} r_2^s(r_1 - r_2) e_s \sum_{n=0}^{\infty} r_2^n t^n \right].$$

For $\alpha^* = \sum_{s=0}^{3} r_1^s e_s$ and $\beta^* = \sum_{s=0}^{3} r_2^s e_s$, from (26) we obtain

$$g_T(t) = \sum_{n=0}^{\infty} (\alpha^* r_1^n + \beta^* r_2^n)\, t^n. \tag{27}$$

Consequently, by the equality of generating function in (7) and (27), we have Binet's formula for $T_{h,n}(x)$ in (23) ■

Theorem 10 *For $m \in \mathbb{Z}$, $n \in \mathbb{N}$, the generating function of the sequence $\{T_{h,m+n}(x)\}$ is as follows*

$$\sum_{n=0}^{\infty} T_{h,m+n}(x) t^n = \frac{T_{h,m}(x) + T_{h,m-1}(x) t}{1 - h(x)t - t^2}.$$

Proof. By using the Binet formula for $T_{h,n}(x)$, we write

$$\sum_{n=0}^{\infty} T_{h,m+n}(x) t^n = \sum_{n=0}^{\infty} (\alpha^* r_1^{m+n} + \beta^* r_2^{m+n}) t^n.$$

Therefore, we get

$$\begin{aligned}\sum_{n=0}^{\infty} T_{h,m+n}(x) t^n &= \alpha^* r_1^m \sum_{n=0}^{\infty} r_1^n t^n + \beta^* r_2^m \sum_{n=0}^{\infty} r_2^n t^n \\ &= \alpha^* r_1^m \frac{1}{1 - r_1 t} + \beta^* r_2^m \frac{1}{1 - r_2 t}.\end{aligned}$$

So, from this and (22) we obtain

$$\sum_{n=0}^{\infty} T_{h,m+n}(x)t^n = \frac{\alpha^* r_1^m + \beta^* r_2^m + \left(\alpha^* r_1^{m-1} + \beta^* r_2^{m-1}\right) t}{1 - h(x)t - t^2}.$$

Consequently, if we recall Binet's formula for $T_{h,n}(x)$, we get the result. ∎

4 Conclusions

In this paper, we introduce $h(x)$-Lucas quaternion polynomials that generalize k-Lucas quaternion numbers that generalize Lucas quaternion numbers. Also we derive the Binet formula and generating function of $h(x)$-Lucas quaternion polynomial sequence. We predict that in which part of science the above-introduced generating function and Binet formula for the $h(x)$-Lucas quaternion polynomials and numbers will have the most effective application.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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References

[1] Adler S. L., Quaternionic Quantum Mechanics and Quantum Fields, Oxford Univ. Press, New York, 1994.

[2] Akyigit M., Kösal H. H. and Tosun M., Split Fibonacci quaternions, Adv. Appl. Clifford Algebras, **23** (2011), 535-545.

[3] Conway J. H., Smith D.A., On Quaternions and Octonions: Their Geometry, Arithmetic and Symmetry, A.K. Peters, 2003.

[4] Coxeter H. S. M., Integral Cayley numbers, Duke Math. J., **13** (1946), 567–578.

[5] Dunlap R. A., The Golden ratio and Fibonacci numbers, World Scientific, 1997.

[6] Gokkaya H. and Uslu K., On The Properties Of New Families Of Pell And Pell-Lucas Numbers, Ars Combinatoria, Volume CXVII (2014), 311-318.

[7] Halici S., On Fibonacci Quaternions, Adv. Appl. Clifford Algebras DOI 0.1007/s00006-011-0317-1, 2011.

[8] Halici S., On Complex Fibonacci Quaternions, Adv. Appl. Clifford Algebras, **23** (2013), 105-112.

[9] Hoggat VE., Fibonacci and Lucas numbers, Palo Alto, California: Houghton-Mifflin; 1969.

[10] Horadam A. F., Complex Fibonacci numbers and Fibonacci quaternions, Amer. Math. Monthly, **70** (1963), 289-291.

[11] Iyer M. R., A Note On Fibonacci Quaternions, The Fib. Quart., **3** (1969), 225-229.

[12] Keçilioğlu O., Akkuş İ., The Fibonacci Octonions, Adv. Appl. Clifford Algebras, DOI 10.1007/s00006-014-0468-y.

[13] Kilic E., Yalciner A., New Sums Identities In Weighted Catalan Triangle With The Powers Of Generalized Fibonacci And Lucas Numbers, Ars Combinatoria, Volume CXV (2014), 391-400.

[14] Kincaid D., Cheney W., Numerical Analysis: Mathematics of Scientific Computing, third ed., American Mathematical Society, 2002.

[15] Koshy T., Fibonacci and Lucas Numbers with Applications, John Wiley & Sons, New York, NY, USA, 2001.

[16] Lando SK., Lectures on generating functions, Student Mathematical Library, vol. 23. Providence, RI: American Mathematical Society; 2003.

[17] Nalli A., Haukkanen P., On generalized Fibonacci and Lucas polynomials, Chaos Solitons Fractals, **42** (5) (2009), 3179–3186

[18] Rosen K. H., Handbook of discrete and combinatorial mathematics, CRC Press, Boca Raton, FL, 2000.

[19] Uslu K., Taskara N. and Kose H., The Generalized k-Fibonacci and k-Lucas Numbers, Ars Combinatoria, Volume XCIX (2011), 25-32.

[20] Uslu K., Taskara N. and Gulec H.H., Combinatorial Sums of Generalized Fibonacci and Lucas Numbers, Ars Combinatoria, Volume XCIX (2011), 139-147.

[21] Vajda S., Fibonacci & Lucas numbers, and the golden section, Theory and Applications. Ellis Horwood Limited; 1989.

[22] Wilf HS. Generatingfunctionology. Second ed. Boston, MA: Academic Press Inc.; 1994.