An algorithm for hamiltonian cycles under implicit degree conditions*

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Abstract: In 1989, Zhu, Li and Deng introduced the definition of implicit degree, denoted by $id(v)$, of a vertex v in a graph G. In this paper, we give a simple method to prove that: If G is a k-connected graph of order n such that the implicit degree sum of any $k+1$ independent vertices is more than $(k+1)(n-1)/2$, then G is hamiltonian. And we give an algorithm according to the proof.

Keywords: Implicit degree; Hamiltonian cycles; Independent set; Graph

1 Notation and Introduction

We will use standard notation and terminology of graph theory. Most of them can be found for example in [4]. In addition, all the graphs considered in this paper are finite, undirected and simple.

Let $G = (V(G), E(G))$ be a graph, with vertex set $V(G)$ and edge set $E(G)$. The order of G is $|V(G)|$. For a subgraph H of G, let $G - H$ be the subgraph in G induced by $V(G) - V(H)$. For a vertex $u \in V(G)$, define $N_H(u) = \{v \in V(H) : uv \in E(G)\}$ and $N_H^2(u) = \{v \in V(H) : d(u,v) = 2\}$, where $d(u,v)$ is the distance from u to v in G. The degree of u in H is denoted by $d_H(u) = |N_H(u)|$. If $H = G$, we use $N(v)$, $d(v)$ and $N^2(v)$ in place of $N_G(v)$, $d_G(v)$ and $N_G^2(v)$, respectively. Define $\sigma_k(G) = \min\{d(u_1) + d(u_2) + \ldots + d(u_k) : u_1, u_2, \ldots, u_k$ are k independent vertices in $G\}$.

For a cycle (path) C in G with a given orientation and a vertex x in C, x^+ and x^- denote the successor and the predecessor of x in C, respectively. And for any $I \subseteq V(C)$, let $I^- = \{x : x^+ \in I\}$ and $I^+ = \{x : x^- \in I\}$. For

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two vertices $x, y \in V(C)$, xCy denotes the subpath of P from x to y. We use $y\bar{C}x$ for the path from y to x in the reversed direction of C.

A graph G is hamiltonian if it contains a hamiltonian cycle, i.e. a cycle containing all vertices of G. Hamiltonian problems are interesting and important in graph theory and have been studied deeply. We have two classic results due to Dirac and Ore respectively.

Theorem 1. *(Dirac [6]) If G is a graph of order $n \geq 3$ such that $\sigma_1(G) \geq n/2$, then G is hamiltonian.*

Theorem 2. *(Ore [8]) If G is a graph of order $n \geq 3$ such that $\sigma_2(G) \geq n$, then G is hamiltonian.*

It is natural to consider sufficient conditions concerning the degree sum of more independent vertices for a graph to be hamiltonian. Bondy [2] investigated the degree sum of $k+1$ independent vertices and obtained the following result.

Theorem 3. *(Bondy [2]) Let G be a k-connected graph of order $n \geq 3$ with $k \geq 2$. If $\sigma_{k+1}(G) > (k+1)(n-1)/2$, then G is hamiltonian.*

In order to generalize Theorems 1 and 2, Zhu, Li and Deng [10] proposed the concept of implicit degrees of vertices.

Definition 1. *(Zhu, Li and Deng [10]) Let v be a vertex of a graph G. If $N^2(v) \neq \emptyset$ and $d(v) \geq 2$, then set $k = d(v) - 1$, $m_2 = \min\{d(u) : u \in N^2(v)\}$ and $M_2 = \max\{d(u) : u \in N^2(v)\}$. Suppose $d_1 \leq d_2 \leq \cdots \leq d_{k+1} \leq \cdots$ is the degree sequence of vertices of $N(v) \cup N^2(v)$. Let*

$$d^*(v) = \begin{cases} m_2, & \text{if } d_k < m_2; \\ d_{k+1}, & \text{if } d_{k+1} > M_2; \\ d_k, & \text{if } d_k \geq m_2 \text{ and } d_{k+1} \leq M_2. \end{cases}$$

Then the implicit degree of v, is defined as $id(v) = \max\{d(v), d^*(v)\}$. If $N^2(v) = \emptyset$ or $d(v) \leq 1$, then we define $id(v) = d(v)$.

It is clear that $id(v) \geq d(v)$ for every vertex v. Define $\sigma_k^*(G) = \min\{id(u_1) + id(u_2) + \ldots + id(u_k) : u_1, u_2, \ldots, u_k$ are k independent vertices in $G\}$. The authors in [10] gave a sufficient condition for a graph to be hamiltonian involving implicit degree condition.

Recently, Li, Ning and Cai [7] used $\sigma_{k+1}^*(G)$ in place of $\sigma_{k+1}(G)$ in Theorem 3 [2], and obtained the following result.

Theorem 4.(Li, Ning and Cai [7]) *Let G be a k-connected graph of order $n \geq 3$ with $k \geq 2$. If $\sigma_{k+1}^*(G) > (k+1)(n-1)/2$, then G is hamiltonian.*

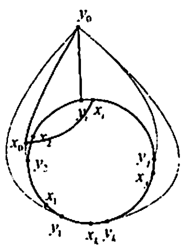

Fig.1

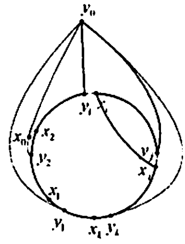

Fig.2

In this paper, we give a simple method to prove Theorem 4 and give an algorithm according to the proof.

2 The proof of Theorem 4

For a path $P = x_1x_2\ldots x_p$ of a graph G, let $l_P(x_1) = \max\{i : x_i \in V(P) \text{ and } x_ix_1 \in E(G)\}$ and $l_P(x_p) = \min\{i : x_i \in V(P) \text{ and } x_ix_p \in E(G)\}$. Set $L_P(x_1) = x_{l_P(x_1)}$ and $L_P(x_p) = x_{l_P(x_p)}$.

Lemma 1.(Li, Ning and Cai [7]) *Let G be a 2-connected graph and $P = x_1x_2\cdots x_p$ with $x_1 = x$ and $x_p = y$ be a path of G connecting x and y. If $xy \notin E(G)$, and $d(u) < id(x)$ for any $u \in N_{G-P}(x) \cup \{x\}$. Then either*
(1) there exists a vertex $x_j \in N_P^-(x)$ such that $d(x_j) \geq id(x)$; or
(2) $N_P^-(x) = N_P(x) \cup \{x\} - \{L_P(x)\}$, $d(x_j) < id(x)$ for any vertex $x_j \in N_P^-(x)$ and $id(x) = \min\{d(v) : v \in N^2(x)\}$.

Proof of Theorem 4 Let G be a graph satisfying the conditions of Theorem 4 and suppose G is non-hamiltonian. Let C be a longest cycle of G and give C a fixed orientation. Then $R = G - C \neq \emptyset$. Let $y_0 \in V(R)$. Since G is k-connected, there are k paths $P_1(y_0, y_1), P_2(y_0, y_2), \ldots, P_k(y_0, y_k)$ from y_0 to C having only y_0 in common pairwise. Let $V(P_i) \cap V(C) = \{y_i\}$ for each $i \in \{1, 2, \ldots, k\}$, and let $y_1, y_2, \ldots, y_k$ occur in this order along C with the given orientation, where the indices are taken modulo k. Let $x_i = y_i^+$ for each $i \in \{1, 2, \ldots, k\}$ and let x_0 be predecessor of y_2 on the path P_2.

Claim 1. $\{x_0, x_1, x_2, \ldots, x_k\}$ and $\{y_0, x_1, x_2, \ldots, x_k\}$ are an independent sets of G.

Proof. If $x_0x_i \in E(G)$ for some $i \in \{1, 2, \ldots, k\}$, then $C' = x_iCy_i\bar{P}_iy_0P_2x_0x_i$ (see Fig.1) is a cycle longer than C, a contradiction. Similarly, $y_0x_i \notin E(G)$.

If $x_ix_j \in E(G)$ for some $i, j \in \{1, 2, \ldots, k\}$ $(i < j)$, then $C' = x_iCy_j\bar{P}_jy_0P_i y_i\bar{C}x_jx_i$ (see Fig.2) is a cycle longer than C, a contradiction. □

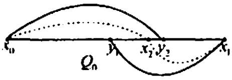

Fig.3

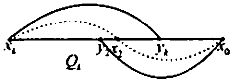

Fig.4

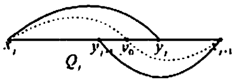

Fig.5

Set $Q_0(x_0, x_1) = x_0\bar{P}_2y_0P_1y_1\bar{C}x_1$; $Q_k(x_k, x_0) = x_kCy_k\bar{P}_ky_0P_2x_0$; and $Q_j(x_j, x_{j+1}) = x_jCy_{j+1}\bar{P}_{j+1}y_0P_jy_j\bar{C}x_{j+1}$ for each $j \in \{1, 2, \dots, k-1\}$. Let $R_j = G - Q_j$ for each $j \in \{0, 1, 2, \dots, k\}$.

Claim 2. $N^-_{Q_j}(x_j) \neq N_{Q_j}(x_j) \cup \{x_j\} - \{L_{Q_j}(x_j)\}$ and $N^+_{Q_j}(x_{j+1}) \neq N_{Q_j}(x_{j+1}) \cup \{x_{j+1}\} - \{L_{Q_j}(x_{j+1})\}$ for each $j \in \{0, 1, 2, \dots, k\}$.

Proof. For the path Q_0, we know $x_0x_2 \notin E(G)$, $x_0y_2 \in E(G)$, $x_1y_1 \in E(G)$ and $x_1x_2 \notin E(G)$. Since x_2 is the predecessor of y_2 on the path Q_0 and y_1 is before x_2 on the path Q_0 (see Fig.3), we have $N^-_{Q_0}(x_0) \neq N_{Q_0}(x_0) \cup \{x_0\} - \{L_{Q_0}(x_0)\}$ and $N^+_{Q_0}(x_1) \neq N_{Q_0}(x_1) \cup \{x_1\} - \{L_{Q_0}(x_1)\}$.

For the path Q_k, we know $x_ky_k \in E(G)$, $x_kx_2 \notin E(G)$ and $x_0y_2 \in E(G), x_0x_2 \notin E(G)$. Since x_2 is before y_k and y_2 is the predecessor of x_2 on the path Q_k (see Fig.4), we have $N^-_{Q_k}(x_k) \neq N_{Q_k}(x_k) \cup \{x_k\} - \{L_{Q_k}(x_k)\}$ and $N^+_{Q_k}(x_0) \neq N_{Q_k}(x_0) \cup \{x_0\} - \{L_{Q_k}(x_0)\}$.

For the path $Q_j, j \in \{1, 2, \dots, k-1\}$, we know $x_jy_j \in E(G), x_jy_0 \notin E(G)$ and $x_{j+1}y_{j+1} \in E(G), x_{j+1}y_0 \notin E(G)$. Since y_0 is before y_j and y_{j+1} is before y_0 on the path Q_j (see Fig.5), we have $N^-_{Q_j}(x_j) \neq N_{Q_j}(x_j) \cup \{x_j\} - \{L_{Q_j}(x_j)\}$ and $N^+_{Q_j}(x_{j+1}) \neq N_{Q_j}(x_{j+1}) \cup \{x_{j+1}\} - \{L_{Q_j}(x_{j+1})\}$. □

By similar argument as in Lemma 1, we have:

Claim 3. If $d(u) < id(x_j)$ for any $u \in N_{R_j}(x_j) \cup \{x_j\}$. Then either
(a) there exists a vertex $v \in N^-_{Q_j}(x_j)$ such that $d(v) \geq id(x_j)$; or
(b) $N^-_{Q_j}(x_j) = N_{Q_j}(x_j) \cup \{x_j\} - \{L_{Q_j}(x_j)\}$, $d(w) < id(x_j)$ for any $w \in N^-_{Q_j}(x_j)$ and $id(x_j) = \min\{d(x) : x \in N^2(x_j)\}$. □

Claim 4. For each $j \in \{0, 1, \dots, k\}$, there exists a path $W_j(w^j_1, w^j_2)$ such that
(i) $V(Q_j) \subseteq V(W_j)$, and
(ii) $d(w^j_1) \geq id(x_j)$ and $d(w^j_2) \geq id(x_{j+1})$.

Proof. For convenience, set $Q_j(x_j, x_{j+1}) = u^j_1u^j_2 \dots u^j_{j_q}$. We have the following two cases to discuss.

Case 1. There is a vertex $u \in N_{R_j}(x_j) \cup \{x_j\}$ such that $d(u) \geq id(x_j)$.

Case 1.1. There is a vertex $v \in N_{R_j}(x_{j+1}) \cup \{x_{j+1}\}$ such that $d(v) \geq$

$id(x_{j+1})$.

In this case, $W_j(w_1^j, w_2^j) = ux_jQ_jx_{j+1}v$ ($w_1^j = u$ and $w_2^j = v$) is the path satisfying (i) and (ii).

Case 1.2. $d(v) < id(x_{j+1})$ for any vertex $v \in N_{R_j}(x_{j+1}) \cup \{x_{j+1}\}$.

By Claims 2 and 3, there exists a vertex $u_l^j \in N_{Q_j}^+(x_{j+1})$ such that $d(u_l^j) \geq id(x_{j+1})$. Then $W_j(w_1^j, w_2^j) = ux_jQ_ju_{l-1}^jx_{j+1}\bar{Q}_ju_l^j$ ($w_1^j = u$ and $w_2^j = u_l^j$) is the path satisfying (i) and (ii).

Case 2. $d(u) < id(x_j)$ for any $u \in N_{R_j}(x_j) \cup \{x_j\}$.

By Claims 2 and 3, there exists a vertex $u_s^j \in N_{Q_j}^-(x_j)$ such that $d(u_s^j) \geq id(x_j)$.

Case 2.1. There exists a vertex $v \in N_{R_j}(x_{j+1}) \cup \{x_{j+1}\}$ such that $d(v) \geq id(x_{j+1})$.

In this case, $W_j(w_1^j, w_2^j) = u_s^j\bar{Q}_jx_ju_{s+1}^jQ_jx_{j+1}v$ ($w_1^j = u_s^j$ and $w_2^j = v$) is the path satisfying (i) and (ii).

Case 2.2. $d(v) < id(x_{j+1})$ for any vertex $v \in N_{R_j}(x_{j+1}) \cup \{x_{j+1}\}$.

Case 2.2.1. $s + 1 \leq l_{Q_j}(x_{j+1})$, where s is the index of u_s^j on Q_j.

Then by Claims 2 and 3, there exists a vertex $u_t^j \in N_{Q_j}^+(x_{j+1})$ such that $d(u_t^j) \geq id(x_{j+1})$. Therefore, $W_j(w_1^j, w_2^j) = u_s^j\bar{Q}_jx_ju_{s+1}^jQ_ju_{t-1}^jx_{j+1}\bar{Q}_ju_t^j$ ($w_1^j = u_s^j$ and $w_2^j = u_t^j$) is the path satisfying (i) and (ii).

Case 2.2.2. $s + 1 > l_{Q_j}(x_{j+1})$.

Set
$$A_j = \{u_i^j : u_i^j \in N_{Q_j}^-(x_{j+1}) \text{ and } i < s\},$$
$$B_j = \{u_i^j : u_i^j \in N_{Q_j}^+(x_{j+1}) \text{ and } i > s+1\}, \text{ and}$$
$$C_j = \{u_i^j : u_i^j \in N_{Q_j}^-(x_{j+1}),\ i \geq s \text{ and } i \text{ is as small as possible}\}.$$

Then $|A_j| = |N(x_{j+1}) \cap V(x_jQ_ju_s^j)|$, $|B_j| = |N(x_{j+1}) \cap V(u_s^jQ_jx_{j+1})|$ and $|C_j| = 1$. Thus, $|A_j| + |B_j| + |C_j| + |N_{R_j}(x_{j+1})| - |\{x_{j+1}\}| \geq d(x_{j+1})$, $u_{l_{Q_j}(x_{j+1})-1}^j \in A_j \cap N^2(x_{j+1})$ and $C_j \subseteq N^2(x_{j+1})$. By Claim 2, there exists a vertex $u_l^j \in (A_j \cup B_j) - \{x_{j+1}\}$ such that $d(u_l^j) \geq id(x_{j+1})$.

When $u_l^j \in B_j - \{x_{j+1}\}$, $W_j(w_1^j, w_2^j) = u_s^j\bar{Q}_jx_ju_{s+1}^jQ_ju_{l-1}^jx_{j+1}\bar{Q}_ju_l^j$ ($w_1^j = u_s^j$ and $w_2^j = u_l^j$) is the path satisfying (i) and (ii).

When $v_h^j \in A_j$, $W_j(w_1^j, w_2^j) = u_s^j \bar{Q}_j u_{l+1}^j x_{j+1} \bar{Q}_j u_{s+1}^j x_j Q_j u_l^j$ ($w_1^j = u_s^j$ and $w_2^j = u_t^j$) is the path satisfying (i) and (ii).

Now we complete Claim 4. □

Since C is the longest cycle of G, for each path $W_j(w_1^j, x_2^j)$ obtained from the above, we have

$$N(w_1^j) \cap N(w_2^j) \cap (V(G) - V(W_j)) = \emptyset,$$

$$N_{W_j}^{-}(w_1^j) \cap N_{W_j}(w_2^j) = \emptyset \text{ and } w_1^j w_2^j \notin E(G).$$

Then,

$$d(w_1^j) + d(w_2^j) \leq |V(G) - V(W_j)| + (|W_j| - 1) = n - 1.$$

Thus

$$\sum_{j=0}^{k} (d(w_1^j) + d(w_2^j)) \leq (k+1)(n-1).$$

But

$$\begin{aligned}\sum_{j=0}^{k} (d(w_1^j) + d(w_2^j)) &\geq \sum_{j=0}^{k} (id(x_j) + id(x_{j+1})) \\ &\geq 2\sum_{j=0}^{k} id(x_j) \\ &\geq 2\sigma_{k+1}^{*}(G) > (k+1)(n-1),\end{aligned}$$

a contradiction. □

3 Algorithm

First, we give an algorithm to determine the implicit degree of a vertex. Let $V(G) = \{v_1, v_2, \ldots, v_n\}$.

a. An algorithm to determine the implicit degree of a vertex

Input: The adjacency matrix $A = (a_{ij})_{n \times n}$ of G.
Output: The implicit degree of a vertex v_i.
Step 1: Let $N(v_i) = \emptyset$ and $N^2(v_i) = \emptyset$.
    for $j = 1$ to n, do
        $d(v_j) = 0$.
Step 2: Determine the degree of each vertex $v \in V(G)$.

for $j = 1$ to n, do
for $k = 1$ to n, do
if $a_{jk} == 1$, do
$d(v_j) = d(v_j) + 1$,
else $d(v_j) = d(v_j)$.

Step 3. Find the neighbors and 2-neighbors of v_i.
for $j = 1$ to n, do
if $a_{ij} == 1$, do
$N(v_i) = N(v_i) \cup \{v_j\}$,
else $N(v_i) = N(v_i)$.
for $k = 1$ to n, do
if $a_{ik} == 0$, do
for $j = 1$ to n, do
if $a_{ij} == 1$ and $a_{jk} == 1$, do
$N^2(v_i) = N^2(v_i) \cup \{v_k\}$.
else $N^2(v_i) = N^2(v_i)$.

Step 4. Let $d_1 \leq d_2 \leq \ldots d_{d(v_i)} \leq \ldots$ be the degree sequence of vertices of $N(v_i) \cup N^2(v_i)$, $m_2 = \min\{d(u) : u \in N^2(v)\}$ and $M_2 = \max\{d(u) : u \in N^2(v_i)\}$.

Step 5: Determine the implicit degree of v_i.

Based on the proof of Claim 4 in Section 2, we give the algorithm to construct the path $W_j(w_1^j, w_2^j)$ for each $j \in \{0, 1, 2, \ldots, k\}$ satisfying (i) and (ii) in Claim 4.

b. An algorithm to construct the path $W_j(w_1^j, w_2^j)$

Input: $Q_j(x_j, x_{j+1})$, $j = 0, 1, 2, \ldots, k$.
Output: $W_j(w_1^j, w_2^j)$ for $j = 0, 1, 2, \ldots, k$.
Step 1: Let $Q_j(x_j, x_{j+1}) = u_1^j u_2^j \ldots u_{j_q}^j$, $R_j = G - Q_j$, $U_1^j = N_{R_j}(x_j) \cup \{x_j\}$, $U_2^j = N_{R_j}(x_{j+1}) \cup \{x_{j+1}\}$, $V_1^j = N_{Q_j}^-(x_j)$ and $V_2^j = N_{Q_j}^+(x_{j+1})$. And let $l_{Q_j}(x_{j+1}) = \min\{i : u_i^j x_{j+1} \in E(G)\}$ for $j = 0, 1, 2, \ldots, k$.
Step 2: for $j = 0$ to k, do
while $U_1^j \neq \emptyset$, do
for $u \in U_1^j$, do
if $d(u) \geq id(x_j)$, do
while $U_2^j \neq \emptyset$, do
for $v \in U_2^j$, do
if $d(v) \geq id(x_{j+1})$, do
Return $W_j(w_1^j, w_2^j) = u x_j Q_j x_{j+1} v$.
else $U_2^j = U_2^j - \{v\}$.

end for
end while
for $w \in V_2^j$, do
if $d(w) \geq id(x_{j+1})$, do
return $W_j(w_1^j, w_2^j) = ux_jQ_jw^- x_{j+1}Q_jw$.
else $V_2^j = V_2^j - \{w\}$.
end for
else $U_1^j = U_1^j - \{u\}$.
end for
end while
while $V_1^j \neq \emptyset$, do
for $x \in V_1^j$, do
if $d(x) \geq id(x_j)$, do
$label(x) = \{i : x = u_i^j \text{ and } u_i^j \in V(Q_j)\}$.
if $l_{Q_{j+1}}(x_{j+1}) \geq label(x) + 1$, do
while $V_2^j \neq \emptyset$, do
for $y \in V_2^j$, do
if $d(y) \geq id(x_{j+1})$, do
return $W_j(w_1^j, w_2^j) = x\bar{Q}_jx_jx^+Q_jy^- x_{j+1}\bar{Q}_jy$.
else $V_2^j = V_2^j - \{y\}$.
end for
else let $A_j = \{u_i^j : u_i^j \in N_{Q_j}^-(x_{j+1}) \text{ and } i < lable(x)\}$,
and $B_j = \{u_i^j : u_i^j \in N_{Q_j}^+(x_{j+1}) \text{ and } i > lable(x)+1\}$.
while $A_j \neq \emptyset$, do
for $z \in A_j$, do
if $d(z) \geq id(x_{j+1})$, do
return $W_j(w_1^j, w_2^j) = x\bar{Q}_jz^+x_{j+1}\bar{Q}_jx^+x_jQ_jz$.
else $A_j = A_j - \{z\}$.
end for
end while
while $B_j \neq \emptyset$, do
for $z \in B_j$, do
if $d(z) \geq id(x_{j+1})$, do
return $W_j(w_1^j, w_2^j) = x\bar{Q}_jx_jx^+Q_jz^- x_{j+1}\bar{Q}_jz$.
else $B_j = B_j - \{z\}$.
end for
end while
else $V_1^j = V_1^j - \{x\}$.
end for
end while
end for

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Decomposition of a $2K_{10t}$ into H_3 Graphs

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Abstract

An H_3 graph is a multigraph on three vertices with double edges between two pairs of distinct vertices and a single edge between the third pair. In this paper, we decompose a complete multigraph $2K_{10t}$ into H_3 graphs.

1 Introduction

A graph can be decomposed into a collection of subgraphs such that every edge of the graph is contained in one of the subgraphs. Decomposing a graph into simple graphs has been well studied in literature. For a well-written survey on the decomposition of a complete graph into simple graphs with small number of points and edges, see [1]. A *multigraph* is a graph where more than one edge between a pair of points is allowed. The decomposition of copies of a complete graph into proper multigraphs has not received much attention yet, see [2, 3, 4, 5, 6, 9, 10]. A complete multigraph λK_v ($\lambda > 1$) is a graph on v points with λ edges between every pair of distinct points.

Definition 1 An H_3 graph is a multigraph on three vertices with double edges between two pairs of distinct vertices and a single edge between the third pair.

If $V = \{a, b, c\}$ and a double edge between a and b and a double edge between b and c, then we denote the H_3 graph as $\langle a, b, c\rangle_{H_3}$ (see figure 1). An $H_3(v, \lambda)$ is a decomposition of a λK_v into H_3 graphs. In particular, an $H_3(10t, 2)$ is a decomposition of a $2K_{10t}$ graph into $\frac{2\times 10t\times(10t-1)}{2\times 5} = 2t(10t-1)$ H_3 graphs.

Hurd and Sarvate [6] show that the necessary condition for existence of an $H_3(v, 2)$ is $v = 5t$ or $v = 5t + 1$. They claim that an $H_3(5t + 1, 2)$ exists for $t \geq 1$, and there does not exist an $H_3(5, 2)$, but an $H_3(10, 2)$ and an $H_3(15, 2)$

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exist. The general case for an $H_3(5t, 2)$ where $t > 3$ was left open. In this paper, we continue to work on this problem and prove that an $H_3(10t, 2)$ (i.e. $H_3(5t, 2)$ for all even integers t) exists. To settle the H_3 decomposition problem completely, one needs to complete the decomposition of $2K_{10t+5}$ into H_3 graphs.

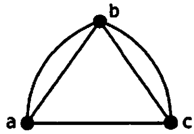

Figure 1: An H_3 Graph

We need the following results.

Definition 2 A *1-factor* of a graph G is a set of pairwise disjoint edges which partition the vertex set. A *1-factorization* of a graph G is the set of 1-factors which partition the edge set of the graph.

A 1-factorization of K_{2n} contains $2n - 1$ 1-factors. In [11], Stanton and Goulden define the *difference partition* $P_1, \ldots, P_n$ of K_{2n} as n disjoint classes, where the edge (i, j) is in P_k if and only if $(i-j) \equiv k (\text{mod } 2n)$ where the vertices are labeled $0, 1, \ldots, 2n - 1$.

Theorem 1 *[7] Consider the set T of triangles $(1 + i, 1 + x + i, 1 + x + y + i)$ for $i = 1, \ldots, 2n$. The set T contains exactly the edges from P_x, P_y, P_{x+y}, where $x + y < n$.*

When $x + y = n$, we observe the following result.

Lemma 1 *The set T of triangles $(1+i, 1+x+i, 1+x+y+i)$ for $i = 1, \ldots, 2n$ contains exactly the edges from $P_x, P_y, 2P_{x+y}$, where $x + y = n$.*

Lemma 2 *[11] The pairs in $P_{2x+1}(2x + 1 < n)$ split into two 1-factors.*

Lemma 3 *[11] If $2x + 1 < n$, then $P_{2x} \cup P_{2x+1}$ splits into four 1-factors.*

Lemma 4 *[11] If n is even, then P_n is a single 1-factor. If n is odd, then $P_{n-1} \cup P_n$ can be split into three 1-factors.*

2 Constructions for $H_3(10t, 2)$s

In this section, we develop certain procedures to be used for the $H_3(10t, 2)$ in general. Notice that a 1-factorization of a λK_v can be obtained by duplicating the