Decomposition of a $2K_{10t}$ into H_3 Graphs

Dinesh G. Sarvate *
Department of Mathematics
College of Charleston
Charleston, SC 29424

Li Zhang †
Department of Mathematics
and Computer Science
The Citadel
Charleston, SC 29409

Abstract

An H_3 graph is a multigraph on three vertices with double edges between two pairs of distinct vertices and a single edge between the third pair. In this paper, we decompose a complete multigraph $2K_{10t}$ into H_3 graphs.

1 Introduction

A graph can be decomposed into a collection of subgraphs such that every edge of the graph is contained in one of the subgraphs. Decomposing a graph into simple graphs has been well studied in literature. For a well-written survey on the decomposition of a complete graph into simple graphs with small number of points and edges, see [1]. A *multigraph* is a graph where more than one edge between a pair of points is allowed. The decomposition of copies of a complete graph into proper multigraphs has not received much attention yet, see [2, 3, 4, 5, 6, 9, 10]. A complete multigraph λK_v ($\lambda > 1$) is a graph on v points with λ edges between every pair of distinct points.

Definition 1 An H_3 graph is a multigraph on three vertices with double edges between two pairs of distinct vertices and a single edge between the third pair.

If $V = \{a, b, c\}$ and a double edge between a and b and a double edge between b and c, then we denote the H_3 graph as $\langle a, b, c\rangle_{H_3}$ (see figure 1). An $H_3(v, \lambda)$ is a decomposition of a λK_v into H_3 graphs. In particular, an $H_3(10t, 2)$ is a decomposition of a $2K_{10t}$ graph into $\frac{2\times 10t\times(10t-1)}{2\times 5} = 2t(10t-1)$ H_3 graphs.

Hurd and Sarvate [6] show that the necessary condition for existence of an $H_3(v, 2)$ is $v = 5t$ or $v = 5t + 1$. They claim that an $H_3(5t + 1, 2)$ exists for $t \geq 1$, and there does not exist an $H_3(5, 2)$, but an $H_3(10, 2)$ and an $H_3(15, 2)$

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exist. The general case for an $H_3(5t, 2)$ where $t > 3$ was left open. In this paper, we continue to work on this problem and prove that an $H_3(10t, 2)$ (i.e. $H_3(5t, 2)$ for all even integers t) exists. To settle the H_3 decomposition problem completely, one needs to complete the decomposition of $2K_{10t+5}$ into H_3 graphs.

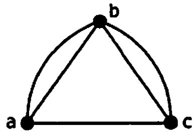

Figure 1: An H_3 Graph

We need the following results.

Definition 2 A *1-factor* of a graph G is a set of pairwise disjoint edges which partition the vertex set. A *1-factorization* of a graph G is the set of 1-factors which partition the edge set of the graph.

A 1-factorization of K_{2n} contains $2n-1$ 1-factors. In [11], Stanton and Goulden define the *difference partition* $P_1, \ldots, P_n$ of K_{2n} as n disjoint classes, where the edge (i, j) is in P_k if and only if $(i-j) \equiv k (\bmod 2n)$ where the vertices are labeled $0, 1, \ldots, 2n-1$.

Theorem 1 *[7] Consider the set T of triangles $(1+i, 1+x+i, 1+x+y+i)$ for $i = 1, \ldots, 2n$. The set T contains exactly the edges from P_x, P_y, P_{x+y}, where $x + y < n$.*

When $x + y = n$, we observe the following result.

Lemma 1 *The set T of triangles $(1+i, 1+x+i, 1+x+y+i)$ for $i = 1, \ldots, 2n$ contains exactly the edges from $P_x, P_y, 2P_{x+y}$, where $x + y = n$.*

Lemma 2 *[11] The pairs in $P_{2x+1} (2x + 1 < n)$ split into two 1-factors.*

Lemma 3 *[11] If $2x + 1 < n$, then $P_{2x} \cup P_{2x+1}$ splits into four 1-factors.*

Lemma 4 *[11] If n is even, then P_n is a single 1-factor. If n is odd, then $P_{n-1} \cup P_n$ can be split into three 1-factors.*

2 Constructions for $H_3(10t, 2)$s

In this section, we develop certain procedures to be used for the $H_3(10t, 2)$ in general. Notice that a 1-factorization of a λK_v can be obtained by duplicating the

1-factors in the 1-factorization of K_v λ times.

Procedure FACTOR-FOR-H-THREE(F, A, n'): Given a 1-factorization (of a multigraph G) $F = \{F_1, \ldots, F_{2n}\}$ where $F_i = F_{i+n}$ $(1 \leq i \leq n)$ and V is the vertex set of G. Let $A = \{1, \ldots, 2n\}$ be a set of $n' = 2n$ points such that $A \cap V = \emptyset$ (notice that n' equals the numbers of 1-factors in F). For $j \in A = 1, \ldots, n$ and each pair (a, b) in F_j and F_{j+n}, we construct H_3 graphs $\langle a, j, b\rangle_{H_3}$ and $\langle a, j + n, b\rangle_{H_3}$. The resulting H_3 graphs contain two edges between any pair of distinct points in V (note that every edge comes exactly once in a 1-factorization and we construct two H_3 graphs on every edge) and two edges between any pair of points where one point is in V and the other point is in A (note that each point in A is used on every edge in a 1-factor of G to construct H_3 graphs and the edges in that 1-factor are disjoint and partition V).

Lemma 5 *If $v \equiv 4(mod\ 10)$, then an $H_3(3v - 2, 2)$ exists. In other words, an $H_3(10t, 2)$ exists if $t \equiv 1(mod\ 3)$.*

Proof: Assume $v \equiv 4(\text{mod } 10) = 10z + 4(z \geq 0)$. Since $10z + 4$ is even, K_{10z+4} has $10z + 3$ 1-factors and $2K_{10z+4}$ has $20z + 6$ 1-factors in F. Perform procedure FACTOR-FOR-H-THREE(F, A, n'), where $n' = 20z+6$. Since $20z+6 \equiv 1(\text{mod } 5)$, an $H_3(20z+6, 2)$ exists. Obtain an $H_3(20z+6, 2)$ on the n' points in A. Combine the H_3 graphs obtained, we have an $H_3((10z+4)+(20z+6), 2) = H_3(30z+10, 2) = H_3(3v-2, 2)$. Since an $H_3(30z+10, 2) = H_3(10(3z+1), 2)$, an $H_3(10t, 2)$ exists if $t \equiv 1(\text{mod } 3)$. □

Theorem 2 *If an $H_3(5t, 2)$ and an $H_3(10t, 2)$ exist, for all $t > 1$, then an $H_3(20t, 2)$ exists.*

Proof: It is known that the complete bipartite graph $K_{5t,5t}$ has $5t$ 1-factors, then $2K_{5t,5t}$ has $10t$ 1-factors in F.

Perform procedure FACTOR-FOR-H-THREE(F, A, n'), where $n' = 10t$. Obtain an $H_3(5t, 2)$ on each of the two vertex sets of $2K_{5t,5t}$, respectively. Obtain an $H_3(10t, 2)$ on the $10t$ points in A. Combine all the H_3 graphs obtained, we have an $H_3(5t + 5t + 10t, 2) = H_3(20t, 2)$. □

Procedure TRIANGLE-TO-H-THREE($2P_x, 2P_y, P_{x+y}, n' = 2n$): Given disjoint classes P_x, P_y and P_{x+y} from a difference partition of K_{2n} where $x + y < n$. By Theorem 1, the set T of $2n$ triangles $(1 + i, 1 + x + i, 1 + x + y + i)$ $(i = 1, \ldots, 2n)$ contains exactly the edges from P_x, P_y, P_{x+y}. Using $P_x, P_x, P_y, P_y, P_{x+y}$, we can construct $2n$ H_3 graphs $\langle 1 + i, 1 + x + i, 1 + x + y + i\rangle_{H_3}$. If $x + y = n$, then we use $P_x, P_x, P_y, P_y, P_{x+y}, P_{x+y}$ to construct $2n$ H_3 graphs (note that if $x + y = n$, the set T of $2n$ triangles contains exactly the edges from $P_x, P_y, P_{x+y}, P_{x+y}$ by Lemma 1). Similarly, TRIANGLE-TO-H-THREE($P_x, 2P_y, 2P_{x+y}, n'$) produces $2n$ H_3 graphs $\langle 1 + i, 1 + x + y + i, 1 + x + i\rangle_{H_3}$ and TRIANGLE-TO-H-THREE($2P_x, P_y, 2P_{x+y}, n'$) produces $2n$ H_3 graphs $\langle 1 + i, 1 + x + y + i, 1 + y + i\rangle_{H_3}$.

Lemma 6 *If $s \equiv 2(mod\ 5)$ and $s > 2$, then an $H_3(6s-12,2)$ exists. In other words, an $H_3(10t,2)$ exists if $t \equiv 0(mod\ 3), t > 0$.*

Proof: Suppose $s \equiv 7(\text{mod } 10)$. We duplicate a difference partition $P_1, \ldots, P_s$ of K_{2s} to obtain two copies of each class. TRIANGLE-TO-H-THREE($2P_1, 2P_{s-1}$, $2P_s, 2s$) produces $2s$ H_3 graphs. By Lemma 3, $P_{2i} \cup P_{2i+1}$ splits into four 1-factors, thus $2P_{2i} \cup 2P_{2i+1}$ splits into eight 1-factors, $i = 1, \ldots, \frac{s-3}{2}$. Let F be the set of these $8 \times \frac{s-3}{2} = 4(s-3)$ 1-factors. Perform FACTOR-FOR-H-THREE($F, A, n' = 4(s-3)$), and then obtain additional H_3 graphs from an $H_3(4(s-3),2)$ on the set of n' new points in A (note that $4(s-3) \equiv 1(\text{mod } 5)$, an $H_3(4(s-3),2)$ exists [6]). Combine all of the H_3 graphs obtained, we have an $H_3(2s+4(s-3),2) = H_3(6s-12,2)$.

Suppose $s \equiv 2(\text{mod } 10)$. Similarly, TRIANGLE-TO-H-THREE ($2P_2, 2P_{s-2}$, $2P_s, 2s$) produces $2s$ H_3 graphs. Each of P_1, P_3 and P_{s-1} splits into two 1-factors by Lemma 2, thus $2P_1$, $2P_3$ and $2P_{s-1}$ split into 12 1-factors. Also, $2P_{2i} \cup 2P_{2i+1}$ splits into eight 1-factors, $i = 2, \ldots, \frac{s-4}{2}$. Let F be the set of $8 \times (\frac{s-4}{2} - 1) + 12 = 4(s-3)$ 1-factors. Perform FACTOR-FOR-H-THREE($F, A, n' = 4(s-3)$), and then obtain additional H_3 graphs from an $H_3(4(s-3),2)$ on the set of n' new points in A. Combine all of the H_3 graphs obtained, we have an $H_3(6s-12,2)$. Let $s = 5z+2 (z > 0)$, then an $H_3(6s-12,2) = H_3(30z,2) = H_3(10(3z),2)$, i.e., an $H_3(10t,2)$ exists if $t \equiv 0(\text{mod } 3), t > 0$. □

Lemma 7 *If $s > 3$ and $s \equiv 2(mod\ 5)$, then an $H_3(6s-22,2)$ exists. In other words, an $H_3(10t,2)$ exists if $t \equiv 2(mod\ 3)$.*

Proof: Suppose $s \equiv 7(\text{mod } 10)$. Observe $2P_1$ and $2P_{s-1} \cup 2P_s$ split into 10 1-factors. TRIANGLE-TO-H-THREE ($2P_2, P_4, 2P_6, 2s$) produces $2s$ H_3 graphs and TRIANGLE-TO-H-THREE ($2P_3, P_4, 2P_7, 2s$) produces $2s$ H_3 graphs. $2P_5$ split into 4 1-factors. Also, $2P_{2i} \cup 2P_{2i+1}$ splits into eight 1-factors, $i = 4, \ldots, \frac{s-3}{2}$. Let F be the set of $8 \times (\frac{s-3}{2} - 3) + 10 + 4 = 4s - 22$ 1-factors. Perform FACTOR-FOR-H-THREE ($F, A, n' = 4s-22$), and then obtain additional H_3 graphs from an $H_3(4s-22,2)$ on the set of n' new points in A (note that $4s-22 \equiv 1(\text{mod } 5)$, an $H_3(4s-22,2)$ exists [6]). Combine all of the H_3 graphs obtained, we have an $H_3(2s+4s-22,2) = H_3(6s-22,2)$.

Suppose $s \equiv 2(\text{mod } 10)$. TRIANGLE-TO-H-THREE ($P_2, 2P_4, 2P_6, 2s$) produces $2s$ H_3 graphs and TRIANGLE-TO-H-THREE($P_2, 2P_8, 2P_{10}, 2s$) produces $2s$ H_3 graphs. Each of $2P_i$, $i = 1, 3, 5, 7, 9, 11$, split into 4 1-factors. $2P_s$ split into 2 1-factors. Also, $2P_{2i} \cup 2P_{2i+1}$ splits into eight 1-factors, $i = 6, \ldots, \frac{s-2}{2}$. Let F be the set of $8 \times (\frac{s-2}{2} - 5) + 4 \times 6 + 2 = 4s - 22$ 1-factors. The rest procedures follow the previous paragraph, so we have an $H_3(2s+4s-22,2) = H_3(6s-22,2)$. Let $s = 5z+7 (z > 0)$, then an $H_3(6s-22,2) = H_3(30z+20,2) = H_3(10(3z+2),2)$ exists, i.e., an $H_3(10t,2)$ exists if $t \equiv 2(\text{mod } 3)$. □

Combining Lemmas 5, 6 and 7, we have the following main result.

Theorem 3 *An* $H_3(10t, 2)$ *exists if* $t \geq 1$.

3 Summary

In this paper, we decompose a complete multigraph $2K_{10t}$ into H_3 graphs. To settle the H_3 decomposition problem completely, one needs to complete the decomposition of $2K_{10t+5}$ into H_3 graphs, which is still an open problem. Examples of such decompositions for $v = 15, 25, 35, 45, 135, 155, 205, 225$ and $10t+5$ for $t \equiv 1 (\text{mod } 21)$ along with some recursive constructions are given in [8] .

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