Hamilton cycles in claw-heavy graphs with Fan-type condition restricted to two induced subgraphs

Xing Huang

011 Base, Aviation Industry Group, Guizhou, 561018, P.R. China

E-mail: bitmathhuangxing@163.com

Abstract

A graph G on $n \geq 3$ vertices is called claw-heavy if every induced claw of G has a pair of nonadjacent vertices such that their degree sum is at least n. We say that a subgraph H of G is f-heavy if $\max\{d(x), d(y)\} \geq \frac{n}{2}$ for every pair of vertices $x, y \in V(H)$ at distance 2 in H. For a given graph R, G is called R-f-heavy if every induced subgraph of G isomorphic to R is f-heavy. For a family $\mathcal{R}$ of graphs, G is called $\mathcal{R}$-f-heavy if G is R-f-heavy for every $R \in \mathcal{R}$. In this paper, we show that every 2-connected claw-heavy graph is hamiltonian if G is $\{P_7, D\}$-f-heavy, or $\{P_7, H\}$-f-heavy, where D is a deer and H is a hourglass. Our result is a common generalization of previous theorems of Broersma et al. and Fan on hamiltonicity of 2-connected graph.

Keywords: Claw-heavy; Hamilton cycle; Fan-type degree condition

1 Introduction

In this paper, we use Bondy and Murty [2] for notation and terminology not defined here, and consider only undirected, finite and simple graphs.

Let G be a graph and H be a subgraph of G. For a vertex $u \in V(G)$, define $N_H(u) = \{v \in V(H) : uv \in E(G)\}$. The degree of u in H is denoted by $d_H(u) = |N_H(u)|$. For $x, y \in V(H)$, the distance between x and y in H, denoted by $d_H(x, y)$, is the length of a shortest (x, y)-path in H. When there is no danger of ambiguity, we can use $N(u)$, $d(u)$ and $d(x, y)$ in place of $N_G(u)$, $d_G(u)$ and $d_G(x, y)$, respectively.

For a subset S of $V(G)$, we use $\langle S\rangle$ to denote the subgraph of G induced by S. A graph H is called an induced subgraph of G if $H = \langle S\rangle$ for some

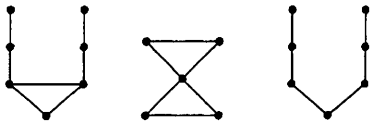

*Fig.*1

$S \subseteq V(G)$. An induced subgraph of G with the vertex set $\{x, u, v, w\}$ and the edge set $\{xu, xv, xw\}$ is called a claw ($K_{1,3}$), with the center x and end vertices u, v, w. For a given graph R, G is called R-free if G contains no induced subgraph isomorphic to R. For a family $\mathcal{R}$ of graphs, G is called $\mathcal{R}$-free if G is R-free for every $R \in \mathcal{R}$.

A vertex v of a graph G on n vertices is called heavy if $d(v) \geq \frac{n}{2}$, and if v is not heavy, we call it light. Following [3], a claw of G is called 2-heavy if at least two of its end vertices are heavy, and G is called 2-heavy if all of its claws are 2-heavy. Following [5], G is called claw-heavy if every induced claw of G has a pair of nonadjacent vertices u and v such that $d(u) + d(v) \geq n$. Clearly, every 2-heavy graph is claw-heavy, but a claw-heavy graph is not necessarily 2-heavy. Following [11], a subgraph H of G is called f-heavy if $\max\{d(x), d(y)\} \geq \frac{n}{2}$ for every pair of vertices $x, y \in V(H)$ at distance 2 in H. For a given graph R, G is called R-f-heavy if every induced subgraph of G isomorphic to R is f-heavy. For a family $\mathcal{R}$ of graphs, G is called $\mathcal{R}$-f-heavy if G is R-f-heavy for every $R \in \mathcal{R}$. Clearly, every R-free graph is also R-f-heavy, and a graph is 2-heavy is equivalent to that it is claw-f-heavy.

A cycle in a graph G is called a Hamilton cycle if it contains all vertices of G. And G is called hamiltonian if it contains a Hamilton cycle. Degree condition is an important type of sufficient conditions for the existence of Hamilton cycles in graphs. The following result due to Fan is well known.

Theorem 1. *([7]) Let G be a 2-connected graph on $n \geq 3$ vertices. If $\max\{d(x), d(y)\} \geq \frac{n}{2}$ for every pair of vertices x and y at distance 2, then G is hamiltonian.*

There is another type of sufficient conditions for hamiltonicity, called forbidden subgraph conditions. The following two results belong to this type, where P_7, D and H are graphs in Fig.1.

Theorem 2. *([4]) Let G be a 2-connected graph. If G is $\{K_{1,3}, P_7, D\}$-free, then G is hamiltonian.*

Theorem 3. *([8]) Let G be a 2-connected graph. If G is $\{K_{1,3}, P_7, H\}$-free, then G is hamiltonian.*

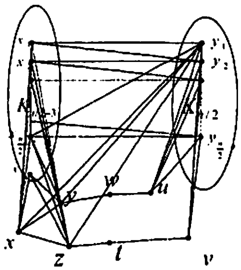

Fig.2

In [3], the authors extended Theorems 2 and 3 to the class of 2-heavy graphs.

Theorem 4. *([3]) Let G be a 2-connected graph. If G is 2-heavy, and moreover $\{P_7, D\}$-free, or $\{P_7, H\}$-free, then G is hamiltonian.*

In 2009, Chen et al. [6] relaxed 2-heavy in Theorem 4 to claw-heavy, and got the following result.

Theorem 5. *([6]) Let G be a 2-connected graph. If G is claw-heavy, and moreover $\{P_7, D\}$-free, or $\{P_7, H\}$-free, then G is hamiltonian.*

By relaxing forbidden subgraph conditions to conditions in which the subgraphs are allowed, but where Fan-type degree condition is imposed on these subgraphs if they appear, Ning extended Theorem 4 as follows.

Theorem 6. *([10]) Let G be a 2-connected graph. If G is $\{K_{1,3}, P_7, D\}$-f-heavy or $\{K_{1,3}, P_7, H\}$-f-heavy, then G is hamiltonian.*

Our objective in this paper is to prove that we can use claw-heavy in place of $K_{1,3}$-f-heavy in Theorem 6.

Theorem 7. *Let G be a 2-connected graph. If G is claw-heavy, and moreover $\{P_7, D\}$-f-heavy or $\{P_7, H\}$-f-heavy, then G is hamiltonian.*

Remark. The graph in Fig.2 shows our result in Theorem 7 does strengthen those in Theorems 5 and 6. Let $n \geq 20$ be an even integer and $K_{\frac{n}{2}} \cup K_{\frac{n}{2}-7}$ denote the union of two complete graphs $K_{\frac{n}{2}}$ and $K_{\frac{n}{2}-7}$. And let $V(K_{\frac{n}{2}}) = \{x_1, x_2, \ldots, x_{\frac{n}{2}}\}$ and $V(K_{\frac{n}{2}-7}) = \{y_1, y_2, \ldots, y_{\frac{n}{2}-7}\}$. We choose a graph G with $V(G) = V(K_{\frac{n}{2}} \cup K_{\frac{n}{2}-7}) \cup \{x, y, z, u, v, w, t\}$ and $E(G) = E(K_{\frac{n}{2}} \cup K_{\frac{n}{2}-7}) \cup \{x_iy_i, x_iy_{i+1} : i = 1, 2, \ldots, \frac{n}{2}-7\} \cup \{x_{\frac{n}{2}-7}y_{\frac{n}{2}-7}, x_{\frac{n}{2}-7}y_1\} \cup \{xy, xz, yz, yw, zt, uw, vt\} \cup \{xx_i, yx_i, zx_i : x_i \in V(K_{\frac{n}{2}})\} \cup \{uy_i, vy_i : y_i \in V(K_{\frac{n}{2}-7})\} \cup \{xy_1, yy_1, zy_1\}$. It is easy to see that G is a hamiltonian graph satisfying the condition of Theorem 7, but not the conditions of Theorems 5 or 6.

2 Proof of Theorem 7

For a cycle C in G with a given orientation and a vertex x in C, x^+ and x^- denote the successor and the predecessor of x in C, respectively. And for any $I \subseteq V(C)$, let $I^- = \{x : x^+ \in I\}$ and $I^+ = \{x : x^- \in I\}$. For two vertices $x, y \in C$, xCy denotes the subpath of C from x to y, and $y\bar{C}x$ denotes the path from y to x in the reversed direction of C. A similar notation is used for paths.

A cycle C is called a heavy cycle if it contains all the heavy vertices of G. We use $E^*(G)$ to denote the set $\{xy : xy \in E(G)$ or $d(x) + d(y) \geq n, x, y \in V(G)\}$. Let $k \geq 3$ be an integer. Following [9], a sequence of vertices $C = x_1x_2 \ldots x_kx_1$ is called an Ore-cycle or briefly, o-cycle of G, if $x_ix_{i+1} \in E^*(G)$ for every $i \in \{1, 2, \ldots, k\}$, where $x_{k+1} = x_1$. The deficit of an o-cycle C is the integer $def(C) = |\{i \in \{1, 2, \ldots, k\} : x_ix_{i+1} \notin E(G)\}|$. Thus, a cycle is an o-cycle of deficit 0. We define an o-path of G similarly.

The technique of the proof of Theorem 7 is motivated by Chen, Zhang and Qiao [6]. And the proof is based on the following lemmas.

Lemma 1. *([1],[12]) Every 2-connected graph contains a heavy cycle.*

Lemma 2. *([9]) Let G be a graph and C be an o-cycle of G. Then there exists a cycle C' of G such that $V(C) \subseteq V(C')$.*

Proof of Theorem 7. Suppose to the contrary that G is not hamiltonian. By Lemma 1, G contains a heavy cycle. Let C be a longest heavy cycle with a given orientation. Then $V(G) \setminus V(C) \neq \emptyset$. Since G is 2-connected, there exists a path P connecting two vertices $x_1 \in V(C)$ and $x_2 \in V(C)$ internally disjoint with C and such that $|V(P)| \geq 3$. Let $P = x_1u_1u_2 \ldots u_rx_2$ be such a path of minimum length.

Claim 1. $u_kx_i^+, u_kx_i^- \notin E^*(G)$ for every $k \in \{1, 2, \ldots, r\}$ and $i = 1, 2$.

Proof. If $u_kx_1^- \in E^*(G)$ for some $k \in \{1, 2, \ldots, r\}$, then $C' = x_1^-u_k\bar{P}x_1Cx_1^-$ is an o-cycle containing all the vertices of C and $|V(C')| > |V(C)|$. By Lemma 2, there exists a heavy cycle longer than C in G, a contradiction. The other assertions can be proved similarly. □

Claim 2. $x_1^-x_1^+ \in E^*(G)$ and $x_2^-x_2^+ \in E^*(G)$.

Proof. If $x_1^-x_1^+ \notin E(G)$, then $\{x_1, x_1^-, x_1^+, u_1\}$ induces a claw. Since G is claw-heavy, $d(x_1^-) + d(x_1^+) \geq n$ by Claim 1. This implies that $x_1^-x_1^+ \in E^*(G)$. Similarly, we can prove that $x_2^-x_2^+ \in E^*(G)$. □

Claim 3. Either $x_1^-x_1^+ \in E(G)$ or $x_2^-x_2^+ \in E(G)$.

Proof. Suppose to the contrary that $x_1^-x_1^+ \notin E(G)$ and $x_2^-x_2^+ \notin E(G)$. By Claim 1, $d(x_1^-)+d(x_1^+) \geq n$ and $d(x_2^-)+d(x_2^+) \geq n$. This implies that $(d(x_1^-)+d(x_2^-)) + (d(x_1^+)+d(x_2^+)) \geq 2n$. Then $d(x_1^-)+d(x_2^-) \geq n$ or $d(x_1^+)+d(x_2^+) \geq n$. Thus $C' = x_1^-\bar{C}x_2\bar{P}x_1Cx_2^-x_1^-$ is an o-cycle containing all the vertices of C and $|V(C')| > |V(C)|$ or $C'' = x_1^+Cx_2\bar{P}x_1\bar{C}x_2^+x_1^+$ is an o-cycle containing all the vertices of C and $|V(C'')| > |V(C)|$. Therefore, by Lemma 2, there exists a heavy cycle longer than C in G, a contradiction. So $x_1^-x_1^+ \in E(G)$ or $x_2^-x_2^+ \in E(G)$. □

Claim 4. $x_1^-x_2^- \notin E^*(G)$, $x_1^+x_2^+ \notin E^*(G)$, $x_ix_{3-i}^- \notin E^*(G)$ and $x_ix_{3-i}^+ \notin E^*(G)$ for $i = 1, 2$.

Proof. By Claim 2, $x_1^-x_1^+ \in E^*(G)$. If $x_1^-x_2^- \in E^*(G)$, then $C' = x_1^-x_2^-\bar{C}x_1Px_2Cx_1^-$ is an o-cycle containing all the vertices of C and $|V(C')| > |V(C)|$. By Lemma 2, there exists a heavy cycle longer than C in G, a contradiction. Then $x_1^-x_2^- \notin E^*(G)$. Similarly, $x_1^+x_2^+ \notin E^*(G)$.

If $x_1x_2^- \in E^*(G)$, then $C' = x_1x_2^-\bar{C}x_1^+x_1^-\bar{C}x_2\bar{P}x_1$ is an o-cycle containing all the vertices of C and $|V(C')| > |V(C)|$. By Lemma 2, there exists a heavy cycle longer than C in G, a contradiction. So $x_1x_2^- \in E^*(G)$. The other assertions can be proved similarly. □

By Claim 4, there is some vertex in $x_i^+Cx_{3-i}^-$ not adjacent to x_i in G for $i = 1, 2$. Let y_i be the first vertex in $x_i^+Cx_{3-i}^-$ not adjacent to x_i in G for $i = 1, 2$. Let u be a vertex in $V(P) \setminus \{x_1, x_2\}$ and let z_i be an arbitrary vertex in $x_i^+Cy_i$ for $i = 1, 2$.

Claim 5. $uz_1, uz_2, z_1x_2, z_2x_1, z_1z_2 \notin E^*(G)$.

Proof. Suppose $uz_1 \in E^*(G)$. By Claim 1, we have $z_1 \neq x_1^+$. Then $x_1z_1^- \in E(G)$ by the choice of y_1. Then $C' = x_1Puz_1Cx_1^-x_1^+Cz_1^-x_1$ is an o-cycle containing all the vertices of C and $|V(C')| > |V(C)|$. By Lemma 2, there exists a heavy cycle longer than C in G, a contradiction. Hence $uz_1 \notin E^*(G)$. Similarly, $uz_2 \notin E^*(G)$.

Suppose $z_1x_2 \in E^*(G)$. By Claim 4, $z_1 \neq x_1^+$. Then by Claim 2, $C' = x_1Px_2z_1Cx_2^-x_2^+Cx_1^-x_1^+Cz_1^-x_1$ is an o-cycle containing all the vertices of C and $|V(C')| > |V(C)|$. By Lemma 2, there exists a heavy cycle longer than C in G, a contradiction. Hence $z_1x_2 \notin E^*(G)$. Similarly, $z_2x_1 \notin E^*(G)$.

Suppose $z_1z_2 \in E^*(G)$. By Claim 4, $z_1 \neq x_1^+$ or $z_2 \neq x_2^+$. Then by Claim 2, $C' = x_1Px_2z_2^-\bar{C}x_2^+x_2^-\bar{C}z_1z_2Cx_1^-x_1^+Cz_1^-x_1$(if $z_1 \neq x_1^+$ and $z_2 \neq x_2^+$) or $x_1Px_2z_2^-\bar{C}x_2^+x_2^-\bar{C}z_1z_2Cx_1$ (if $z_1 = x_1^+$ and $z_2 \neq x_2^+$) or $x_1Px_2\bar{C}z_1z_2Cx_1^-x_1^+Cz_1^-x_1$(if $z_1 \neq x_1^+$ and $z_2 = x_2^+$) is an o-cycle containing all the vertices of C and $|V(C')| > |V(C)|$. By Lemma 2, there exists a heavy cycle longer than C in G, a contradiction. Hence $z_1z_2 \notin E^*(G)$. □

By Claim 3, without loss of generality, we may assume $x_1^-x_1^+ \in E(G)$.

Claim 6. $x_1x_2 \in E(G)$.

Proof. Suppose $x_1x_2 \notin E(G)$. Now by the choice of P and Claim 5, we have $\{y_1, y_1^-, x_1, u_1, u_2, \ldots, u_r, x_2, y_2^-, y_2\}$ induces P_{r+6}, where $r \geq 1$. Since G is P_7-f-heavy, G is also P_{r+6}-f-heavy. By the choice of C, u_1 and u_r are light. It follows that y_1^- and y_2^- are heavy. Thus $y_1^-y_2^- \in E^*(G)$, contradicting Claim 5. □

Case1. $r = 1$.

Note that G is D-f-heavy or H-f-heavy. If G is D-f-heavy, then by Claims 5 and 6 and the choice of y_1 and y_2, $\{y_1, y_1^-, x_1, u_1, x_2, y_2^-, y_2\}$ induces a D. Since u_1 is light, y_1^- and y_2^- are heavy. It follows that $y_1^-y_2^- \in E^*(G)$, contradicting Claim 5.

Next, we assume G is H-f-heavy. Then by Claims 1, 4 and 6, $\{u_1, x_1, x_2, x_1^-, x_1^+\}$ induces an H and $\{u_1, x_1, x_2, x_2^-, x_2^+\}$ induces an H. Since u_1 is light, we have x_1^- and x_2^- are heavy. This implies that $x_1^-x_2^- \in E^*(G)$, contradicting Claim 4.

Case 2. $r \geq 2$.

Claim 7. $x_2^-x_2^+ \in E(G)$.

Proof. Suppose to the contrary that $x_2^-x_2^+ \notin E(G)$. Then $d(x_2^-)+d(x_2^+) \geq n$ by Claim 2. Without loss of generality, suppose $d(x_2^+) \geq n/2$. By Claims 1 and 4, we have $d(u_r) + d(x_2^+) < n$ and $d(x_1) + d(x_2^+) < n$, respectively. Since $d(u_r) < n/2 \leq d(x_2^+)$, $d(u_r) + d(x_1) < n$. Then $\{x_2, x_1, u_r, x_2^+\}$ induces a claw such that there is no pair of nonadjacent vertices with degree sum at least n, contradicting the hypothesis of Theorem 7. So $x_2^-x_2^+ \in E(G)$. □

Claim 8. $d(x_1) > \frac{n}{2}$ and $d(x_2) > \frac{n}{2}$.

Proof. By the choice of P, we have $u_1x_2 \notin E(G)$ and $u_rx_1 \notin E(G)$. By Claim 4, we obtain $x_1x_2^+ \notin E^*(G)$ and $x_1^-x_2 \notin E^*(G)$. And by Claim 1, we get $u_1x_1^- \notin E^*(G)$ and $u_rx_2^+ \notin E^*(G)$. Note that $x_1x_2 \in E(G)$ by Claim 6. Thus $\{x_1, x_1^-, u_1, x_2\}$ induces a claw and $\{x_2, x_2^+, u_r, x_1\}$ induces a claw. Since G is claw-heavy, $d(u_1) + d(x_2) \geq n$ and $d(u_r) + d(x_1) \geq n$. Since

$d(u_1) < \frac{n}{2}$ and $d(u_r) < \frac{n}{2}$, we have $d(x_1) > \frac{n}{2}$ and $d(x_2) > \frac{n}{2}$. □

By Claim 8, $d(x_1)+d(x_2) > n$. By the choice of P, we have $N_{G-C}(x_1) \cap N_{G-C}(x_2) = \emptyset$. Thus, $|N_C(x_1)| + |N_C(x_2)| > |V(C)|$. Since $x_1x_2 \in E(G)$ by Claim 6, by the choice of y_1 and y_2 and Claims 4 and 5, we have $|N_{C[x_1^-,y_1^-]}(x_1)| + |N_{C[x_1^-,y_1^-]}(x_2)| = |V(C[x_1^-, y_1^-])|$ and $|N_{C[x_2^-,y_2^-]}(x_1)| + |N_{C[x_2^-,y_2^-]}(x_2)| = |V(C[x_2^-, y_2^-])|$. Moreover, by the choice of y_1, y_2 and Claim 5, we have $x_1y_1, x_2y_1, x_1y_2, x_2y_2 \notin E(G)$. Thus,

$$|N_{C[y_1^+,x_2^{-2}]}(x_1)| + |N_{C[y_1^+,x_2^{-2}]}(x_2)| + |N_{C[y_2^+,x_1^{-2}]}(x_1)| + |N_{C[y_2^+,x_1^{-2}]}(x_2)|$$

$$> |V(C[y_1^+, x_2^{-2}])| + |(C[y_2^+, x_1^{-2}])| + 2.$$

This implies that either

$$|N_{C[y_1^+,x_2^{-2}]}(x_1)| + |N_{C[y_1^+,x_2^{-2}]}(x_2)| > |V(C[y_1^+, x_2^{-2}])| + 1$$

or

$$|N_{C[y_2^+,x_1^{-2}]}(x_1)| + |N_{C[y_2^+,x_1^{-2}]}(x_2)| > |(C[y_2^+, x_1^{-2}])| + 1.$$

Without loss of generality, we may assume that

$$|N_{C[y_1^+,x_2^{-2}]}(x_1)| + |N_{C[y_1^+,x_2^{-2}]}(x_2)| > |V(C[y_1^+, x_2^{-2}])| + 1.$$

Then there is a vertex $v \in C[y_1^+, x_2^{-2}]$ such that $x_1v \in E(G)$ and $x_2v^- \in E(G)$. Now $C' = x_1^+Cv^-x_2\bar{P}x_1vCx_2^1x_2^+Cx_1^-x_1^+$ is a cycle containing all the vertices of C and $|V(C)| < |V(C')|$, contradicting the choice of C. Now we complete the proof of Theorem 7. □

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