# A note on Lagrangians of 4-uniform hypergraphs

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**Abstract**

In 1989, Frankl and Füredi [1] conjectured that the $r$-uniform hypergraph with $m$ edges formed by taking the first $m$ sets in the colex ordering of $\mathbb{N}^{(r)}$ has the largest Lagrangian of all $r$-uniform hypergraphs of size $m$. For 2-graphs, Motzkin-Straus theorem implies this conjecture is true. For 3-uniform hypergraphs, it was proved by Talbot in 2002 that the conjecture is true while $m$ in certain range. In this paper, we prove that the 4-uniform hypergraphs with $m$ edges formed by taking the first $m$ sets in the colex ordering of $\mathbb{N}^{(r)}$ has the largest Lagrangian of all 4-uniform hypergraphs with $t$ vertices and $m$ edges which satisfying $\binom{t-1}{4} \leq m \leq \binom{t-1}{4} + \binom{t-2}{3} - 17\binom{t-2}{2} + 1$.

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# 1 Introduction

A hypergraph is a generalization of a graph in which an edge can connect any number of nodes. Exactly, a hypergraph $H$ is a pair $(V, E)$, where $V$ is a finite set of elements called nodes or vertices, and $E$ is a set of nonempty subsets of V called edges or links. Moreover, $E$ is a subset of $2^V \setminus \emptyset$, where $2^V$ is the power set of $V$. An edge $e = \{a_1, a_2, \cdots, a_r\}$ will be simply denoted by $a_1a_2\cdots a_r$ in this paper. An $r$-uniform hypergraph is a hypergraph such that all its edges have same size $r$. That is to say, an $r$-uniform hypergraph is a collection of sets of size $r$. So a 2-uniform hypergraph is a general sense graph, a 3-uniform hypergraph is a collection

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of triples, and so on. For a positive integer $r$, let $V^{(r)}$ denote the set consists of all $r$-subsets of $V$. Let $K_t^{(r)}$ denote the complete $r$-uniform hypergraph on $t$ vertices, that is the $r$-uniform hypergraph on $t$ vertices containing all possible edges. A complete $r$-uniform hypergraph on $t$ vertices is also called a clique with order $t$. A clique is said to be maximal if there is no other clique containing it, while it is called maximum if it has maximum cardinality. Let $\mathbb{N}$ be the set of all positive integers. For an integer $n \in \mathbb{N}$, let $[n]$ denote the set $\{1, 2, 3, \cdots, n\}$. Let $[n]^{(r)}$ represent the complete $r$-graph on the vertex set $[n]$.

For $r$-uniform hypergraphs $H = (V, E)$, denote the $(r-1)$-neighborhood of a vertex $i \in V$ by $E_i = \{A \in V^{(r-1)} : A \cup \{i\} \in E\}$. Similarly, denote the $(r-2)$-neighborhood of a pair of vertices $i, j \in V$ by $E_{ij} = \{B \in V^{(r-2)} : B \cup \{i, j\} \in E\}$. Denote the complement of $E_i$ by $E_i^c = \{A \in V^{(r-1)} : A \cup \{i\} \in V^{(r)} \backslash E\}$. Also, denote the complement of $E_{ij}$ by $E_{ij}^c = \{B \in V^{(r-2)} : B \cup \{i, j\} \in V^{(r)} \backslash E\}$ and $E_{i \backslash j} = E_i \cap E_j^c$.

**Definition 1.1** *For an $r$-uniform hypergraph $H = ([n], E(H))$ and a vector $\vec{x} = (x_1, \cdots, x_n) \in R^n$, define*

$$\lambda(H, \vec{x}) = \sum_{i_1 i_2 \cdots i_r \in E(H)} x_{i_1} x_{i_2} \cdots x_{i_r}.$$

*Let $S = \{\vec{x} = (x_1, x_2, \cdots, x_n) : \sum_{i=1}^{n} x_i = 1, x_i \geq 0$ for $i = 1, 2, \cdots, n\}$. The Lagrangian*[1] *of $H$, denoted by $\lambda(H)$, is the maximum of the above homogeneous function over the standard simplex $S$. Precisely,*

$$\lambda(H) = \max\{\lambda(H, \vec{x}) : \vec{x} \in S\}.$$

The vector $\vec{x} = (x_1, x_2, \cdots, x_n) \in R^n$ is called a feasible weighting for $H$ if $\vec{x} \in S$. The value $x_i$ is called the weight of the vertex $i$. A vector $\vec{y} \in S$ is called an optimal weighting for $H$ if $\lambda(H, \vec{y}) = \lambda(H)$.

In [4], Motzkin and Straus established a remarkable connection between the clique number and the Lagrangian of a graph.

**Theorem 1.1** [4] If $H$ is a 2-uniform graph in which a largest clique has order $t$ then $\lambda(H) = \lambda(K_t^{(2)}) = \frac{1}{2}(1 - \frac{1}{t})$.

The obvious generalization of Motzkin and Straus' result to hypergraphs is false because there are many examples of hypergraphs that do not achieve their Lagrangian on any proper subhypergraph. Lagrangian of hypergraphs

[1]Let us note that this use of the name Lagrangian is at odds with the tradition. Indeed, names as Laplacian, Hessian, Gramian, Grassmanian, etc., usually denote a structured object like matrix, operator, or manifold, and not just a single number.

has been proved to be a useful tool in hypergraph extremal problems. Applications of Lagrangian method can be found in [1], [2], [3], [5] , [8]and [9]. In most applications, an upper bound is needed. Frankl and Füredi [1] asked the following question. Given $r \geq 3$ and $m \in \mathbb{N}$ how large can the Lagrangian of an $r$-uniform hypergraph with $m$ edges be? For distinct $A, B \in \mathbb{N}^{(r)}$ we say that $A$ is less than $B$ in the *colex ordering* if $max(A \triangle B) \in B$, where $A \triangle B = (A \setminus B) \cup (B \setminus A)$. For example, the first $\binom{t}{r}$ $r$-tuples in the colex ordering of $\mathbb{N}^{(r)}$ are the edges of $[t]^{(r)}$. The following conjecture of Frankl and Füredi (if it is true) proposes a solution to the question mentioned above.

**Conjecture 1.2** [1] The $r$-uniform hypergraph with $m$ edges formed by taking the first $m$ sets in the colex ordering of $\mathbb{N}^{(r)}$ has the largest Lagrangian of all $r$-uniform hypergraphs with $m$ edges. In particular, the $r$-uniform hypergraph with $\binom{t}{r}$ edges and the largest Lagrangian is $[t]^{(r)}$.

This conjecture is true when $r = 2$ by Theorem 1.1. For the case $r = 3$, Talbot in [10] proved the following.

**Theorem 1.3** [10] Let $m$ and $t$ be integers satisfying $\binom{t-1}{3} \leq m \leq \binom{t-1}{3} + \binom{t-2}{2} - (t-1)$. Then Conjecture 1.2 is true for $r = 3$ and this value of $m$.

Although the obvious generalization of Motzkin and Straus' result to hypergraphs is false, we attempt to explore the relationship between the Lagrangian of a hypergraph and the size of its maximum cliques for hypergraphs when the number of edges is in certain range. In [7], it is conjectured that the following Motzkin and Straus type results are true for hypergraphs.

**Conjecture 1.4** [7] Let $t$, $m$, and $r \geq 3$ be positive integers satisfying $\binom{t-1}{r} \leq m \leq \binom{t-1}{r} + \binom{t-2}{r-1}$. Let $H$ be an $r$-uniform hypergraph with $m$ edges and $H$ contain a clique of order $t-1$. Then $\lambda(H) = \lambda([t-1]^{(r)})$.

**Conjecture 1.5** [7] Let $t$, $m$, and $r \geq 3$ be positive integers satisfying $\binom{t-1}{r} \leq m \leq \binom{t-1}{r} + \binom{t-2}{r-1}$. Let $H$ be an $r$-uniform hypergraph with $m$ edges without containing a clique of order $t-1$. Then $\lambda(H) < \lambda([t-1]^{(r)})$.

Note that the upper bound $\binom{t-1}{r} + \binom{t-2}{r-1}$ in Conjecture 1.4 is the best possible (see [7]). Conjecture 1.4 is confirmed when $r = 3$ in [7]. Let $C_{r,m}$ denote the $r$-uniform hypergraph with $m$ edges formed by taking the first $m$ sets in the colex ordering of $\mathbb{N}^{(r)}$. The following result was given in [10].

**Lemma 1.6** [10] For any integers $m, t$, and $r$ satisfying

$$\binom{t-1}{r} \leq m \leq \binom{t-1}{r} + \binom{t-2}{r-1},$$

we have $\lambda(C_{r,m}) = \lambda([t-1]^{(r)})$.

In [6], the following result is obtained for $r$-uniform hypergraphs.

**Theorem 1.7** [6] Let $t$,$m$ and $r$ be positive integers satisfying

$$\binom{t-1}{r} \leq m \leq \binom{t-1}{r} + \binom{t-2}{r-1} - (2^{r-3}-1)(\binom{t-2}{r-2} - 1).$$

Let $H$ be an $r$-uniform hypergraph with $t$ vertices and $m$ edges and contain a clique of order $t-1$. Then $\lambda(H) = \lambda([t-1]^{(r)})$.

Some other results on Conjecture 1.2, 1.4 and 1.5 can also be seen from [11] and [12]. In this paper, we first show the following.

**Theorem 1.8** Let $m$ and $t$ be integers satisfying

$$\binom{t-1}{4} \leq m \leq \binom{t-1}{4} + \binom{t-2}{3} - 17\binom{t-2}{2} + 1.$$

Let $H$ be a 4-uniform hypergraph with $t$ vertices and $m$ edges without containing a clique of order $t-1$. Then $\lambda(H) < \lambda([t-1]^{(4)})$.

Then, combing Theorems 1.7 (in the case when $r = 4$) and Theorem 1.8, we have the following result immediately.

**Corollary 1.9** Let $m$ and $t$ be integers satisfying

$$\binom{t-1}{4} \leq m \leq \binom{t-1}{4} + \binom{t-2}{3} - 17\binom{t-2}{2} + 1.$$

Let $H$ be a 4-uniform hypergraph with $t$ vertices and $m$ edges. Then $\lambda(H) \leq \lambda([t-1]^{(4)})$.

Note that Theorem 1.8 and Corollary 1.9 provide evidence for Conjecture 1.5 and Conjecture 1.2 respectively. The proof of Theorem 1.8 will be given in Section 2. Other related results will be discussed in Section 3.

# 2 Proof of Theorem 1.8

We will impose one additional condition on any optimal weighting $\vec{x} = (x_1, x_2, \cdots, x_n)$ for an $r$-uniform hypergraph $H$:

$$|\{i : x_i > 0\}| \text{ is minimal, i.e. if } \vec{y} \text{ is a feasible weighting for } H \text{ satisfying}$$
$$|\{i : y_i > 0\}| < |\{i : x_i > 0\}|, \text{ then } \lambda(H, \vec{y}) < \lambda(H). \quad (1)$$

When the theory of Lagrange multipliers is applied to find the optimum of $\lambda(H, \vec{x})$, subject to $\sum_{i=1}^{n} x_i = 1$, notice that $\lambda(E_i, \vec{x})$ corresponds to the partial derivative of $\lambda(H, \vec{x})$ with respect to $x_i$. The following lemma gives some necessary conditions of an optimal weighting for $H$.

**Lemma 2.1** [2] Let $H = (V, E)$ be an $r$-uniform hypergraph on the vertex set $[n]$ and $\vec{x} = (x_1, x_2, \cdots, x_n)$ be an optimal weighting for $H$ with $k$ $(\leq n)$ non-zero weights $x_1$, $x_2, \cdots, x_k$ satisfying condition (1). Then for every $\{i, j\} \in [k]^{(2)}$, (a) $\lambda(E_i, \vec{x}) = \lambda(E_j, \vec{x}) = r\lambda(H)$, (b) there is an edge in $E$ containing both $i$ and $j$.

**Definition 2.1** An $r$-uniform hypergraph $H = (V, E)$ on the vertex set $[n]$ is left-compressed if $j_1 j_2 \cdots j_r \in E$ implies $i_1 i_2 \cdots i_r \in E$ whenever $i_k \leq j_k, 1 \leq k \leq r$. Equivalently, an $r$-uniform hypergraph $H = (V, E)$ on the vertex set $[n]$ is left-compressed if $E_{j \setminus i} = \emptyset$ for any $1 \leq i < j \leq n$.

**Remark 2.2** (a) In Lemma 2.1, part(a) implies that $x_j \lambda(E_{ij}, \vec{x}) + \lambda(E_{i\setminus j}, \vec{x}) = x_i \lambda(E_{ij}, \vec{x}) + \lambda(E_{j\setminus i}, \vec{x})$. In particular, if $H$ is left-compressed, then $(x_i - x_j)\lambda(E_{ij}, \vec{x}) = \lambda(E_{i\setminus j}, \vec{x})$ for any $i, j$ satisfying $1 \leq i < j \leq k$ since $E_{j\setminus i} = \emptyset$.

(b) If $H$ is left-compressed, then for any $i, j$ satisfying $1 \leq i < j \leq k$,

$$x_i - x_j = \frac{\lambda(E_{i\setminus j}, \vec{x})}{\lambda(E_{ij}, \vec{x})} \tag{2}$$

holds. If $H$ is left-compressed and $E_{i\setminus j} = \emptyset$,for any $i, j$ satisfying $1 \leq i < j \leq k$, then $x_i = x_j$.

(c) By (2), if $H$ is left-compressed, then an optimal weighting $\vec{x} = (x_1, x_2, \cdots, x_n)$ for $H$ must satisfy $x_1 \geq x_2 \geq \cdots \geq x_n \geq 0$.

Denote $\lambda^{r-}_{(m,t-1,t)} = \max\{\lambda(H) : H$ is an $r$-uniform hypergraph with $t$ vertices and $m$ edges not containing a clique of order $t-1\}$. The following lemma implies that we only need to consider left-compressed 4-uniform hypergraphs $H$ when we prove Theorem 1.8.

**Lemma 2.3** Let $m$ and $t$ be integers satisfying

$$\binom{t-1}{4} \leq m \leq \binom{t-1}{4} + \binom{t-2}{3} - 17\binom{t-2}{2} + 1.$$

There exists a left-compressed 4-uniform hypergraph $H$ on vertex set $[t]$ with $m$ edges without containing $[t-1]^{(4)}$ such that $\lambda(H) = \lambda^{4-}_{(m,t-1,t)}$.

In the proof of Lemma 2.3, we need to define some partial order relation. An $r$-tuple $i_1 i_2 \cdots i_r$ is called a *descendant* of an $r$-tuple $j_1 j_2 \cdots j_r$ if $i_s \leq j_s$ for each $1 \leq s \leq r$, and $i_1 + i_2 + \cdots + i_r < j_1 + j_2 + \cdots + j_r$. In this case, the $r$-tuple $j_1 j_2 \cdots j_r$ is called an *ancestor* of $i_1 i_2 \cdots i_r$. The $r$-tuple $i_1 i_2 \cdots i_r$ is called a *direct descendant* of $j_1 j_2 \cdots j_r$ if $i_1 i_2 \cdots i_r$ is a descendant of $j_1 j_2 \cdots j_r$ and $j_1 + j_2 + \cdots + j_r = i_1 + i_2 + \cdots + i_r + 1$.

We say that $i_1i_2\cdots i_r$ has lower hierarchy than $j_1j_2\cdots j_r$ if $i_1i_2\cdots i_r$ is a descendant of $j_1j_2\cdots j_r$. This is a partial order on the set of all $r$-tuples.

*Proof of Lemma 2.3.* Let $H$ be a 4-uniform hypergraph with $t$ vertices and $m$ edges without containing a clique of order $t-1$ such that $\lambda(H) = \lambda^{4-}_{(m,t-1,t)}$. We call $H$ an extremal 4-uniform hypergraph for $m$, $t-1$ and $t$. Let $\vec{x} = (x_1, x_2, \cdots, x_t)$ be an optimal weighting of $H$. We can assume that $x_i \geq x_j$ when $i < j$ since otherwise we can just relabel the vertices of $H$ and obtain another extremal 4-uniform hypergraph for $m$, $t-1$ and $t$ with an optimal weighting $\vec{x} = (x_1, x_2, \cdots, x_t)$ satisfying $x_i \geq x_j$ when $i < j$. Next we obtain a new 4-uniform hypergraph $H$ from $H$ by performing the following:

1. If $(t-4)(t-3)(t-2)(t-1) \in E(H)$, then there is at least one 4-tuple in $[t-1]^{(4)} \setminus E(H)$, we replacing $(t-4)(t-3)(t-2)(t-1)$ by this 4-tuple;

2. If an edge in $H$ has a descendant other than $(t-4)(t-3)(t-2)(t-1)$ that is not in $E(H)$, then replace this edge by a descendant other than $(t-4)(t-3)(t-2)(t-1)$ with the lowest hierarchy. Repeat this until there is no such an edge.

Then $H$ satisfies the following properties:

1. The number of edges in $H$ is the same as the number of edges in $H$.

2. $\lambda(H) = \lambda(H, \vec{x}) \leq \lambda(H, \vec{x}) \leq \lambda(H)$.

3. $(t-4)(t-3)(t-2)(t-1) \notin E(H)$.

4. For any edge in $E(H)$, all its descendants other than $(t-4)(t-3)(t-2)(t-1)$ will be in $E(H)$.

If $H$ is not left-compressed, then there is an ancestor of $(t-4)(t-3)(t-2)(t-1)$ $uvwz$ such that $uvwz \in E(H)$. Hence $(t-4)(t-3)(t-2)t$ and all the descendants of $(t-4)(t-3)(t-2)t$ except $(t-4)(t-3)(t-2)(t-1)$ will be in $E(H)$. Then

$$m \geq \binom{t-1}{4} - 1 + \binom{t-2}{3} > \binom{t-1}{4} + \binom{t-2}{3} - 17\binom{t-2}{2} + 1$$

which is a contradiction. $H$ does not contain $[t-1]^{(4)}$ since $H$ does not contain $(t-4)(t-3)(t-2)(t-1)$. Clearly $H$ is on vertex set $[t]$. ■

In the following three lemmas, Lemma 2.4 implies the maximum weight of $H$ can not be too large if $\lambda(H) \geq \lambda([t-1]^{(4)})$, and Lemma 2.6 implies $H$ contains most of the first $\binom{t-2}{4}$ edges in colex ordering of $\mathbb{N}^{(4)}$ if $\lambda(H) \geq \lambda([t-1]^{(4)})$, while Lemma 2.5 implies $H$ also contains most of the next $\binom{t-2}{3}$ edges.

**Lemma 2.4** (a) Let $H$ be a 4-uniform hypergraph on vertex set $[t]$. Let $\vec{x} = (x_1, x_2, \cdots, x_t)$ be an optimal weighting for $H$ satisfying $x_1 \geq x_2 \geq \cdots \geq x_t \geq 0$. Then $x_1 < x_{t-5} + x_{t-4}$ or $\lambda(H) \leq \frac{1}{24}\frac{(t-3)^2}{(t-2)(t-1)} < \lambda([t-1]^{(4)})$.

(b) Let $H$ be a 4-uniform hypergraph on vertex set $[t]$. Let $\vec{x} = (x_1, x_2, \cdots, x_t)$ be an optimal weighting for $H$ satisfying $x_1 \geq x_2 \geq \cdots \geq x_t \geq 0$. Then

$$x_1 < 2(x_{t-3} + x_{t-2}) \textit{ or } \lambda(H) \leq \frac{1}{24}\frac{(t-3)^2}{(t-2)(t-1)} < \lambda([t-1]^{(4)}).$$

*Proof.* (a) If $x_1 \geq x_{t-5} + x_{t-4}$, then $4x_1 + x_2 + \cdots + x_{t-6} \geq x_1 + x_2 + \cdots + x_{t-4} + x_{t-3} + x_{t-2} + x_{t-1} + x_t = 1$. Recalling that $x_1 \geq x_2 \geq \cdots \geq x_{t-6}$, we have $x_1 \geq \frac{1}{t-3}$. Using Lemma 2.1, we have $\lambda(H) = \frac{1}{4}\lambda(E_1, x)$. Note that $E_1$ is a 3-uniform hypergraph with $t-1$ vertices and total weights at most $1 - \frac{1}{t-3}$. Hence by Theorem 1.3.

$$\begin{aligned}\lambda(H) &= \frac{1}{4}\lambda(E_1, x) \leq \frac{1}{4}\binom{t-1}{3}\left(\frac{1-\frac{1}{t-3}}{t-1}\right)^3 \\ &= \frac{1}{24}\frac{(t-2)(t-4)^3}{(t-3)^2(t-1)^2} < \frac{(t-4)(t-3)(t-2)}{24(t-1)^3} = \lambda([t-1]^{(4)}).\end{aligned} \quad (3)$$

(b) If $x_1 \geq 2(x_{t-3} + x_{t-2})$, then $2x_1 + x_2 + \cdots + x_{t-4} \geq x_1 + x_2 + \cdots + x_{t-4} + x_{t-3} + x_{t-2} + x_{t-1} + x_t = 1$. Recalling that $x_1 \geq x_2 \geq \cdots \geq x_{t-4}$, we have $x_1 \geq \frac{1}{t-3}$. The rest of the proof is the same as that in part (a), we omit the computation details here. ∎

**Lemma 2.5** Let $H$ be a left-compressed 4-uniform hypergraph in the vertex set $[t]$, then

$$|[t-2]^{(3)} \backslash E_{t-1}| \leq 8|E_{(t-1)t}| \textit{ or } \lambda(H) < \lambda([t-1]^{(4)}).$$

*Proof.* Let $\vec{x} = (x_1, x_2, \cdots, x_t)$ be an optimal weighting for $H$. Since $H$ is left-compressed, by Remark 2.2 (a), $x_1 \geq x_2 \geq \cdots \geq x_t \geq 0$. If $x_t = 0$, then $\lambda(H) < \lambda([t-1]^{(4)})$. So we assume that $x_t > 0$.

Consider a new weighting for $H$, $\vec{y} = (y_1, y_2, \cdots, y_t)$ given by $y_i = x_i$ for $i \neq t-1, t$, $y_{t-1} = x_{t-1} + x_t$ and $y_t = 0$. By Lemma 2.1 (a), $\lambda(E_{t-1}, \vec{x}) = \lambda(E_t, \vec{x})$, so

$$\begin{aligned}\lambda(H, \vec{y}) - \lambda(H, \vec{x}) &= x_t(\lambda(E_{t-1}, \vec{x}) - x_t\lambda(E_{(t-1)t}, \vec{x})) - x_t(\lambda(E_t, \vec{x}) \\ &\quad - x_t\lambda(E_{(t-1)t}, \vec{x})) - x_{t-1}x_t\lambda(E_{(t-1)t}, \vec{x})) \\ &= x_t(\lambda(E_{t-1}, \vec{x}) - \lambda(E_t, \vec{x})) - x_t^2\lambda(E_{(t-1)t}, \vec{x}) \\ &= -x_t^2\lambda(E_{(t-1)t}, \vec{x}).\end{aligned} \quad (4)$$

Assume that $|[t-2]^{(3)} \backslash E_{t-1}| > 8|E_{(t-1)t}|$. If $\lambda(H) < \lambda([t-1]^{(4)})$ we are done. Otherwise if $\lambda(H) \geq \lambda([t-1]^{(4)})$ we will show that there exists a set of edges $F \subset [t-1]^{(4)} \setminus E$ satisfying

$$\lambda(F, \vec{y}) > x_t^2 \lambda(E_{(t-1)t}, \vec{x}). \tag{5}$$

Then using (4) and (5), the 4-uniform hypergraph $H^* = ([t], E^*)$, where $E^* = E \cup F$, satisfies $\lambda(H^*, \vec{y}) > \lambda(H)$. Since $\vec{y}$ has only $t-1$ positive weights, then $\lambda(H^*, \vec{y}) \leq \lambda([t-1]^{(4)})$, and consequently, $\lambda(H) < \lambda([t-1]^{(4)})$.

We now construct the set of edges $F$. Let $C = [t-2]^{(3)} \setminus E_{t-1}$. Then by the assumption, $|C| > 8|E_{(t-1)t}|$ and $\lambda(C, \vec{x}) \geq 8|E_{(t-1)t}| x_{t-4} x_{t-3} x_{t-2}$.

Let $F$ consist of those edges in $[t-1]^{(4)} \setminus E$ containing the vertex $t-1$. Since $\lambda(H) \geq \lambda([t-1]^{(4)})$ then $x_{t-5} \geq \frac{x_1}{2}$ by Lemma 2.4 (a) and $x_{t-4} \geq x_{t-3} > \frac{x_1}{4}$ by Lemma 2.4 (b). Hence

$$\begin{aligned}
\lambda(F, \vec{y}) &= (x_{t-1} + x_t)\lambda(C, \vec{x}) > 2x_t \cdot 8|E_{(t-1)t}| x_{t-4} x_{t-3} x_{t-2} \\
&\geq x_t^2 |E_{(t-1)t}| (x_1)^2 \geq x_t^2 \sum_{i_1 i_2 \in E_{(t-1)t}} x_{i_1} x_{i_2} \\
&= x_t^2 \lambda(E_{(t-1)t}, \vec{x}).
\end{aligned} \tag{6}$$

Hence $F$ satisfies (5). This proves Lemma 2.5. ■

**Lemma 2.6** Let $H$ be a left-compressed 4-uniform hypergraph on the vertex set $[t]$, then

$$|[t-2]^{(4)} \backslash E| \leq 8|E_{(t-1)t}| \text{ or } \lambda(H) < \lambda([t-1]^{(4)}).$$

*Proof.* Let $\vec{x} = (x_1, x_2, \cdots, x_t)$ be an optimal weighting for $H$. Since $H$ is left-compressed, by Remark 2.2 (a), $x_1 \geq x_2 \geq \cdots \geq x_t \geq 0$. If $x_t = 0$, then $\lambda(H) < \lambda([t-1]^{(4)})$. So we assume that $x_t > 0$.

Consider a new weighting for $H$, $\vec{y} = (y_1, y_2, \cdots, y_t)$ given by $y_i = x_i$ for $i \neq t-1, t$, $y_{t-1} = x_{t-1} + x_t$ and $y_t = 0$. By Lemma 2.1 (a), $\lambda(E_{t-1}, \vec{x}) = \lambda(E_t, \vec{x})$, similar to (4), we have

$$\lambda(H, \vec{y}) - \lambda(H, \vec{x}) = -x_t^2 \lambda(E_{(t-1)t}, \vec{x}). \tag{7}$$

Assume that $|[t-2]^{(4)} \backslash E| > 8|E_{(t-1)t}|$. If $\lambda(H) < \lambda([t-1]^{(4)})$ we are done. Otherwise if $\lambda(H) \geq \lambda([t-1]^{(4)})$ we will show that there exists a set of edges $F \subset [t-2]^{(4)} \setminus E$ satisfying

$$\lambda(F, \vec{y}) > x_t^2 \lambda(E_{(t-1)t}, \vec{x}). \tag{8}$$

Then using (7) and (8), the 4-uniform hypergraph $H^* = ([t], E^*)$, where $E^* = E \cup F$, satisfies $\lambda(H^*, \vec{y}) > \lambda(H)$. Since $\vec{y}$ has only $t-1$ positive weights, then $\lambda(H^*, \vec{y}) \leq \lambda([t-1]^{(4)})$, and consequently, $\lambda(H) < \lambda([t-1]^{(4)})$. This is a contradiction.

We now construct the set of edges $F$. Let $D = [t-2]^{(4)} \backslash E$. Then by the assumption, $|D| > 8|E_{(t-1)t}|$ and $\lambda(D, \vec{x}) \geq 8|E_{(t-1)t}|x_{t-5}x_{t-4}x_{t-3}x_{t-2}$.

Let $F = D$. Since $\lambda(H) \geq \lambda([t-1]^{(4)})$ then $x_{t-5} \geq \frac{x_1}{2}$ by Lemma 2.4 (a) and $x_{t-4} \geq x_{t-3} > \frac{x_1}{4}$ by Lemma 2.4 (b). Hence

$$\begin{aligned} \lambda(F, \vec{y}) &= \lambda(D, \vec{x}) > 8|E_{(t-1)t}|x_{t-5}x_{t-4}x_{t-3}x_{t-2} \geq x_t^2|E_{(t-1)t}|(x_1)^2 \\ &\geq x_t^2 \sum_{i_1 i_2 \in E_{(t-1)t}} x_{i_1}x_{i_2} = x_t^2 \lambda(E_{(t-1)t}, \vec{x}). \end{aligned} \tag{9}$$

Hence $F$ satisfies (8). This proves Lemma 2.6. ■

Now we are ready to prove Theorem 1.8.

*Proof of Theorem 1.8.* Let $m$ and $t$ be integers satisfying $\binom{t-1}{4} \leq m \leq \binom{t-1}{4} + \binom{t-2}{3} - 17\binom{t-2}{2} + 1$. Let $H$ be a 4-uniform hypergraph with $t$ vertices and $m$ edges without containing a clique of order $t-1$ such that $\lambda(H) = \lambda^{4-}_{(m,t-1,t)}$. Then by Lemma 2.3, we can assume that $H$ is left-compressed and does not contain $[t-1]^{(4)}$. Let $\vec{x} = (x_1, x_2, \cdots, x_t)$ be an optimal weighting for $H$. Since $H$ is left-compressed, by Remark 2.2 (a), $x_1 \geq x_2 \geq \cdots \geq x_t \geq 0$. If $x_t = 0$, then $\lambda(H) < \lambda([t-1]^{(4)})$ since $H$ does not contain $[t-1]^{(4)}$. So we assume that $x_t > 0$.

If $\lambda(H) < \lambda([t-1]^{(4)})$ we are done. Otherwise $|[t-2]^{(3)} \backslash E_{t-1}| \leq 8|E_{(t-1)t}|$ by Lemma 2.5 and $|[t-2]^{(4)} \backslash E| \leq 8|E_{(t-1)t}|$ by Lemma 2.6. Recalling that $H$ does not contain $[t-1]^{(4)}$, we have

$$0 < |[t-1]^{(4)} \backslash E| = |[t-2]^{(3)} \backslash E_{t-1}| + |[t-2]^{(4)} \backslash E| \leq 16|E_{(t-1)t}| \leq 16\binom{t-2}{2}.$$

Let $E^* = E \bigcup [t-1]^{(4)}$ and $H^* = ([t], E^*)$. Denote the edges of $H^*$ as $m^*$, then $\binom{t-1}{4} \leq m^* \leq \binom{t-1}{4} + \binom{t-2}{3} - \binom{t-2}{2} + 1$. So $\lambda(H^*) = \lambda([t-1]^{(4)})$ by Theorem 1.7. Clearly, $\lambda(H^*, \vec{x}) - \lambda(H, \vec{x}) > 0$ since $x_1 \geq x_2 \geq \cdots \geq x_t > 0$ and $|[t-1]^{(4)} \backslash E| > 0$. Hence $\lambda(H) = \lambda(H, \vec{x}) < \lambda(H^*, \vec{x}) \leq \lambda(H^*) = \lambda([t-1]^{(4)})$. This completes the proof of Theorem 1.8. ■

# 3 Remarks and Conclusions

We would like to point out the following result for $r$-uniform hypergraphs.

**Theorem 3.1** [10] For any $r \geq 4$ there exists constants $\gamma_r$ and $\kappa_0(r)$ such that if $m$ satisfies

$$\binom{t-1}{r} \leq m \leq \binom{t-1}{r} + \binom{t-2}{r-1} - \gamma_r(t-1)^{r-2},$$

with $t \geq \kappa_0(r)$, let $H$ be an $r$-uniform hypergraph on $t$ vertices with $m$ edges, then $\lambda(H) \leq \lambda([t-1]^{(r)})$.

Note that, in the proof of Theorem 3.1, we see that $\gamma_r = 2^{2^r}$ and $t \geq \kappa_0(r)$ (a sufficently large number). Now carrying out the major proof of Theorem 3.1 with more detailed computation in the case when $r = 4$, as a comparison, we have an improved result (Theorem 3.5) that will be stated below. We first need a few lemmas.

Let $c_q$ and $d_q$ be defined as follow for positive integer $q$:

$$c_0 = 1, c_{q+1} = \sum_{i=1}^{c_q}(d_q + i + 1),\ d_0 = 1, d_{q+1} = c_q + d_q.$$

Denote $\lambda^r_{(m,t)} = \max\{\lambda(H) : H$ is an $r$-uniform hypergraph with $t$ vertices and $m$ edges$\}$. The following lemmas are proved in [10].

**Lemma 3.2** [10] Let $H$ be a left-compressed $r$-uniform hypergraph with $t$ vertices and $m$ edges such that $\lambda(H) = \lambda^r_{(m,t)}$. Let $\vec{x} = (x_1, x_2, \cdots, x_t)$ be an optimal weighting for $H$ with $k$ nonzero weights. Then

$$|[k - (d_{r-2} + 1)]^{(r)} \backslash E| \leq c_{r-2}|E_{(k-1)k}|.$$

**Lemma 3.3** [10] Let $H$ be a left-compressed $r$-uniform hypergraph with $t$ vertices and $m$ edges such that $\lambda(H) = \lambda^r_{(m,t)}$. Let $\vec{x} = (x_1, x_2, \cdots, x_t)$ be an optimal weighting for $H$ with $k$ nonzero weights. Then

$$|[k - (d_{r-2} + 1)]^{(r-1)} \backslash E_k| \leq c_{r-2}|E_{(k-1)k}|.$$

**Lemma 3.4** [10] There exists a left-compressed $r$ graph $H$ with $m$ edges and $t$ vertices such that $\lambda(H) = \lambda^r_{(m,t)}$.

Using Lemmas 3.2, 3.3 and 3.4 we have the following result for $r = 4$.

**Theorem 3.5** Let $m$ and $t$ be integers satisfying $\binom{t-1}{4} \leq m \leq \binom{t-6}{4} + 6\binom{t-6}{3} - 105\binom{t-1}{2} - 1$. Let $H$ be a 4-uniform hypergraph with $t$ vertices and $m$ edges. Then $\lambda(H) \leq \lambda([t-1]^{(4)})$.

*Proof.* Let $m$ and $t$ be integers satisfying $\binom{t-1}{4} \leq m \leq \binom{t-6}{4} + 6\binom{t-6}{3} - 105\binom{t-1}{2} - 1$. Let $H$ be a 4-uniform hypergraph with $t$ vertices and $m$ edges such that $\lambda(H) = \lambda^4_{(m,t)}$. Then by Lemma 3.4, we can assume that $H$ is left-compressed. Let $\vec{x} = (x_1, x_2, \cdots, x_t)$ be an optimal weighting for $H$ with $k$ nonzero weights. Next we show that $k \leq t-1$. So $\lambda(H) \leq \lambda([t-1]^{(4)})$. Assume that $k = t$ for a contradiction.

Set $r = 4$ in Lemma 3.2 and 3.3 and note that $d_2 = 5$ and $c_2 = 15$, we have

$$|[k-6]^{(4)} \backslash E| \leq 15|E_{(k-1)k}| \text{ and } |[k-6]^{(3)} \backslash E_k| \leq 15|E_{(k-1)k}|. \quad (10)$$

Since $H$ is left-compressed, then $|[k-6]^{(3)} \backslash E_{k-i}| \leq 15|E_{(k-1)k}|$ for $0 \leq i \leq 5$. If $k = t$, then

$$\begin{aligned} m = |E| &= |E \cap [k-6]^{(4)}| + \sum_{i=0}^{5} |[k-6]^{(3)} \bigcap E_{k-i}| \\ &\geq \binom{t-6}{4} + 6\binom{t-6}{3} - 105\binom{t-1}{2}. \end{aligned} \quad (11)$$

This contradicts to the assumption that

$$m \leq \binom{t-6}{4} + 6\binom{t-6}{3} - 105\binom{t-1}{2} - 1.$$

Hence $k \leq t-1$ *and* $\lambda(H) \leq \lambda([t-1]^{(4)})$. ■

Note that

$$\begin{aligned} &\binom{t-6}{4} + 6\binom{t-6}{3} - 105\binom{t-1}{2} - 1 \\ &= \binom{t-1}{4} + \binom{t-2}{3} + 4\binom{t-3}{2} + 3\binom{t-4}{2} + 2\binom{t-5}{2} + \binom{t-6}{2} \\ &-105\binom{t-1}{2} - 1. \end{aligned} \quad (12)$$

Clearly, this value is less than

$$\binom{t-1}{4} + \binom{t-2}{3} - 17\binom{t-2}{2} + 1. \quad (13)$$

Hence the upper bound and approach in Corollary 1.9 are better than that in Theorem 3.5. Also, in Theorem 3.5, we get rid of the restriction that the number of vertex set $t$ is sufficiently large, which is imposed in Theorem 3.1 when $r = 4$.

A natural question is to generalize Theorem 1.8 to general case when $r \geq 5$. Unfortunately, for general case when $r \geq 5$, we are not able to obtain similar results since we are not able to obtain $x_1 < \gamma(x_{t-2} + x_{t-3})$ for some $\gamma$ as we did in Lemma 2.4. Another challenge in the future study is how to prove similar results as Theorem 1.8 without restriction of exactly set $t$ vertices. This will be considered in the future work.

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