On the existence and the number of (2-d)-kernels in graphs

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Abstract

In this paper we study (2-d)-kernels in graphs. We shall show that the problem of the existence of (2-d)-kernels is $\mathcal{NP}$-complete for a general graph. We also give some results related to the problem of counting of (2-d)-kernels in graphs. For special graphs we show that the number of (2-d)-kernels is equal to the Fibonacci numbers.

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1 Introduction and preliminary results

In general we use the standard terminology and notation of combinatorics and graph theory see [1, 2]. Only simple, undirected graphs are considered. A path P_n, $n \geq 2$ is a graph with $V(P_n) = \{x_1, \ldots, x_n\}$ and $E(P_n) = \{x_i x_{i+1} : i = 1, \ldots n-1\}$. In what follows G stands for a graph with the vertex set $V(G)$, $|V(G)|$ denotes the cardinality of $V(G)$.

A subset $S \subseteq V(G)$ is an *independent set* of G if no two vertices of S are adjacent in G. An independent set of G is *maximal* if there is no independent set of G containing S as a proper subset.

A subset $Q \subseteq V(G)$ is a *dominating set* of G if each vertex from $V(G) \backslash Q$ has a neighbour in Q. A subset J is a *kernel* of G if J is independent and dominating. The concept of kernels was introduced by Neumann and Morgenstern in digraphs in the context of game theory and kernels were studied in the next decades see [11, 12, 15]. H. Galeana-Sánchez played an important role in studying the existence of kernels and their generalizations in digraphs, see for example papers [4, 5, 8]. Recently interesting results for kernels are obtained by C. Hernández-Cruz, see [7, 8].

In [13] A. Włoch introduced a new type of kernels by considering dominating sets with additional restrictions. We recall this definition.

A subset $J \subset V(G)$ is a *2-dominating kernel* of G if J is independent and 2-dominating i.e each vertex from $V(G) \setminus J$ has at least two neighbours in J.

For convenience instead of 2-dominating kernel we will write (2-d)-kernel.

The definition of (2-d)-kernel implies that a connected graph with (2-d)-kernel J has an order at least 3 and $|J| \geq 2$. If G is totally disconnected then $V(G)$ is a (2-d)-kernel. In this paper only connected graphs will be studied. Necessary and sufficient conditions for the existence of (2-d)-kernels in graphs were given in [13].

We shall prove that the problem of the existence of (2-d)-kernels is $\mathcal{NP}$-complete for general graphs.

Theorem 1.1. *(2-d)-kernel is $\mathcal{NP}$-complete.*

Proof. Let G be a graph. Given a subset $K \subseteq V(G)$, it can be verified in polynomial time whether K is a (2-d)-kernel. Hence (2-d)-kernel is in $\mathcal{NP}$.

In order to prove $\mathcal{NP}$-hardness (and hence $\mathcal{NP}$-completeness), we reduce an instance G of the well-know $\mathcal{NP}$-complete problem 3-coloring to an instance H of (2-d)-kernel such that G is 3-colorable if and only if H has a (2-d)-kernel, and the encoding length of H is polynomially bounded in terms of the encoding length of G.

Let G be an instance of 3-coloring. Let us assume that G is connected. We construct H as follows. For every vertex u of G we create vertices $x_u, x'_u, y_u, y'_u, z_u, z'_u, w_u, w'_u$ and edges such that $X_u = \{x_u, x'_u\}$, $Y_u = \{y_u, y'_u\}$, $Z_u = \{z_u, z'_u\}$ are the parts of a complete 3-partite graph, and w_u is adjacent to the rest of the vertices, as shown in Figure 1.

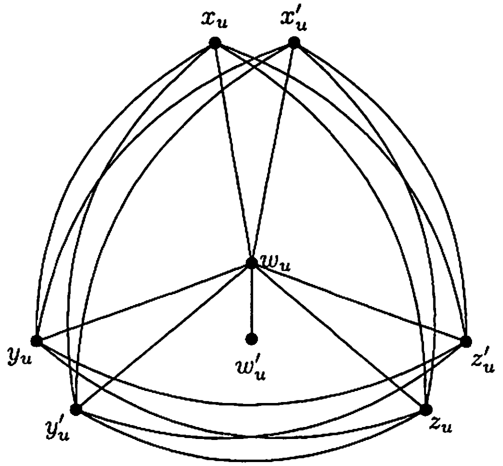

Figure 1: Gadget for every vertex u.

For every edge $uv \in E(G)$ we will add all possible edges between X_u, Y_u, Z_u and X_v, Y_v, Z_v, respectively, as shown in Figure 2.

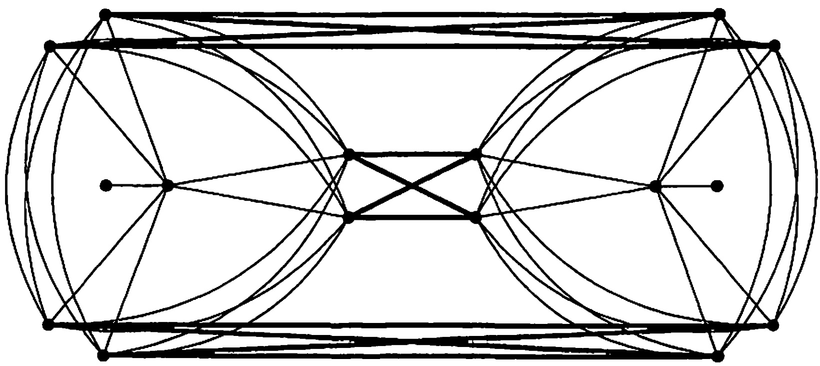

Figure 2: Gadget for every edge.

Clearly, $|V(H)| = 8|V(G)|$ and $|E(H)| = 19|V(G)| + 12|E(G)|$. It is direct to observe that, for every $u \in V(G)$, w'_u must belong to every (2-d)-kernel of H, if any exists. We also claim the following statements to hold.

Claim 1. *Let $u \in V(G)$ be an arbitrary vertex. If K is a (2-d)-kernel of H, then*

- $x_u \in K$ *if and only if* $x'_u \in K$,
- $y_u \in K$ *if and only if* $y'_u \in K$,
- $z_u \in K$ *if and only if* $z'_u \in K$.

Proof of Claim 1. Observe that $N(x_u) = N(x'_u)$. If $x_u \in K$, then $N(x_u) \cap K = \varnothing = N(x'_u) \cap K$. Thus, $x'_u \in K$. An analogous argument shows the remaining implication. □

Claim 2. *Suppose that H has a (2-d)-kernel K. For every $u \in V(G)$, then exactly one of the following statements holds:*

- $X_u \subseteq K$,
- $Y_u \subseteq K$,
- $Z_u \subseteq K$.

Proof of Claim 2. Since $w'_u \in K$, we have $w_u \notin K$. But w_u must be 2-dominated by K, hence, $X_u \cap K \neq \varnothing$, or $Y_u \cap K \neq \varnothing$, or $Z_u \cap K \neq \varnothing$. Suppose without loss of generality that $X_u \cap K \neq \varnothing$; it follows from Claim 1 that $X_u \subseteq K$. Recall that $H[X_u \cup Y_u \cup Z_u]$ is a complete 3-partite graph, thus, $Y_u \cap K = \varnothing = Z_u \cap K$. □

Suppose that H has a (2-d)-kernel K. Define $c : V(G) \to \{X, Y, Z\}$ to be the function such that $c(u) = C$ if and only if $C_u \subseteq K$. It follows from Claim 2 that c is well defined. The independence of K and the construction of H imply that, if $uv \in E(G)$, then $c(u) \neq c(v)$. Thus, c is a 3-coloring of G.

Let $c : V(G) \to \{X, Y, Z\}$ be a 3-coloring of G. Define K to be the set $K = \bigcup_{u \in V(G)} \{w'_u\} \cup \bigcup_{u \in V(G)} c(u)_u$. Since c is a 3-coloring of G, if $uv \in E(G)$, then $c(u) \neq c(v)$, and hence, K is an independent set. Also, for every $u \in V(G)$, the vertices in $(X_u \cup Y_u \cup Z_u \cup \{w_u\}) \setminus c(u)_u$ are 2-dominated by $c(u)_u$. Hence K is a (2-d)-kernel of H.

Hence, G has a 3-coloring if and only if H has a (2-d)-kernel; moreover, there is a bijection between the 3-colorings of G and the (2-d)-kernels of H. Since the encoding length of H is linearly bounded in terms of the encoding length of G, we conclude that (2-d)-kernel is $\mathcal{NP}$-complete. □

2 The number of (2-d)-kernels in graphs

In this section we give some results which concerns the problem of counting of (2-d)-kernels in graphs. Let $\sigma_{(2-d)}(G)$ denote the number of (2-d)-kernels in graphs. For an arbitrary integer $n \geq 1$ we define the sequence of graphs $G_1, \ldots, G_n$ as follows: $G_1 = K_1$ and for $n \geq 2$,
$V(G_n) = \{u_1, \ldots, u_{n-1}, v_1, \ldots, v_{n-1}, x_1, \ldots, x_{n-1}, y_1, \ldots, y_{n-1}\}$,
$E(G_2) = \{x_1y_1, y_1u_1, u_1v_1, v_1x_1\}$ and for $n \geq 3$,
$E(G_n) = E(G_2) \cup \bigcup_{i=2}^{n-1}\{x_iy_i, y_iu_i, u_iv_i, v_ix_i, y_{i-1}x_i\}$.

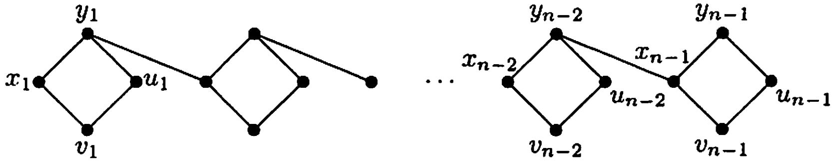

Figure 3: Graph G_n, $n \geq 3$

Theorem 2.1. *For an arbitrary integer n, $n \geq 1$, we have $\sigma_{(2-d)}(G_n) = n$.*

Proof. Let $n \geq 1$ be an arbitrary integer. For $n = 1, 2$ it immediately follows that $\sigma_{(2-d)}(G_1) = 1$, $\sigma_{(2-d)}(G_2) = 2$. Let $n \geq 3$. The definition of the graph G_i implies that for each $3 \leq i \leq n$ the graph G_i has exactly $i-1$ cycles C_4 as induced subgraphs. Let $V(C_4^i) = \{x_i, y_i, u_i, v_i\}$, $E(C_4^i) = \{x_iy_i, y_iu_i, u_iv_i, v_ix_i\}$.

To calculate the number of (2-d)-kernels in the graph G_n we define the family $\mathcal{F}(G_n)$ of (2-d)-kernels i.e $\mathcal{F}(G_n) = \{J \subseteq V(G_n) : J$ is a (2-d)-kernel of $G_n\}$. We consider two subfamilies $\mathcal{F}_x(G_n) = \{J \in \mathcal{F}(G_n) : x_{n-1} \in J\}$ and $\mathcal{F}_{-x}(G_n) = \{J \in \mathcal{F}(G_n) : x_{n-1} \notin J\}$. Consequently $\sigma_{(2-d)}(G_n) = |\mathcal{F}(G_n)| = |\mathcal{F}_x(G_n)| + |\mathcal{F}_{-x}(G_n)|$. Let $|\mathcal{F}_x(G_n)| = \sigma_x(G_n)$ and $|\mathcal{F}_{-x}(G_n)| = \sigma_{-x}(G_n)$. Assume that J is a (2-d)-kernel of G_n and consider the following cases:

1. $x_{n-1} \in J$. Then $x_i, u_i \in J$ for all $1 \leq i \leq n-2$, otherwise the set J is not independent. This means that $\sigma_x(G_n) = 1$,

2. $x_{n-1} \notin J$. Then $y_{n-1}, v_{n-1} \in J$. If $x_{n-2} \notin J$ then $\{J \setminus \{y_{n-1}, v_{n-1}\}\} = \mathcal{F}_{-x}(G_{n-1})$. If $x_{n-2} \in J$ then $\{J \setminus \{y_{n-1}, v_{n-1}\}\} = \mathcal{F}_x(G_{n-1})$

Therefore the number $\sigma_{(2-d)}(G_n)$ of all (2-d)-kernels of graph G_n is given by the relation $\sigma_{(2-d)}(G_n) = \sigma_{(2-d)}(G_{n-1}) + 1$.

Thus the theorem is proved. □

Corollary 2.2. *Let $n \geq 2$ be a fixed integer. Then for an arbitrary $m \geq 4n-4$ there exists a graph G of order m such that $\sigma_{(2-d)}(G) = n$.*

Proof. Let $n \geq 2$ be a fixed integer. We will define the graph G of order $m \geq 4n-4$ such that $\sigma_{(2-d)}(G) = n$. If $m = 4n-4$ then $G = G_n$ for all $n \geq 2$. Let $m > 4n-4$ and $m-4n+4 = p$, $p \geq 1$. We construct a graph G from the graph G_n as follows. Let $1 \leq i \leq n-1$ be a fixed 4-cycle of G_n induced by the set of vertices $\{x_i, y_i, u_i, v_i\}$. Then $V(G) = V(G_n) \cup \{t, t_1, \ldots, t_{p-1}\}$ and $E(G) = E(G_n) \cup \{tx_i, ty_i, tu_i, tv_i, tt_1, \ldots, tt_{p-1}\}$. Clearly $|V(G)| = m$. Let J be an arbitrary (2-d)-kernel of G. It is obvious that $t \notin J$, otherwise vertices u_i, v_i are not 2-dominated by J. Moreover $\{t_1, \ldots, t_{p-1}\} \subset J$. This implies that $\sigma_{(2-d)}(G) = n$. □

For a large n the graph G_n has a large number of vertices. For this reason it is interesting to find other sequences of graphs which realize a fixed number of (2-d)-kernels.

In various counting problems the solution is given by the Fibonacci numbers F_n. They are defined by the linear recurrence equation $F_n = F_{n-1} + F_{n-2}$, for $n \geq 2$ with initial terms $F_0 = F_1 = 1$. The Lucas numbers L_n are the cyclic version of the Fibonacci numbers and they are defined by $L_n = L_{n-1} + L_{n-2}$, for $n \geq 2$ with $L_0 = 2$ and $L_1 = 1$. Actually, Fibonacci numbers and the like are studied, also in graphs and different combinatorial problems, for example in the context of the Merrifield-Simmons index, Hosoya index, number partitions and others [3, 6, 9, 10, 14].

For an arbitrary integer $n \geq 1$ we define the sequence of graphs $H_1, \ldots, H_n$ as follows: $H_1 = K_1$ and for $n \geq 2$,
$V(H_n) = \{u_1, \ldots, u_{n-1}, v_1, \ldots, v_{n-1}, x_1, \ldots, x_{n-1}, y_1, \ldots, y_{n-1}\}$,
$E(H_n) = \{x_1y_1, y_1u_1, u_1v_1, v_1x_1\} \cup \bigcup_{i=2}^{n-1} \{x_iy_i, y_iu_i, u_iv_i, v_ix_i, x_iu_{i-1}\}$

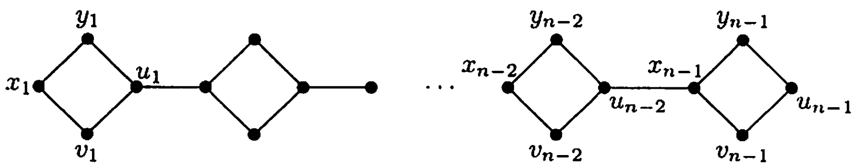

Figure 4: Graph H_n, $n \geq 2$

Theorem 2.3. *For an arbitrary integer n, $n \geq 1$, we have $\sigma_{(2-d)}(H_n) = F_n$.*

Proof. Let $n \geq 1$ be an arbitrary integer. Then for $H_1 = K_1$ and $H_2 = C_4$ it immediately follows that $\sigma_{(2-d)}(H_1) = 1 = F_1$ and $\sigma_{(2-d)}(H_2) = 2 = F_2$. Let $n \geq 3$.

The definition of the graph H_i implies that for each $3 \leq i \leq n$ the graph H_i has exactly $i-1$ cycles C_4 as induced subgraphs. Let $V(C_4^i) = \{x_i, y_i, u_i, v_i\}$ and $E(C_4^i) = \{x_iy_i, y_iu_i, u_iv_i, v_ix_i\}$. Then $\bigcup_{i=1}^{n-1} V(C_4^i) = V(H_n)$ and the definition of H_n immediately implies labeling of vertices of H_n.

To calculate the number of (2-d)-kernels in the graph H_n we define the family $\mathcal{F}(H_n)$ of (2-d)-kernels i.e $\mathcal{F}(H_n) = \{J \subseteq V(H_n) : J$ is a (2-d)-kernel of $H_n\}$.

Let $\mathcal{F}_x(H_n) = \{J \in \mathcal{F}(H_n) : x_{n-1} \in J\}$ and $\mathcal{F}_{-x}(H_n) = \{J \in \mathcal{F}(H_n) : x_{n-1} \notin J\}$. Moreover let $\sigma_{(2-d)}(H_n) = |\mathcal{F}(H_n)| = |\mathcal{F}_x(H_n)| + |\mathcal{F}_{-x}(H_n)|$. Let $|\mathcal{F}_x(H_n)| = \sigma_x(H_n)$ and $|\mathcal{F}_{-x}(H_n)| = \sigma_{-x}(H_n)$. Assume that J is a (2-d)-kernel of H_n and consider the following cases:

1. $x_{n-1} \notin J$ and $x_{n-2} \in J$. Then $y_{n-1}, v_{n-1} \in J$ and this means that $J \setminus \{x_{n-2}, u_{n-2}, v_{n-1}, y_{n-1}\} \in \mathcal{F}(H_{n-2})$ and further $\{J \setminus \{x_{n-2}, u_{n-2}, v_{n-1}, y_{n-1}\}\} = \mathcal{F}(H_{n-2})$,

2. $x_{n-1}, x_{n-2} \in J$. Then $J \setminus \{x_{n-1}\} \in \mathcal{F}_x(H_{n-1})$ and further $\{J \setminus \{x_{n-1}\}\} = \mathcal{F}_x(H_{n-1})$,

3. $x_{n-1} \in J$ and $x_{n-2} \notin J$. Then $J \setminus \{x_{n-1}\} \in \mathcal{F}_{-x}(H_{n-1})$ and further $\{J \setminus \{x_{n-1}\}\} = \mathcal{F}_{-x}(H_{n-1})$.

Therefore the number $\sigma_{(2-d)}(H_n)$ of all (2-d)-kernels of graph H_n is given by the relation $\sigma_{(2-d)}(H_n) = \sigma_x(H_{n-1}) + \sigma_{-x}(H_{n-1}) + \sigma_{(2-d)}(H_{n-2})$ and finally $\sigma_{(2-d)}(H_n) = \sigma_{(2-d)}(H_{n-1}) + \sigma_{(2-d)}(H_{n-2})$, for $n \geq 3$. By the initial conditions it immediately follows that $\sigma_{(2-d)}(H_n) = F_n$. □

Corollary 2.4. *Let $n \geq 2$ be a fixed integer. Then for an arbitrary $m \geq 4n-4$ there exists a graph H of order m such that $\sigma_{(2-d)}(H) = F_n$.*

Proof. Starting from the graph H_n, by the same construction as in Corollary 2.2, for a 4-cycle of H_n, we obtain the graph H such that $\sigma_{(2-d)}(H) = F_n$. □

Corollary 2.5. *For an arbitrary integer n, $n \geq 1$, we have $\sigma_{(2-d)}(G_{F_n}) = \sigma_{(2-d)}(H_n)$*

Now we define the sequence of graphs $H_1^*, \ldots, H_n^*$ as follows: $H_1^* = C_4$, $H_2^* = K_4 - e$, where e is an arbitrary edge of the graph K_4, $H_3^* = H_3$ and for $4 \leq i \leq n$ the graph H_i^* is constructed from graph H_i by adding the edge $x_1 u_{i-1}$.

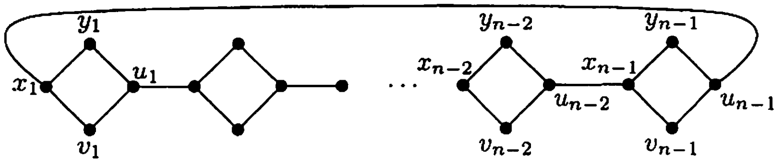

Figure 5: Graph H_n^*, $n \geq 4$

Theorem 2.6. *For an arbitrary integer n, $n \geq 1$, we have $\sigma_{(2-d)}(H_n^*) = L_n$.*

Proof. Let $n \geq 1$ be an arbitrary integer. Then for H_i^*, $i = 1, 2, 3$ it immediately follows that $\sigma_{(2-d)}(H_1^*) = 2 = L_1$, $\sigma_{(2-d)}(H_2^*) = 1 = L_2$ and $\sigma_{(2-d)}(H_3^*) = 3 = L_3$.

Let $\mathcal{F}(H_n) = \{J \subseteq V(H_n) : J \text{ is a (2-d)-kernel of } H_n\}$ and $\mathcal{L}(H_n^*) = \{J \subseteq V(H_n^*) : J \text{ is (2-d)-kernel of } H_n^*\}$. We consider the following cases:

1. $y_i \in J$. This means that $J \setminus \{y_i, v_i\} \in \mathcal{F}$ (H_{n-1}) and further $\{J \setminus \{y_i, v_i\}\} = \mathcal{F}(H_{n-1})$,

2. $y_i \notin J$. Then $x_i, u_i \in J$ and $y_{i-1}, v_{i-1,}, y_{i+1}, v_{i+1} \in J$. This means that $J \setminus \{x_i, u_i, y_{i-1}, v_{i-1,}, y_{i+1}, v_{i+1}\} \in \mathcal{F}(H_{n-3})$ and further $\{J \setminus \{x_i, u_i, y_{i-1}, v_{i-1,}, y_{i+1}, v_{i+1}\}\} = \mathcal{F}(H_{n-3})$.

Therefore the number $\sigma_{(2-d)}(H_n^*)$ of all (2-d)-kernels of graph H_n^* is given by the relation $\sigma_{(2-d)}(H_n^*) = \sigma_{(2-d)}(H_{n-1}) + \sigma_{(2-d)}(H_{n-3})$. By Theorem 2.3 it follows that $\sigma_{(2-d)}(H_n^*) = L_n$. □

Proving in the same way as in Corollary 2.2 for the graph H^* we have:

Corollary 2.7. *Let $n \geq 2$ be a fixed integer. Then for an arbitrary $m \geq 4n - 4$ there exists a graph H^* of order m such that $\sigma_{(2-d)}(H^*) = L_n$.*

Corollary 2.8. *For an arbitrary integer n, $n \geq 1$, we have $\sigma_{(2-d)}(G_{L_n}) = \sigma_{(2-d)}(H_n^*)$*

For an arbitrary integer $n \geq 0$ we define the sequence of graphs $R_0, \ldots, R_n$ as follows: $R_0 = P_3$ with $V(P_3) = \{x_1, x_2, x_3\}$ and for $n \geq 1$
$V(R_n) = \{x_1, x_2, x_3, y_1, \ldots, y_n, u_1, \ldots, u_n, v_1, \ldots v_n, t_1, \ldots, t_n\}$,
$E(R_n) = \{x_1x_2, x_2x_3\} \cup \bigcup_{i=1}^{n} \{x_2y_i, y_iu_i, u_it_i, t_iv_i, v_iy_i\}$.

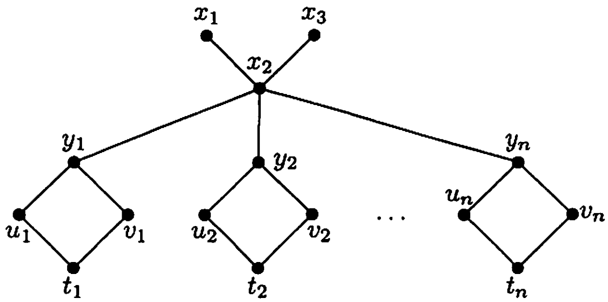

Figure 6: Graph R_n, $n \geq 1$

Theorem 2.9. *For an arbitrary integer n, $n \geq 0$, we have $\sigma_{(2-d)}(R_n) = 2^n$.*

Proof. We use induction for our proof. It is obvious that $\sigma_{(2-d)}(R_0) = \sigma_{(2-d)}(P_3) = 1 = 2^0$. Let $n \geq 1$ and assume that for an arbitrary R_n we have $\sigma_{(2-d)}(R_n) = 2^n$. We shall show that $\sigma_{(2-d)}(R_{n+1}) = 2^{n+1}$. By the induction hypothesis $\sigma_{(2-d)}(R_n) = 2^n$. Let $\mathcal{J}(R_n) = \{J_1, \ldots, J_{2^n}\}$ be the family of all (2-d)-kernels of R_n. The definition of the graph R_n immediately gives that $\{x_1, x_3\} \subset J_i$, for $i = 1, \ldots, 2^n$. Moreover for every (2-d)-kernel J of the graph R_{n+1} there exists $1 \leq i \leq 2^n$ such that $J_i \subset J$. Since the graph C_4 has exactly two (2-d)-kernels it is obvious that $\sigma_{(2-d)}(R_{n+1}) = \sigma_{(2-d)}(R_n) \cdot 2 = 2^{n+1}$, by induction hypothesis.

Thus the theorem is proved. □

Corollary 2.10. *Let $n \geq 1$ be a fixed integer. Then for an arbitrary $m \geq 4n + 3$ there exists a graph R of order m such that $\sigma_{(2-d)}(R) = 2^n$.*

Proof. If $m = 4n + 3$ then $R = R_n$. Let $m > 4n + 3$ and $m - 4n - 3 = p$. Then it is suffices to add p leaves to the vertex x_2 in the graph R_n. In other words $V(R) = V(R_n) \cup \{t_1, ..., t_p\}$ and $E(R) = E(R_n) \cup \bigcup_{i=1}^{p} x_2 t_i$. Then the result is obvious. □

Corollary 2.11. *For an arbitrary integer n, $n \geq 0$, we have $\sigma_{(2-d)}(G_{2^n}) = \sigma_{(2-d)}(R_n)$*

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