The result on (3,1)*-choosability of graphs of nonnegative characteristic without 4-cycles and intersecting triangles[1]

Haihui Zhang[2]

(*School of Mathematical Science, Huaiyin Normal University, 111 Changjiang West Road, Huaian, Jiangsu, 223300, China*)

Abstract

A graph G is called $(k,d)^*$-choosable if for every list assignment L satisfying $|L(v)| \geq k$ for all $v \in V(G)$, there is an L-coloring of G such that each vertex of G has at most d neighbors colored with the same color as itself. In this paper, it is proved that every graph of nonnegative characteristic without 4-cycles and intersecting triangles is $(3,1)^*$-choosable.

Keywords: Graph, Defective choosability, Characteristic, Intersecting cycles

AMS 2000 Subject Classifications: 05C15, 05C78;

1 Introduction

Graphs considered in this paper are finite, simple and undirected. Let $G = (V, E, F)$ be a graph, where V, E and F denote the set of vertices, edges and faces of G, respectively. For the used but undefined terminology and notation, we refer the reader to the book by Bondy and Murrty [1].

A proper k-coloring of G is a mapping ϕ from $V(G)$ to a color set $1, 2, \cdots, k$ such that $\phi(x) \neq \phi(y)$ for any adjacent vertices x and y. A graph is k-colorable if it has a proper k-coloring. Cowen, Cowen, and Woodall [6] considered defective colorings of graphs. A graph G is said to be d-improper k-colorable, or simply, $(k,d)^*$-colorable, if the vertices of G can be colored with k colors in such a way that each vertex has at most d neighbors receiving the same color as itself. Obviously, a $(k,0)^*$-coloring is an ordinary proper k-coloring.

A list assignment of G is a function L that assigns a list $L(v)$ of colors to each vertex $v \in V(G)$. An L-coloring with impropriety d for integer $d \geq 0$,

[1]Research was supported by the NSFC Tianyuan foundation (Grant No. 11226285) and the Natural Science Foundation of Jiangsu (Grant No. BK20140451).

[2]E-mail: hhzh@hytc.edu.cn

or simply $(L,d)^*$-coloring, is a mapping ϕ that assigns a color $\phi(v) \in L(v)$ to each vertex $v \in V(G)$ such that v has at most d neighbors colored with $\phi(v)$. For integers $m \geq d \geq 0$, a graph is called $(m,d)^*$-choosable, if G admits an $(L,d)^*$-coloring for every list assignment L with $|L(v)| = m$ for all $v \in V(G)$. An $(m,0)^*$-choosable graph is simply called m-choosable.

The notion of list improper coloring was introduced independently by Škrekovski [12] and Eaton and Hull [9]. They proved that every planar graph is $(3,2)^*$-choosable and every outerplanar graph is $(2,2)^*$-choosable. Škrekovski proved in [13] that every planar graph without 3-cycles is $(3,1)^*$-choosable, and in [14] that every planar graph G is $(2,1)^*$-choosable if its girth $g(G) \geq 9$, $(2,2)^*$-choosable if $g(G) \geq 7$, $(2,3)^*$-choosable if $g(G) \geq 6$, and $(2,d)^*$-choosable if $g(G) \geq 5$ and $d \geq 4$. Lih et al. [10] proved that every planar graph without 4-cycles and l-cycles for some $l \in \{5,6,7\}$ is $(3,1)^*$-choosable. Dong and Xu [8] shown that it is also true for some $l \in \{8,9\}$. Cushing and Kierstead [7] constructively proved that every planar graph is $(4,1)^*$-choosable which perfectly solved the last remaining question left open in [9, 12]. In [5], Chen and Raspaud proved that every planar graph without 4-cycles adjacent to 3- and 4-cycles is $(3,1)^*$-choosable, as a corollary, every planar graph graph without 4-cycles is $(3,1)^*$-choosable. Wang and Xu proved every planar graph without cycles of length 4 is $(3,1)^*$-choosable in [16].

For other classes of graphs, Zhang [19] proved that every graph G embeddable on the torus without 5- and 6-cycles is $(3,1)^*$-choosable. Xu and Zhang [18] proved that every toroidal graph without adjacent triangles is $(4,1)^*$-choosable. Chen *et al.* [4] proved that every graph embeddable in a surface of nonnegative characteristic without a 5-cycle with a chord or a 6-cycle with a chord is $(4,1)^*$-choosable, and every graph embeddable in a surface of nonnegative characteristic without chordal k-cycles for all $k \in \{4,5,6\}$ is $(3,1)^*$-choosable.

Let $\triangle_1$ and $\triangle_2$ be two triangles of a graph G. Define the distance between $\triangle_1$ and $\triangle_2$ be the length of a shortest path connecting a vertex of $\triangle_1$ to a vertex of $\triangle_2$. Let $d(\triangle)$ denote the least distance between two triangles. Lam et al [3] showed that every planar graph G with $d(\triangle) \geq 2$ is 4-choosable. Xu [17] proved that every plane graph in which no two triangles share a common vertex, that is $d(\triangle) \geq 1$, is 4-choosable. Wang and Lih also independently proved this result in [15]. The condition $d(\triangle) \geq$

1 is essential because there exists non-4-choosable planar graphs G with $d(\triangle) = 0$ (see [11]).

In [16], Wang and Xu conjectured every planar graph without intersecting triangles is $(3,1)^*$-choosable. We consider this problem with a relaxed condition. In fact, this paper investigates improper choosability for graphs of nonnegative characteristic without 4-cycles and intersecting triangles. The Euclidean plane, the projective plane, the torus, and the Klein bottle are all the sufaces of nonnegative characteristic.

Let $\mathcal{G}$ denote the family of graphs with nonnegative characteristic containing no 4-cycles and intersecting triangles. The main result is to show that every graph in $\mathcal{G}$ is $(3,1)^*$-choosable. In order to prove the main theorem, we use the technique of discharging to obtain several forbidden configurations for the graphs in $\mathcal{G}$ and state as a theorem below.

Theorem 1 *For every graph $G \in \mathcal{G}$, one of the following conditions holds:*

(1) $\delta(G) < 3$.

(2) G contains two adjacent 3^--vertices.

(3) G contains a $(4^-, 4^-, 4^-)$-face.

(4) G contains an even $(3^-, 4^-, \cdots\cdots, 3^-, 4^-)$-$2n$-face, where $n \geq 2$.

As a consequence of Theorem 1, we derive the following Theorem 2.

Theorem 2 *Every graph of nonnegative characteristic without 4-cycles and intersecting triangles is $(3,1)^*$-choosable.*

2 Notation

We use $N_G(v)$ and $d_G(v)$ to denote the set and number of vertices adjacent to a vertex v, respectively, and use $\delta(G)$ to denote the minimum degree of G. A face of an embedded graph is said to be incident with all edges and vertices on its boundary. Two faces are adjacent if they share a common edge. The degree of a face f of G, denoted also by $d_G(f)$, is the number of edges incident with it, where each cut-edge is counted twice. When no confusion may occur, we write $N(v), d(v), d(f)$ instead of $N_G(v), d_G(v), d_G(f)$. A k-vertex (or k-face) is a vertex (or face) of degree k, a k^--vertex (or k^--face) is a vertex (or face) of degree at most k, and a k^+-vertex (or k^+-face) is a vertex (or face) of degree at least k. For $f \in F(G)$, we write

$f = [u_1 u_2 \cdots u_n]$ if u_1, u_2, $\cdots$, u_n are the vertices clockwisely lying on the boundary of f. An n-face $[u_1 u_2 u_3 \cdots u_n]$ is called an $(m_1, m_2, m_3, \cdots, m_n)$-face if $d(u_i) = m_i$ for $i = 1, 2, 3, \cdots, n$. A k-cycle is a cycle with k edges. Two cycles are adjacent if they share at least one common edge. Two cycles or faces are intersecting if they share at least one common (boundary) vertex.

3 Proof of Theorem 1

In the proof of Theorem 1, we use the technique of discharging. In the beginning, each vertex v is assigned a charge $\omega(v) = (k-1) \cdot d_G(v) - 2k$ if $v \in V(G)$, and $\omega(f) = d_G(f) - 2k$ if $f \in F(G)$. Using the Euler-Poincare formula $|V(G)| - |E(G)| + |F(G)| \geq 0$ and the well-known relation $\sum_{v \in V(G)} d(v) = \sum_{f \in F(G)} d(f) = 2|E(G)|$, we have

$$\sum_{v \in V(G)} \{(k-1) \cdot d_G(v) - 2k\} + \sum_{f \in F(G)} \{d_G(f) - 2k\} = -2k. \qquad (1)$$

By the discharging rules stated in the following, we will redistribute the charges for the vertices and faces so that the total sum of the weights is kept constant while the transferring is in progress. However, once the transferring is finished, we get the new charges are nonnegative, moreover, there exists some $x \in V(G) \cup F(G)$ such that $w'(x) > 0$, then

$$0 < \sum_{x \in V(G) \bigcup F(G)} w'(x) = \sum_{x \in V(G) \bigcup F(G)} w(x) = -2k. \qquad (2)$$

This contradiction completes the proof of Theorem 1.

Let $k = 3$ in formula (1).

Assume to the contrary that the theorem does not hold. Let G be such a connected graph in $\mathcal{G}$. Let w be a weight on $V(G) \cup F(G)$ by defining $w(v) = 2d(v) - 6$ if $v \in V(G)$, and $w(f) = d(f) - 6$ if $f \in F(G)$ as above. For two elements x and y of $V(G) \cup F(G)$, we use $\tau(x \to y)$ to denote the charge transferred from x to y.

By the choice of G, we have:

(O1) $\delta(G) \geq 3$;

(O2) Every 3-vertex is adjacent to only 4^+-vertices;
(O3) G contains no $(4^-, 4^-, 4^-)$-face;
(O4) G contains no intersecting triangles;
(O5) G contains no even $(3, 4, \cdots\cdots, 3, 4)$-$2n$-face.

Let $m_i(v)$ be the number of i-faces incident with v and $n_j(v)$ be the number of j-vertices adjacent to v. Let $n_i(f)$ denote the number of i-vertices incident with f. We have:

Claim 1 *For each vertex $v \in V(G)$, $|m_3(v)| \leq 1$.*

Claim 2 *For each face $f \in F(G)$, $n_3(f) \leq \lfloor \frac{d(f)}{2} \rfloor$.*

Let v be a d-vertex and f be an l-face incident with v. The new charge function $w'(x)$ is obtained by the discharging rules given below:
(R1) For $d = 4$, $\tau(v \to f) = 1$ for $l = 3$.
(R2) For $d \geq 4$, $\tau(v \to f) = \frac{1}{3}$ if $l = 5$.
(R3) For $d \geq 5$, $\tau(v \to f) = 2$ for $l = 3$.

We now verify that $w'(x) \geq 0$ for any $x \in V(G) \cup F(G)$.

Let f be an h-face of G. The proof is divided into three cases according to the value of h.

Case 1. $h \geq 6$. Then $w'(f) = w(f) \geq 0$.

Case 2. $h = 5$. Then $n_3(f) \leq 2$ by Claim 2. So $n_{4^+}(f) \geq 3$, then $w'(f) \geq w(f) + 3 \cdot \frac{1}{3} = 0$ by (R2).

Case 4. $h = 3$. Then $w(f) = 3 - 6 = -3$. We write $f = [v_1 v_2 v_3]$. By Claim 2, we have $n_3(f) \leq 1$.

Subcase 4.1. If $n_3(f) = 0$, then $w'(f) \geq w(f) + 3 \cdot 1 = 0$ by (R1) and (R3).

Subcase 4.2. If $n_3(f) = 1$, then there must have a 5^+-vertex incident with f by (O2) and (O3). If f is a $(3, 4, 5^+)$-face, then $w'(f) \geq w(f) + 1 \cdot 1 + 1 \cdot 2 = 0$ by (R1) and (R3). If f is a $(3, 5^+, 5^+)$-face, then

$$w'(f) \geq w(f) + 2 \cdot 2 = 1 > 0. \tag{3}$$

by (R1), (R3).

Let v be a d-vertex of G.

If $d = 3$, $\omega'(v) = \omega(v) = 0$.

If $d = 4$, then by (R1) and (R2), v just transfers to the incident 3- or 5-faces. If no 3-face is incident to v, then

$$\omega'(v) \geq \omega(v) - \frac{1}{3} \cdot 4 > 0. \tag{4}$$

In addition, by Claim 2, $m_3(v) = 1$, then $\omega'(v) \geq \omega(v) - 1 - \frac{1}{3} \cdot 3 = 0$.

If $d \geq 5$, then

$$\omega'(v) \geq \omega(v) - 2 - \frac{1}{3} \cdot (d(v) - 1) = \frac{5 \cdot d(v) - 23}{3} > 0. \tag{5}$$

by (R1), (R2) and by Claim 1.

Now, we get that $\omega'(x) \geq 0$ for each $x \in V(G) \cup F(G)$. It follows that

$$0 \leq \sum\{w'(x) \mid x \in V(G) \cup F(G)\} = \sum\{w(x) \mid x \in V(G) \cup F(G)\} \leq 0. \tag{6}$$

If $\sum_{x \in V(G) \cup F(G)} w'(x) > 0$, we are done. Assume that $\sum_{x \in V(G) \cup F(G)} w'(x) = 0$, so we have no 5^+-vertices and 7^+-faces by the above proof.

Claim 3 *G contains no 3-faces.*

Proof. Let G be a graph in $\mathcal{G}$. We have G contains no 5^+-vertices by equation 5, so the vertices incident with any 3-faces must be 4^--vertices, we get it by G contains no $(4^-, 4^-, 4^-)$-faces. ▮

Claim 4 *G contains no 4-vertices.*

Proof. Let v be a 4-vertex in $V(G)$, then $m_3(v) = 0$ by Claim 3, we can easily get it by equation 4. ▮

By Claim 4, we have $d(v) = 3$ for every vertex in $V(G)$, this contradiction completes the proof of Theorem 1. ■

4 Proof of Theorem 2

Suppose Theorem 2 is false. Let $G = (V, E)$ be a counterexample to Theorem 2 with the smallest $|V| + |E|$. Clearly G is connected. Embedding G into the surface with nonnegative characteristic. Let $L = \{L(v) \| L(v)| \geq 3$ for all $v \in V(G)\}$ be a list assignment such that G has no L-coloring in the sense that every vertex has at most one neighbor colored the same color as itself. By the minimality of G. Lemma 1 is straightforward.

Lemma 1 *(See [10])*
(1) $\delta(G) \geq 3$;
(2) G has no adjacent 3*-vertices;*
(3) There is no $(3,4,4)$*-face.*

If G contains a $(4^-,4^-,4^-)$-face, then it is a $(4,4,4)$-face by Lemma 1. This configuration $(4,4,4)$-face has been proved to be reducible by Chen and Raspaud in [5], we omit it here.

Let G contains an even $(3,4,\cdots,3,4)$-$2n$-face f, where $n \geq 2$. Let L be a 3-list assignment of G, and $f = [v_1v_2\cdots v_{2k}]$ be such a $(3,4,\cdots,3,4)$-$2n$-face with the degree condition. By the assumption, there exists an $(L,1)^*$-coloring ϕ of $G - V(f)$. Let $L'(v)$ be the color list of v after removing the colors used by the neighbors of v used in ϕ. Consider the coloring ϕ' of all the vertices in $V(f)$, we can easily extend the coloring ϕ to the graph G by colored the vertices v_{2k} firstly with the remaining one color in its list for $k = 1,\cdots,n$, and the vertices v_j with the color from $L(v_j)\backslash\phi'(v_{j-1})$ for $j = 3,5,7,\cdots,2k-1$ in order, finally color v_1 by one color of in $L'(v_1)\backslash\phi'(v_{2k})$.

References

[1] J. A. Bondy and U. S. R. Murty, Graph Theory, *Springer, New York*, 2008.

[2] P. C. B. Lam, B. Xu and J. Liu, The 4-choosability of plane graphs without 4-cycles, *J. Combin. Theory Ser (B)*, **76**(1999)117-126.

[3] P. C. B. Lam, W. C. Shiu, B. Xu, On structure of some plane graphs with application to choosability, *J. Combin. Theory Ser (B)*, **82**(2001)285-297.

[4] Y. Chen, W. Zhu and W. Wang, Improper choosability of graphs of nonnegative characteristic, *Comput. Math. Appl.*, **56**(2008)2073-2078.

[5] M. Chen, A. Raspaud, On $(3,1)^*$-choosability of planar graphs without adjacent short cycles, *Discrete Appl. Math.* , **162**(2014)159-166.

[6] L. Cowen, R. Cowen, D. Woodall, Defective colorings of graphs in surfaces: partitions into subgraphs of bounded valency, *J. Graph Theory*, **10**(1986)187-195.

[7] W. Cushing and H. A. Kierstead, Planar graphs are 1-relaxed, 4-choosable, *European J. Combin.*, **31**(2009)1385-1397.

[8] W. Dong and B. Xu, A note on list improper coloring of plane graphs, *Discrete Appl. Math.* **28**(2009)433-436.

[9] N. Eaton and T. Hull, Defective list colorings of planar graphs, *Bull. Inst. Comb. Appl.*, **25**(1999)79-87.

[10] K. Lih, Z. Song, W. Wang and K. Zhang, A note on list improper coloring planar graphs, *Appl. Math. Lett.*, **14**(2001)269-273.

[11] M. Voigt. List colorings of planar graphs. *Discrete Math.*, **120**(1993)215-219.

[12] R. Škrekovski, List improper colorings of planar graphs, *Comb. Prob. Comp.*, **8**(1999)293-299.

[13] R. Škrekovski, A Grötzsch-type theorem for list colorings with impropriety one, *Comb. Prob. Comp.*, **8**(1999)493-507.

[14] R. Škrekovski, list improper colorings of planar graphs with prescribed girth, *Discrete Math.*, **214**(2000)221-233.

[15] W. Wang, K. Lih, choosability and edge choosability of planar graphs without intersecting triangles, *Siam J. Discrete Math.* **15**(4)(2002)538-545.

[16] Y. Wang, L. Xu, Improper choosability of planar graphs without 4-cycle, *Siam J. Discrete Math.* **27**(4)(2013)2029-2037.

[17] B. Xu. (4m,m)-Choosability of Plane Graphs. *Journal of Systems Science and Complexity*, 14(2001)174-178.

[18] B. Xu and H. Zhang, Every torodial graph without adjacent triangles is $(4,1)^*$-choosable, *Discrete Appl. Math.*, **155**(2007)74-78.

[19] L. Zhang, A $(3,1)^*$-choosable theorem on toroidal graphs, *Discrete Appl. Math.*, **160**(2012)332-338.