On α - Incidence Energy and α - Distance Energy of a Graph

Rao Li
Dept. of mathematical sciences
University of South Carolina Aiken
Aiken, SC 29801
Email: raol@usca.edu

Abstract

The α - incidence energy of a graph is defined as the sum of αth powers of the signless Laplacian eigenvalues of the graph, where α is a real number such that $\alpha \neq 0$ and $\alpha \neq 1$. The α - distance energy of a graph is defined as the sum of αth powers of the absolute values of the eigenvalues of the distance matrix of the graph, where α is a real number such that $\alpha \neq 0$. In this note, we present some bounds for the α - incidence energy of a graph. We also present some bounds for the α - distance energy of a tree.

Keywords : α – *Incidence Energy*, α – *Distance Energy*.

We consider only finite undirected graphs without loops or multiple edges. Notation and terminology not defined here follow that in [1]. For a square matrix M, we use $Det(M)$ and M^t to denote its determinant and transpose respectively. Let G be a graph with n vertices and e edges. We assume that the vertices in G are ordered such that $\Delta = d_1 \geq d_2 \geq ... \geq d_n$, where d_i, $1 \leq i \leq n$, is the degree of vertex v_i in G. We define $\Sigma_k(G)$ as $\Sigma_{i=1}^n d_i^k$. For each vertex v_i, $1 \leq i \leq n$, m_i is defined as the sum of degrees of vertices that are adjacent to v_i. Obviously, $\Sigma_{i=1}^n m_i = \Sigma_{i=1}^n d_i^2 = \Sigma_2(G)$. We also let $T_1(G) = 2\sqrt{\frac{\Sigma_{i=1}^n d_i^2}{n}}$ and $T_2(G) = \sqrt{\frac{\Sigma_{i=1}^n (d_i^2+m_i)}{\Sigma_{i=1}^n d_i^2}}$. Let $A(G)$ be the adjacency matrix of G and let $D(G)$ be the diagonal matrix of the degree sequence of G. The eigenvalues $\lambda_1(G) \geq \lambda_2(G) \geq ... \geq \lambda_{n-1} \geq \lambda_n(G) = 0$ of $L(G) := D(G) - A(G)$ of G are called the Laplacian eigenvalues of the graph G. The eigenvalues $\tau_1(G) \geq \tau_2(G) \geq ... \geq \tau_{n-1} \geq \tau_n(G)\ (\geq 0)$ of $Q(G) := D(G) + A(G)$ of G are called the signless Laplacian eigenvalues of the graph G. Notice that $IE(G) := \sum_{i=1}^n (\tau_i(G))^{\frac{1}{2}}$ is an equivalent definition of the incidence energy of the graph G (see [5] and [11]). Motivated by the equivalent definition for the incidence energy of the graph G, we define the α - incidence energy, denoted $IE_\alpha(G)$, of G as $\sum_{i=1}^n (\tau_i(G))^\alpha$, where α is a real number such that $\alpha \neq 0$ and $\alpha \neq 1$. Notice that $\sum_{i=1}^n \tau_i(G) = 2|E(G)|$. So we don't allow $\alpha = 1$ in our definition of $IE_\alpha(G)$. The distance matrix, denoted $D(G)$, of a connected G is the a square matrix of order n such that its (i,j) - entry is d_{ij}, the number of edges of the shortest path between vertices v_i and v_j. The eigenvalues $\nu_1(G) \geq \nu_2(G) \geq ... \geq \nu_{n-1} \geq \nu_n(G)$ of $D(G)$ of G are called the distance eigenvalues of the graph G. Notice that $d_{ii} = 0$, $\sum_{i=1}^n \nu_i = tr(D(G)) = 0$, and $J(G) := \sum_{i=1}^n \nu_i^2 = tr((D(G))^2) = \sum_{i=1}^n \sum_{j=1}^n d_{ij}^2$. In [6], $DE(G) := \sum_{i=1}^n |\nu_i(G)|$ is called the distance energy of the graph G. Motivated by the above definition for $DE(G)$, we define the α - distance energy, denoted $DE_\alpha(G)$, of G as $\sum_{i=1}^n |\nu_i(G)|^\alpha$, where α is a real number such that $\alpha \neq 0$.

Zhou in [8] and [10] obtained several results on the lower or upper bounds for the sum of powers of the Laplacian eigenvalues of a graph. Using the ideas and proof techniques in [8], we obtain some bounds for the α - incidence energy of a graph and

α - distance energy of a tree.

Theorem 1 Let G be a connected graph with $n \geq 3$ vertices, e edges, and maximum degree Δ.

(1) If $\alpha < 0$ or $\alpha > 1$, then

$$(a) \ \ IE_\alpha > (1+\Delta)^\alpha + \frac{(2e-1-\Delta)^\alpha}{(n-1)^{\alpha-1}}$$

and the above inequality cannot become an equality.

$$(b) \ \ IE_\alpha \geq T_1^\alpha + \frac{(2e-T_1)^\alpha}{(n-1)^{\alpha-1}}$$

with equality if and only if G is K_n.

$$(c) \ \ IE_\alpha \geq T_2^\alpha + \frac{(2e-T_2)^\alpha}{(n-1)^{\alpha-1}}$$

with equality if and only if G is K_n.

(2) If $0 < \alpha < 1$, then

$$(a) \ \ IE_\alpha < (1+\Delta)^\alpha + \frac{(2e-1-\Delta)^\alpha}{(n-1)^{\alpha-1}}$$

and the above inequality cannot become an equality.

$$(b) \ \ IE_\alpha \leq T_1^\alpha + \frac{(2e-T_1)^\alpha}{(n-1)^{\alpha-1}}$$

with equality if and only if G is K_n.

$$(c) \ \ IE_\alpha \leq T_2^\alpha + \frac{(2e-T_2)^\alpha}{(n-1)^{\alpha-1}}$$

with equality if and only if G is K_n.

Clearly, (2) in Theorem 1 has the following corollary.

Corollary 1 Let G be a connected graph with $n \geq 3$ vertices, e edges, and maximum degree Δ. Then

$$(a)\ \ IE(G) < \sqrt{1+\Delta} + \sqrt{(2e-1-\Delta)(n-1)}$$

and the above inequality cannot become an equality.

$$(b)\ \ IE(G) \leq \sqrt{T_1} + \sqrt{(n-1)(2e-T_1)}$$

with equality if and only if G is K_n.

$$(c)\ \ IE(G) \leq \sqrt{T_2} + \sqrt{(n-1)(2e-T_2)}$$

with equality if and only if G is K_n.

Notice that (a) and (b) in Corollary 2 are Theorem 3.6 and Theorem 3.5 in [5] respectively and when G is a connected (c) in Corollary 2 is Theorem 3.7 in [5].

Theorem 2 Let T be a tree of $n \geq 2$ vertices. Then

(1) If $\alpha > 1$, then

$$DE_\alpha \geq (\sqrt{J/2}\,)^\alpha (1 + \frac{1}{(n-1)^{\alpha-1}}).$$

with equality if and only $G = K_2$.

If $\alpha < 0$, then

$$DE_\alpha \geq (\sqrt{J(n-1)/n}\,)^\alpha (1 + \frac{1}{(n-1)^{\alpha-1}}).$$

with equality if and only $G = K_2$.

(2) If $0 < \alpha < 1$, then

$$DE_\alpha \leq (\sqrt{J(n-1)/n})^\alpha (1 + \frac{1}{(n-1)^{\alpha-1}}).$$

with equality if and only $G = K_2$.

(3) If $\alpha > 0$, then

$$DE_\alpha \geq (\sqrt{J/2})^\alpha + (n-1)(\frac{(n-1)2^{n-2}}{\sqrt{J/2}})^{\frac{\alpha}{n-1}}.$$

with equality if and only $G = K_2$.

(4) If $n \geq 3$ and $\alpha > 0$, then

$$DE_\alpha > ((n-1)^\alpha + (n-1))2^{\frac{\alpha(n-2)}{2}}.$$

We need the following lemmas to prove Theorem 1 and Theorem 2. Lemma 1 below is from [4] by Grone and Merris.

Lemma 1 Let G be a graph with $n \geq 2$ vertices and $e \geq 1$ edges. Then $\lambda_1(G) \geq \Delta + 1$. If G is connected, the equality holds if and only if $\Delta = n - 1$.

Lemma 2 below is from Lemma 2 in [7].

Lemma 2 Let G be a graph. Then $\lambda_1 \leq \tau_1$, the equality holds if and only if G is a bipartite graph.

Lemma 3 below is Theorem 3.6 in [2].

Lemma 3 Let G be a connected graph with diameter $d(G)$. If $Q(G)$ has exactly k distinct eigenvalues, then $d(G) + 1 \leq k$.

Lemma 4 below is from Theorem 2.1 and Corollary 2.2 in [3].

Lemma 4 Let T be a tree of n vertices with distance eigenvalues $\nu_1 \geq \nu_2 \geq ... \geq \nu_{n-1} \geq \nu_n$. Then $\nu_1 > 0 > \nu_2 \geq ... \geq \nu_{n-1} \geq \nu_n$ and $\nu_1 \nu_2 ... \nu_n = (-1)^{n-1}(n-1)2^{n-2}$.

Lemma 5 below is from Theorem 3 in [9].

Lemma 5 Let T be a tree of $n \geq 3$ vertices with distance eigenvalues $\nu_1 \geq \nu_2 \geq ... \geq \nu_{n-1} \geq \nu_n$. Then $\nu_1 > (n-1)2^{\frac{n-2}{n}}$ and the inequality cannot become an equality.

Lemma 6 below follows from Theorem 3 in [12].

Lemma 6 Let T be a tree of $n \geq 2$ vertices with distance eigenvalues $\nu_1 \geq \nu_2 \geq ... \geq \nu_{n-1} \geq \nu_n$. Then $\nu_1 \leq \sqrt{J(n-1)/n}$ with equality if and only $G = K_2$.

Lemma 7 below follows from Theorem 4 in [12].

Lemma 7 Let T be a tree of $n \geq 2$ vertices with distance eigenvalues $\nu_1 \geq \nu_2 \geq ... \geq \nu_{n-1} \geq \nu_n$. Then $\nu_1 \geq \sqrt{J/2}$ with equality if and only $G = K_2$.

Proof of (1) in Theorem 1. Notice that x^α is concave up when $x > 0$ and $\alpha < 0$ or $\alpha > 1$. Thus

$$(\sum_{i=2}^{n} \frac{1}{n-1}\tau_i)^\alpha \leq \sum_{i=2}^{n} \frac{1}{n-1}\tau_i^\alpha.$$

Hence

$$\sum_{i=2}^{n} \tau_i^\alpha \geq \frac{1}{(n-1)^{\alpha-1}}(\sum_{i=2}^{n} \tau_i)^\alpha$$

with equality holds if and only if $\tau_2 = ... = \tau_n$. Therefore

$$IE_\alpha = \sum_{i=1}^{n} \tau_i^\alpha \geq \tau_1^\alpha + \frac{1}{(n-1)^{\alpha-1}}(\sum_{i=2}^{n} \tau_i)^\alpha = \tau_1^\alpha + \frac{(2e-\tau_1)^\alpha}{(n-1)^{\alpha-1}}.$$

Now consider the function $f(x) = x^\alpha + \frac{(2e-x)^\alpha}{(n-1)^{\alpha-1}}$. It can be easily verified that $f(x)$ is increasing when $x > \frac{2e}{n}$.

From Lemmas 1 and 2, we have $\tau_1 \geq \lambda_1 \geq \Delta + 1 > \Delta \geq \frac{2e}{n}$. Thus

$$IE_\alpha = \sum_{i=1}^{n} \tau_i^\alpha \geq f(\tau_1) \geq f(\lambda_1) \geq f(\Delta + 1)$$

$$= (1+\Delta)^\alpha + \frac{(2e-1-\Delta)^\alpha}{(n-1)^{\alpha-1}}.$$

Suppose that the above inequality becomes an equality. Then $\tau_1 = \lambda_1 = \Delta + 1$ and $\tau_2 = ... = \tau_n$. From Lemmas 1 and 2 again, we have that $\tau_1 = \lambda_1 = n$ and G is a bipartite graph. From $\sum_{i=1}^{n} \tau_i = 2e$, we have that $\tau_2 = ... = \tau_n = \frac{2e-n}{n-1}$. Notice that $\tau_2 = ... = \tau_n = \frac{2e-n}{n-1} \neq \tau_1 = n$, otherwise we have that $n(n-1) = 2e - n \leq n(n-1) - n$, a contradiction. Thus G has two distinct eigenvalues. By Lemma 3, we have that G is complete graph, a contradiction. Therefore we complete the proof of (a) in (1) of Theorem 1.

From the proof of Theorem 3.7 in [5], we have that $\tau_1 \geq T_2 \geq T_1 \geq \frac{4e}{n} > \frac{2e}{n}$. Therefore

$$IE_\alpha = \sum_{i=1}^{n} \tau_i^\alpha \geq f(\tau_1) \geq f(T_2) = T_2^\alpha + \frac{(2e-T_2)^\alpha}{(n-1)^{\alpha-1}}$$

$$\geq f(T_1) = T_1^\alpha + \frac{(2e-T_1)^\alpha}{(n-1)^{\alpha-1}}.$$

Suppose that $IE_\alpha = \sum_{i=1}^{n} \tau_i^\alpha = T_1^\alpha + \frac{(2e-T_1)^\alpha}{(n-1)^{\alpha-1}}$. Then $\tau_1 = T_1 \geq \frac{4e}{n}$ and $\tau_2 = ... = \tau_n$. Notice that $\tau_1 \neq \tau_2 = ... = \tau_n$, otherwise we have that $2e = \sum_{i=1}^{n} \tau_i \geq 4e$, a contradiction. Thus G has two distinct eigenvalues. By Lemma 3, we have that G is a complete graph.

Suppose that $IE_\alpha = \sum_{i=1}^{n} \tau_i^\alpha = T_2^\alpha + \frac{(2e-T_2)^\alpha}{(n-1)^{\alpha-1}}$. Then $\tau_1 = T_2 \geq T_1 \geq \frac{4e}{n}$ and $\tau_2 = ... = \tau_n$. Notice that $\tau_1 \neq \tau_2 = ... = \tau_n$, otherwise we have that $2e = \sum_{i=1}^{n} \tau_i \geq 4e$, a contradiction. Thus G has two distinct eigenvalues. By Lemma 3, we have that G is a complete graph.

Suppose that G is K_n. A simple calculation shows that $\sum_{i=1}^{n} \tau_i^\alpha = f(\tau_1) = f(T_2) = f(T_1) = (2(n-1))^\alpha + (n-1)(n-2)^\alpha$.

Therefore we complete the proofs of (b) and (c) in (1) of Theorem 1.

Proof of (2) in Theorem 1. Notice that x^α is concave down when $x > 0$ and $0 < \alpha < 1$ we have that

$$\sum_{i=2}^{n} \tau_i^\alpha \leq \frac{1}{(n-1)^{\alpha-1}} (\sum_{i=2}^{n} \tau_i)^\alpha$$

with equality holds if and only if $\tau_2 = ... = \tau_n$. Notice further that the function $f(x) = x^\alpha + \frac{(2e-x)^\alpha}{(n-1)^{\alpha-1}}$ is decreasing when $x > \frac{2e}{n}$. By similar arguments as the ones in Proof of (1) in Theorem 1, we can prove that (a), (b), and (c) in (2) of Theorem 1 are true.

Proof of (1) in Theorem 2. Notice that x^α is concave up when $x > 0$ and $\alpha < 0$ or $\alpha > 1$. Thus

$$(\sum_{i=2}^{n} \frac{1}{n-1} |\nu_i|)^\alpha \leq \sum_{i=2}^{n} \frac{1}{n-1} |\nu_i|^\alpha.$$

Hence

$$\sum_{i=2}^{n} |\nu_i|^\alpha \geq \frac{1}{(n-1)^{\alpha-1}} (\sum_{i=2}^{n} |\nu_i|)^\alpha.$$

From Lemma 4, we have that $\nu_1 > 0 > \nu_2 \geq ... \geq \nu_{n-1} \geq \nu_n$. Since $\sum_{i=1}^{n} \nu_i = 0$, we have that $\nu_1 = |\nu_1| = \sum_{i=2}^{n} |\nu_i|$. Hence

$$DE_\alpha = \sum_{i=1}^{n} |\nu_i|^\alpha \geq \nu_1^\alpha + \frac{1}{(n-1)^{\alpha-1}} (\sum_{i=2}^{n} |\nu_i|)^\alpha$$

$$= \nu_1^\alpha(1 + \frac{1}{(n-1)^{\alpha-1}}).$$

If $\alpha > 1$, by Lemma 7, we have that $\nu_1 \geq \sqrt{J/2}$. Thus

$$DE_\alpha \geq \nu_1^\alpha(1 + \frac{1}{(n-1)^{\alpha-1}}) \geq (\sqrt{J/2})^\alpha(1 + \frac{1}{(n-1)^{\alpha-1}}).$$

When the above inequality becomes an equality, we have that $\nu_1 = \sqrt{J/2}$ and Lemma 7 implies that $T = K_2$. A simple computation shows that the above inequality becomes an equality when $T = K_2$.

If $\alpha < 0$, by Lemma 6, we have that $\nu_1 \leq \sqrt{J(n-1)/n}$. Thus

$$DE_\alpha \geq \nu_1^\alpha(1 + \frac{1}{(n-1)^{\alpha-1}}) \geq (\sqrt{J(n-1)/n})^\alpha(1 + \frac{1}{(n-1)^{\alpha-1}}).$$

When the above inequality becomes an equality, we have that $\nu_1 = \sqrt{J(n-1)/n}$ and Lemma 6 implies that $T = K_2$. A simple computation shows that the above inequality becomes an equality when $T = K_2$.

Hence we complete the proof of (1) in Theorem 2.

Proof of (2) in Theorem 2. Notice that x^α is concave down when $x > 0$ and $0 < \alpha < 1$. Thus

$$\sum_{i=2}^{n} |\nu_i|^\alpha \leq \frac{1}{(n-1)^{\alpha-1}}(\sum_{i=2}^{n} |\nu_i|)^\alpha.$$

From Lemma 6, we have that

$$DE_\alpha = \sum_{i=1}^{n} |\nu_i|^\alpha \leq \nu_1^\alpha + \frac{1}{(n-1)^{\alpha-1}}(\sum_{i=2}^{n} |\nu_i|)^\alpha =$$

$$\nu_1^\alpha(1 + \frac{1}{(n-1)^{\alpha-1}}) \leq (\sqrt{J(n-1)/n})^\alpha(1 + \frac{1}{(n-1)^{\alpha-1}}).$$

When the above inequality becomes an equality, we have that $\nu_1 = \sqrt{J(n-1)/n}$ and Lemma 6 implies that $T = K_2$. A simple computation shows that the above inequality becomes an equality when $T = K_2$.

Hence we complete the proof of (2) in Theorem 2.

Proof of (3) in Theorem 2. From Lemma 4, we have that $|\nu_1|\,|\nu_2|...|\nu_n| = C := (n-1)2^{n-2}$. By AM - GM inequality, we have that

$$DE_\alpha = \sum_{i=1}^{n} |\nu_i|^\alpha = |\nu_1|^\alpha + |\nu_2|^\alpha + ... + |\nu_n|^\alpha$$

$$\geq |\nu_1|^\alpha + (n-1)(|\nu_2|^\alpha \ldots |\nu_n|^\alpha)^{\frac{1}{n-1}}$$
$$= \nu_1^\alpha + (n-1)(\frac{C}{\nu_1})^{\frac{\alpha}{n-1}}.$$

Now consider the function $f(x) = x^\alpha + (n-1)(\frac{C}{x})^{\frac{\alpha}{n-1}}$. It can be easily verified that $f(x)$ is non - decreasing when $x \geq C^{\frac{1}{n}}$. From Hadamard inequality and AM - GM inequality, we have that

$$C^2 = Det(D(T))Det(D(T)^t) = Det(D(T)D(T)^t) \leq$$

$$\prod_{i=1}^{n} \sum_{j=1}^{n} d_{i,j}^2 \leq (\frac{\sum_{i=1}^{n}\sum_{j=1}^{n} d_{i,j}^2}{n})^n = (J/n)^n \leq (J/2)^n.$$

We, by Lemma 7, have that $\nu_1 \geq \sqrt{J/2} \geq C^{\frac{1}{n}}$. Therefore

$$DE_\alpha = \sum_{i=1}^{n} |\nu_i|^\alpha \geq f(\nu_1) \geq f(\sqrt{J/2})$$

$$= (\sqrt{J/2})^\alpha + (n-1)(\frac{(n-1)2^{n-2}}{\sqrt{J/2}})^{\frac{\alpha}{n-1}}.$$

When the above inequality becomes an equality, we have that $\nu_1 = \sqrt{J/2}$ and Lemma 7 implies that $T = K_2$. A simple computation shows that the above inequality becomes an equality when $T = K_2$.

Hence we complete the proof of (3) in Theorem 2.

Proof of (4) in Theorem 2. By Lemma 5, we have that $\nu_1 > (n-1)2^{\frac{n-2}{n}} > C^{\frac{1}{n}}$. From the proof of (3) in Theorem 2, we have

$$DE_\alpha > f((n-1)2^{\frac{n-2}{n}}) = ((n-1)^\alpha + (n-1))2^{\frac{\alpha(n-2)}{2}},$$

where $f(x) = x^\alpha + (n-1)(\frac{C}{x})^{\frac{\alpha}{n-1}}$ and $C := (n-1)2^{n-2}$.

Hence we complete the proof of (4) in Theorem 2.

The author would like to thank the referee for his or her suggestions. The author also would like to thank Prof. Bo Zhou for providing his manuscripts.

References

[1] J. A. Bondy and U. S. R. Murty, Graph Theory with Applications, Macmillan, London and Elsevier, New York (1976).

[2] D. Cvetković, New theorems for signless Laplacian eigenvalues, Bull. Acad. Serbe Sci. Arts, Cl. Sci. Math. Natur., Sci. Math. 137(33) (2008) 131 - 146.

[3] M. Edelberg, M. Garey, and R. Graham, On the distance matrix of a tree, *Discrete Math.* **14** (1976) 23 – 39.

[4] R. Grone and R. Merris, The Laplacian spectrum of a graph II, *SIAM J. Discrete Math.* **7** (1994) 221 – 229.

[5] I. Gutman, D. Kiani, M. Mirzakhah, B. Zhou, On incidence energy of a graph, *Linear Algebra and its Applications* **431** (2009) 1223 - 12333.

[6] G. Indulal, I. Gutman, and A. Vijaykumar, On the distance energy of a graph, *MATCH Commun. Math. Comput. Chem.* **60** (2008) 461 - 472.

[7] J. Shu, Y. Hong, and W. Kai, A sharp upper bound on the largest eigenvalue of the Laplacian matrix of a graph, *Linear Algebra and its Applications* **347** (2002) 123 - 129.

[8] B. Zhou, On the sum of powers of the Laplacian eigenvalues of graphs, *Linear Algebra and its Applications* **429** (2008) 2239 - 2246.

[9] B. Zhou, On the largest eigenvalue of the distance matrix of a tree, *MATCH Commun. Math. Comput. Chem.* **58** (2007) 657 - 662.

[10] B. Zhou, On sum of powers of Laplacian eigenvalues and Laplacian Estrada index of graphs, *MATCH Commun. Math. Comput. Chem.* **62** (2009) 611 - 619.

[11] B. Zhou, More upper bounds for the incidence energy, *MATCH Commun. Math. Comput. Chem.* **64** (2010) 123 - 128.

[12] B. Zhou and N. Trinajstić, On the largest eigenvalue of the distance matrix of a connected graph, *Chem. Phys. Lett.* **447** (2007) 384 - 387.