Spectral Invariants and Some Stable Properties of a Graph

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Abstract

For an integer $k \geq 0$, a graphical property P is said to be *k-stable* if whenever $G + uv$ has property P and $d_G(u) + d_G(v) \geq k$, where $uv \notin E(G)$, then G itself has property P. In this note, we present spectral sufficient conditions for several stable properties of a graph.

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1 Introduction

We consider only finite undirected graphs without loops or multiple edges. Notation and terminology not defined here follow that in [2]. We use n to denote the order of a graph. Let $G = (V(G), E(G))$ be a graph with vertex set $V(G)$ and edge set $E(G)$. The complement of G is denoted by $\bar{G} := (V(G), E'(G))$, where $E'(G) = \{xy : x, y \in V(G), xy \notin E(G)\}$.

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The *independence number*, denoted by $\alpha(G)$, of G is the cardinality of a maximum independent set of vertices. We use $C(n,k)$ to denote the number of k-combinations of a set with n distinct elements, K_n to denote the complete graph on n vertices, C_m to denote a cycle of order m, and $K_{m,n}$ to denote the complete bipartite graph with two parts having m and n vertices, respectively.

Let $G+uv$ be the graph obtained from G by adding the edge $uv \notin E(G)$, where $u,v \in V(G)$. For an integer $k \geq 0$, a graphical property P is said to be *k-stable* if whenever $G+uv$ has property P and $d_G(u)+d_G(v) \geq k$, where $uv \notin E(G)$, then G itself has property P. The *k-closure of a graph* G, denoted by $C_k(G)$, is the graph obtained from G by successively joining pairs of nonadjacent vertices whose degree sum is at least k until no such pairs remain. Notice that $d_{C_k(G)}(u)+d_{C_k(G)}(v) \leq k-1$ for any pair of nonadjacent vertices u and v of $C_k(G)$.

We use $D(G) = \mathrm{diag}\,(d_G(v_1), d_G(v_2), \ldots, d_G(v_n))$ to denote the degree matrix of G of order n, where v_i is a vertex in G and $d_G(v_i)$ denotes the degree of the vertex v_i in the graph G, where $i = 1, 2, \ldots, n$. The *adjacency matrix* of G is defined to be a matrix $A(G) = [a_{ij}]$ of order n, where $a_{ij} = 1$ if v_i is adjacent to v_j, and $a_{ij} = 0$ otherwise. The *Laplacian matrix* of G is defined by $L(G) = D(G) - A(G)$. The largest eigenvalue of $A(G)$, denoted by $\mu(G)$, is said to be the spectral radius of G. Let $0 = \lambda_1(G) \leq \lambda_2(G) \leq \cdots \leq \lambda_n(G)$ be the eigenvalues of $L(G)$. Defined $\Sigma_2(G) = \sum_{i=1}^{n} \lambda_i^2(G)$.

A graph G is Hamiltonian if G has a Hamiltonian cycle, a cycle containing all the vertices of G. A graph G is traceable if G has a Hamiltonian path, a path containing all the vertices of G. A graph G is Hamilton-connected if for each pair of vertices in G there exists a Hamiltonian path between them. For an integer $s \geq 2$, a tree T is called a s-tree if the maximum degree of T is at most s. In particular, a Hamilton path of a graph is nothing but its spanning 2-tree.

Recently, several authors established the spectral conditions for the Hamiltonian properties of graphs (see, for example, [3, 6, 7, 9, 10, 11, 12, 15, 16, 17, 18]). In particular, a spectral sufficient condition for a k-connected graph to be Hamiltonian was obtained in [12]. One may naturally ask the following question. What are the spectral sufficient conditions for a k-connected graph to be traceable or Hamilton-connected? To answer the question, we will present respectively spectral sufficient conditions for a k-connected graph to be traceable and Hamilton-connected in Section 2 and Section 3 below. Moreover, in Section 4, we will present spectral sufficient conditions for a graph of order n contains a C_4 or $K_{2,s}$ with $2 \leq s \leq n-2$. In Section 5, we will present spectral sufficient conditions for a graph to be k-connected. In Section 6, we will present spectral sufficient conditions for a k-connected graph to have a spanning s-tree with $s \geq 2$.

We will use the following result in our proofs.

LEMMA 1.1 *Let G be a graph of order n. Suppose that P is a $r(n,P)$-stable property and K_n has property P. Moreover, if $|E(G)| > e(n,P)$, then G has property P.*

(i) *If*

$$\mu(\bar{G}) < \sqrt{\frac{(2n - r(n,P) - 1)(C(n,2) - e(n,P))}{n}},$$

then G has property P.

(ii) *If*

$$\Sigma_2(\bar{G}) < (2n - r(n,P) + 1)(C(n,2) - e(n,P)),$$

then G has property P.

Since the proof of Lemma 1.1 is almost the same as the proof of Theorem 5 in [13] (which is listed as Theorem 1.2 below), We omit the proof of Lemma 1.1 here.

THEOREM 1.2 [13] *Let G be a graph of order n. Suppose that P is a $r(n,P)$-stable property and K_n has property P. Moreover, if $|E(G)| \geq e(n,P)$, then G has property P.*

(i) *If*

$$\mu(\bar{G}) \leq \sqrt{\frac{(2n - r(n,P) - 1)(C(n,2) - e(n,P))}{n}},$$

then G has property P.

(ii) *If*

$$\Sigma_2(\bar{G}) \leq (2n - r(n,P) + 1)(C(n,2) - e(n,P)),$$

then G has property P.

We will also use the following result (which is from Theorem 6 in [13]) in our proof.

LEMMA 1.3 [13] *Let G be a graph of order n. Suppose that P is a $r(n,P)$-stable property and K_n has property P.*

(i) *If*

$$\mu(\bar{G}) < \sqrt{\frac{(2n-r(n,P)-1)(2n-r(n,P)-2)}{n}},$$

then G has property P.

(ii) *If*

$$\Sigma_2(\bar{G}) < (2n-r(n,P)+1)(2n-r(n,P)-2),$$

then G has property P.

2 Spectral conditions for a k-connected graph to be traceable

LEMMA 2.1 [1] *Let G be a graph of order n. The property that G is traceable is $(n-1)$-stable.*

LEMMA 2.2 [5] *Let G be a k-connected graph. If*

$$\alpha(G) \le k+1,$$

then G is a traceable graph.

We use some ideas from [4] to prove the following Lemma 2.3 and Theorem 2.4.

LEMMA 2.3 *Let G be a non-traceable graph of order n. If $G = C_{n-1}(G)$ and $2 \le m \le \alpha(G)$, then*

$$|E(\bar{G})| \ge \begin{cases} \frac{m}{2}(n-m+1) & \text{for } n \text{ even,} \\ \frac{m}{2}(n-m)+m-1 & \text{for } n \text{ odd.} \end{cases}$$

Proof. Let $I = \{v_1, v_2, \cdots, v_m\}$ be a set of independent vertices of G. Obviously, if the $(n-m)$ vertices in $G-I$ are deleted, the resulting graph is disconnected. Let $E(\bar{G}_I)$ be the edge set which represents the edges in $\bar{G}$ that are incident with at least one vertex of I. Then $E(\bar{G}_I)$ contains

$C(m,2)$ edges with both endpoints in I and $\sum_{i=1}^{m}(n-m-d_G(v_i))$ edges with exactly one endpoint in I. It follows that

$$|E(\bar{G})| \geq |E(\bar{G}_I)| = C(m,2) + \sum_{i=1}^{m}(n-m-d_G(v_i))$$

$$= C(m,2) + m(n-m) - \sum_{i=1}^{m} d_G(v_i).$$

By $G = C_{n-1}(G)$, $m \geq 2$ and the definition of $C_{n-1}(G)$, we have $d_G(v_i) + d_G(v_j) \leq n-2$ for $i \neq j, 1 \leq i,j \leq m$. So when n is even, it follows that $\sum_{i=1}^{m} d_G(v_i)$ is maximized when $d_G(v_i) = \frac{n-2}{2}$ for $1 \leq i \leq m$, and when n is odd, it follows that $\sum_{i=1}^{m} d_G(v_i)$ is maximized when $d_G(v_1) = \frac{n-1}{2}$ and $d_G(v_i) = \frac{n-1}{2} - 1$ for $2 \leq i \leq m$. Then, for even n,

$$|E(\bar{G})| \geq C(m,2) + m(n-m) - m(\frac{n-2}{2}) = \frac{m}{2}(n-m+1),$$

and for odd n,

$$|E(\bar{G})| \geq C(m,2)+m(n-m)-(\frac{n-1}{2}+(m-1)(\frac{n-1}{2}-1)) = \frac{m}{2}(n-m)+m-1.$$

■

THEOREM 2.4 *Let G be a k-connected graph of order n. If*

$$|E(G)| > C(n,2) - \frac{k+2}{2}(n-k-1),$$

then G is traceable.

Proof. Assume that G is non-traceable and let $H = C_{n-1}(G)$. Then H is k-connected and, by Lemma 2.1, non-traceable. Using Lemma 2.2, we have $\alpha(H) \geq k+2$. According to Lemma 2.3, $|E(\bar{H})| \geq \frac{k+2}{2}(n-k-1)$, and it follows that

$$|E(G)| \leq |E(H)| = C(n,2) - |E(\bar{H})| \leq C(n,2) - \frac{k+2}{2}(n-k-1),$$

contradicting the known condition and completing the proof. ■

THEOREM 2.5 *Let G be a k-connected graph of order n.*

(i) *If*

$$\mu(\bar{G}) < \sqrt{\frac{k+2}{2}(n-k-1)},$$

then G is traceable.

(ii) *If*

$$\Sigma_2(\bar{G}) < \frac{(n+2)(k+2)}{2}(n-k-1),$$

then G is traceable.

Proof. If G is K_n, then G is traceable. Let $r(n,P) = n-1$, $e(n,P) = C(n,2) - \frac{k+2}{2}(n-k-1)$ in Lemma 1.1, we complete the proof. ■

3 Spectral conditions for a k-connected graph to be Hamilton-connected

LEMMA 3.1 [1] *Let G be a graph of order n. The property that G is Hamilton-connected is* $(n+1)$*-stable.*

LEMMA 3.2 [5] *Let G be a k-connected graph. If*

$$\alpha(G) \le k-1,$$

then G is a Hamilton-connected graph.

Again some ideas from [4] are used when we prove the following Lemma 3.3 and Theorem 3.4.

LEMMA 3.3 *Let G be a non-hamilton-connected graph of order n. If* $G = C_{n+1}(G)$ *and* $2 \le m \le \alpha(G)$*, then*

$$|E(\bar{G})| \ge \begin{cases} \frac{m}{2}(n-m-1) & \text{for } n \text{ even,} \\ \frac{m}{2}(n-m)-1 & \text{for } n \text{ odd.} \end{cases}$$

Proof. Let $I = \{v_1, v_2, \cdots, v_m\}$ be a set of independent vertices of G. By the similar arguments as the ones in the proof of Lemma 2.3, we obtain that

$$|E(\bar{G})| \ge |E(\bar{G}_I)| = C(m,2) + m(n-m) - \sum_{i=1}^{m} d_G(v_i).$$

From $G = C_{n+1}(G)$, $m \ge 2$ and the definition of $C_{n+1}(G)$, we have $d_G(v_i) + d_G(v_j) \le n$ for $i \ne j, 1 \le i,j \le m$. So when n is even, it follows that

$\sum_{i=1}^{m} d_G(v_i)$ is maximized when $d_G(v_i) = \frac{n}{2}$ for $1 \le i \le m$, and when n is odd, it follows that $\sum_{i=1}^{m} d_G(v_i)$ is maximized when $d_G(v_1) = \frac{n+1}{2}$ and $d_G(v_i) = \frac{n+1}{2} - 1$ for $2 \le i \le m$. Then, for even n,

$$|E(\bar{G})| \ge C(m,2) + m(n-m) - m\frac{n}{2} = \frac{m}{2}(n-m-1),$$

and for odd n,

$$|E(\bar{G})| \ge C(m,2) + m(n-m) - (\frac{n+1}{2} + (m-1)(\frac{n+1}{2} - 1)) = \frac{m}{2}(n-m) - 1.$$

■

THEOREM 3.4 *Let G be a k-connected graph of order n, where $k \ge 2$. If*

$$|E(G)| > C(n,2) - \frac{k}{2}(n-k-1),$$

then G is Hamilton-connected.

Proof. Assume that G is non-hamilton-connected. Let $H' = C_{n+1}(G)$, then H' is k-connected and, by Lemma 3.1, non-hamilton-connected. Using Lemma 3.2, we have $\alpha(H') \ge k$. According to Lemma 3.3, $|E(\bar{H'})| \ge \frac{k}{2}(n-k-1)$, and it follows that

$$|E(G)| \le |E(H')| = C(n,2) - |E(\bar{H'})| \le C(n,2) - \frac{k}{2}(n-k-1),$$

contradicting the known condition and completing the proof. ■

THEOREM 3.5 *Let G be a k-connected graph of order $n \ge 3$, where $k \ge 2$.*

(i) *If*

$$\mu(\bar{G}) < \sqrt{\frac{k(n-2)}{2n}(n-k-1)},$$

then G is Hamilton-connected.

(ii) *If*

$$\Sigma_2(\bar{G}) < \frac{nk}{2}(n-k-1),$$

then G is Hamilton-connected.

Proof. If G is K_n, then G is Hamilton-connected. Let $r(n,P) = n+1$,

$$e(n,P) = C(n,2) - \frac{k}{2}(n-k-1)$$

in Lemma 1.1, then we complete the proof. ■

4 Spectral conditions for a graph of order n containing C_4 or $K_{2,s}$ with $2 \leq s \leq n-2$

LEMMA 4.1 [1] Let G be a graph of order n. The property that G contains C_4 is $(2n-5)$-stable.

LEMMA 4.2 [14] Let G be a graph of order n. If $|E(G)| > \frac{n}{4}(1+\sqrt{4n-3})$, then G contains C_4.

Let $e(n,P) = \frac{n}{4}(1+\sqrt{4n-3})$ and $r(n,P) = 2n-5$ in Lemma 1.1, then we have the following result for a graph contains C_4.

THEOREM 4.3 *Let G be a graph of order n.*

(i) *If*

$$\mu_n(\bar{G}) < \sqrt{2n-3-\sqrt{4n-3}},$$

then G contains C_4.

(ii) *If*

$$\Sigma_2(\bar{G}) < \frac{3}{2}(2n^2-3n-n\sqrt{4n-3}),$$

then G contains C_4.

LEMMA 4.4 [1] Let G be a graph of order n. The property P that G contains $K_{2,s}$ is $(n+s-2)$-stable, where $2 \leq s \leq n-2$.

The following lemma is Exercise 7.3.4 on Page 111 in [2].

LEMMA 4.5 [2] Let G be a graph of order n. If $|E(G)| > \frac{n\sqrt{n(s-1)}}{2} + \frac{n}{4}$, then G contains $K_{2,s}$, where $s \geq 2$.

Let $e(n,P) = \frac{n\sqrt{n(s-1)}}{2} + \frac{n}{4}$ and $r(n,P) = n+s-2$ in Lemma 1.1, then we have the following result for a graph contains $K_{2,s}$.

THEOREM 4.6 *Let G be a graph of order n and let s be an integer such that $2 \leq s \leq n-2$.*

(i) *If*

$$\mu_n(\bar{G}) < \sqrt{\frac{(n-s+1)(2n-2\sqrt{n(s-1)}-3)}{4}},$$

then G contains $K_{2,s}$.

(ii) *If*

$$\Sigma_2(\bar{G}) < \frac{n(n-s+3)(2n-2\sqrt{n(s-1)}-3)}{4},$$

then G contains $K_{2,s}$.

5 Spectral conditions for a graph to be k-connected

LEMMA 5.1 [1] *Let G be a graph of order n. The property that G is k-connected is* $(n+k-2)$*-stable, where* $k \geq 1$.

Let $r(n,P) = n+k-2$ in Lemma 1.3, then we have the following result for a graph to be k-connected.

THEOREM 5.2 *Let G be a graph of order n.*

(i) *If*

$$\mu_n(\bar{G}) < \sqrt{\frac{(n-k+1)(n-k)}{n}},$$

then G is k-connected graph, where $k \geq 1$.

(ii) *If*

$$\Sigma_2(\bar{G}) < (n-k+3)(n-k),$$

then G is k-connected graph, where $k \geq 1$.

Next, we will show that the conditions in Theorem 5.2 also imply that a connected graph is traceable and a 3-connected graph is Hamilton-connected.

THEOREM 5.3 *Let G be a connected graph of order* $n \geq 4$ *and* $1 \leq k \leq n-1$.

(i) *If*

$$\mu(\bar{G}) < \sqrt{\frac{(n-k+1)(n-k)}{n}},$$

then G is traceable.

(ii) *If*

$$\Sigma_2(\bar{G}) < (n-k+3)(n-k),$$

then G is traceable.

Proof. If G is K_n, then G is traceable. If G is not K_n, and is k-connected graph, then $1 \le k \le n-2$.

(i) Let $f(k) = 2(n-k+1)(n-k) - n(k+2)(n-k-1) = (n+2)k^2 - (n^2+n+2)k + 4n$. We can easily find that when

$$\frac{n^2+n+2-\sqrt{n^4+2n^3-11n^2-28n+4}}{2(n+2)} \le k$$

$$\le \frac{n^2+n+2+\sqrt{n^4+2n^3-11n^2-28n+4}}{2(n+2)},$$

$f(k) \le 0$. We further have that if $n \ge 4$,

$$\frac{n^2+n+2-\sqrt{n^4+2n^3-11n^2-28n+4}}{2(n+2)} \le 1$$

and

$$\frac{n^2+n+2+\sqrt{n^4+2n^3-11n^2-28n+4}}{2(n+2)} \ge n-2.$$

So, we have

$$\sqrt{\frac{(n-k+1)(n-k)}{n}} \le \sqrt{\frac{k+2}{2}(n-k-1)}$$

when $n \ge 4$.

Hence, G is traceable by Theorem 2.5 (i).

(ii) Let $g(k) = 2(n-k+3)(n-k) - (n+2)(k+2)(n-k-1) = (n+4)k^2 - (n^2+3n)k + 4n + 4$. We again can easily find that when

$$\frac{n^2+3n-\sqrt{n^4+6n^3-7n^2-80n-64}}{2(n+4)} \le k$$

$$\le \frac{n^2+3n+\sqrt{n^4+6n^3-7n^2-80n-64}}{2(n+4)},$$

$g(k) \le 0$. We further have that if $n \ge 4$,

$$\frac{n^2+3n-\sqrt{n^4+6n^3-7n^2-80n-64}}{2(n+4)} \le 1$$

and

$$\frac{n^2+3n+\sqrt{n^4+6n^3-7n^2-80n-64}}{2(n+4)} \ge n-2.$$

So, we have

$$(n-k+3)(n-k) \leq \frac{(n+2)(k+2)}{2}(n-k-1)$$

when $n \geq 4$.

Hence, G is traceable by Theorem 2.5 (ii). ■

THEOREM 5.4 *Let G be a 3-connected graph of order $n \geq 6$ and $3 \leq k \leq n-1$.*

(i) *If*

$$\mu(\bar{G}) < \sqrt{\frac{(n-k+1)(n-k)}{n}},$$

then G is Hamilton-connected.

(ii) *If*

$$\Sigma_2(\bar{G}) < (n-k+3)(n-k),$$

then G is Hamilton-connected.

Proof. If G is K_n, then G is Hamilton-connected. If G is not K_n, and is k-connected graph, then $3 \leq k \leq n-2$.

(i) Let $f(k) = 2(n-k+1)(n-k) - k(n-2)(n-k-1) = nk^2 - (n^2 + n + 4)k + 2n^2 + 2n$. We can easily find that when

$$\frac{n^2+n+4-\sqrt{n^4-6n^3+n^2+8n+16}}{2n} \leq k$$

$$\leq \frac{n^2+n+4+\sqrt{n^4-6n^3+n^2+8n+16}}{2n},$$

$f(k) \leq 0$. We further have that if $n \geq 6$,

$$\frac{n^2+n+4-\sqrt{n^4-6n^3+n^2+8n+16}}{2n} \leq 3$$

and

$$\frac{n^2+n+4+\sqrt{n^4-6n^3+n^2+8n+16}}{2n} \geq n-2.$$

So, we have

$$\sqrt{\frac{(n-k+1)(n-k)}{n}} \leq \sqrt{\frac{k(n-2)}{2n}(n-k-1)}$$

when $n \geq 6$.

Hence, G is Hamilton-connected by Theorem 3.5 (i).

(ii) Let $g(k) = 2(n-k+3)(n-k) - nk(n-k-1) = (n+2)k^2 - (n^2 + 3n + 6)k + 2n^2 + 6n$. We again can easily find that when

$$\frac{n^2+3n+6-\sqrt{n^4-2n^3-19n^2-12n+36}}{2(n+2)} \le k$$

$$\le \frac{n^2+3n+6+\sqrt{n^4-2n^3-19n^2-12n+36}}{2(n+2)},$$

$g(k) \le 0$. We further have if $n \ge 6$,

$$\frac{n^2+3n+6-\sqrt{n^4-2n^3-19n^2-12n+36}}{2(n+2)} \le 3$$

and

$$\frac{n^2+3n+6+\sqrt{n^4-2n^3-19n^2-12n+36}}{2(n+2)} \ge n-2.$$

So, we have

$$(n-k+3)(n-k) \le \frac{nk}{2}(n-k-1)$$

when $n \ge 6$.

Hence, G is Hamilton-connected by Theorem 3.5 (ii). ■

6 Spectral conditions for a k-connected graph to have a spanning s-tree with $s \ge 2$

LEMMA 6.1 [8] *Let G be a k-connected graph of order n. The property that G has a spanning s-tree is $(n-1-(s-2)k)$-stable, where $k \ge 1$, $s \ge 2$.*

Let $r(n, P) = n - 1 - (s-2)k$ in Lemma 1.3, then we have the following result for a graph to have a spanning s-tree with $s \ge 2$.

THEOREM 6.2 *Let G be a k-connected graph of order n.*

(i) *If*

$$\mu_n(\bar{G}) < \sqrt{\frac{(n+(s-2)k)(n-1+(s-2)k)}{n}},$$

then G has a spanning s-tree, where $k \ge 1$, $s \ge 2$.

(ii)*If*

$$\Sigma_2(\bar{G}) < (n+2+(s-2)k)(n-1+(s-2)k),$$

then G has a spanning s-tree, where $k \ge 1$, $s \ge 2$.

Conflicts of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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