Bipolar Fuzzy Competition Graphs

Noura Omair Alshehri[1] and Muhammad Akram[2]
1. Department of Mathematics, Faculty of Sciences(Girls), King Abdulaziz University, Jeddah, Saudi Arabia.
E-mail: nalshehrie@kau.edu.sa
2. Department of Mathematics, University of the Punjab, New Campus, P.O. Box No. 54590, Lahore, Pakistan.
E-mail: m.akram@pucit.edu.pk, makrammath@yahoo.com

Abstract

Fuzzy graph theory is finding an increasing number of applications in modeling real time systems where the level of information inherent in the system varies with different levels of precision. Fuzzy models are becoming useful because of their aim in reducing the differences between the traditional numerical models used in engineering and sciences, and the symbolic models used in expert systems. A bipolar fuzzy model is a generalized soft computing model of a fuzzy model that gives more precision, flexibility, and compatibility to a system when compared with systems that are designed using fuzzy models. In this research article, we introduce certain types of bipolar fuzzy competition graphs including bipolar fuzzy k-competition, bipolar fuzzy p-competition and bipolar fuzzy m-competition. We investigate some properties of these new concepts.

Keywords: Bipolar fuzzy k-competition, bipolar fuzzy p-competition and bipolar fuzzy m-competition
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1 Introduction

Graph theory is an extremely useful tool in solving combinatorial problems in different areas including geometry, algebra, number theory, topology, operations research, optimization, computer science, engineering, physical, biological and social systems. Point-to-point interconnection net- works for parallel and distributed systems are usually modeled by directed graphs (or digraphs). A digraph is a graph whose edges have directions and are called arcs (edges). Arrows on the arcs are used to encode the directional information: an arc from vertex (node) x to vertex y indicates that one may move from x to y but not from y to x. In 1968, Cohen [11] introduced the notion of competition graphs in connection with a problem in ecology. Let $D^* = (V, E)$ be a digraph, which corresponds to a food web. A vertex $x \in V$ represents a specie in the food web and an arc $(x, s) \in E$ means that x preys on the species s. If two species x and y have a common prey s, they will compete for the prey s. Based on this analogy, Cohen defined a graph which represents the relations of competition among the species in the food web. The competition graph $C(D^*)$ of a digraph $D^* = (V, E)$ is an undirected graph which has the same vertex set V and has an edge between two distinct vertices $x, y \in V$ if there exists a vertex $s \in V$ and arcs $(x, s), (y, s) \in E$. That is, if x and y have a common neighbour, then there is an edge between x and y. This idea can be extended for more vertices. For example, if the vertices x and y have, say p, common neighbours, then the vertices are joined by an edge. If a graph is constructed from the digraph D^* based on this rule, then the resulted graph $C(D^*)$ is called p-competition graph. There is another technique to construct other type of competition graph. If two vertices x and v are adjacent, i.e., neighbour, then x and v are called 1-step neighbour. If v is 1-step neighbour of both x and y, i.e., x and v are adjacent, and also y and v are adjacent, then v is called 1-step common neighbour of x and y. If x and v are connected through a path containing $m + 1$ vertices (including x and v), then x and v are called m-step neighbour. If both x and y are m-step neighbour of v, then v is called m-step common neighbour of x and y. Suppose $D^* = (V, E)$ is a digraph. Let a graph $C_m(D^*) = (V, E)$ be constructed from D^*, where the set of vertices of D^* and $C_m(D^*)$ are same and E is defined as $E = \{(x, y) :$ there exists $v \in V$ such that v is m-step common neighbour of x, $y\}$. The graph $C_m(D^*)$ is called m-step competition graph. This graph is an undirected graph with the underlying graph D^*. The m-step competition graph has applications in real life problems. After Cohen's introduction of competition graph, several variations of it are found in literature. These are p-competition graphs of digraphs [14, 18], common enemy graphs of digraphs [22], competition-common enemy graphs of digraphs [30]. Cho, Kim and Nam [10] defined another generalization as m-step competition graph of a digraph. In these representations the graphs are crisp and these representations do not include all competitions of real field. The competition graphs described about the common prey and related species. But, the crisp competition graphs are not sufficient to represent the degree of dependence on a common prey compared to the other species. All problems of prey-predator can not be modeled using fuzzy competition graph. For such problems, p-competition fuzzy graph or m-step fuzzy competition graphs are successfully used. One generalization of fuzzy competition graph, called m-step fuzzy competition graph [27, 28].

In 1994, Zhang [34] initiated the concept of bipolar fuzzy sets as a generalization of fuzzy sets [32]. Bipolar fuzzy sets are an extension of fuzzy sets [32] whose membership degree range is $[-1, 1]$. In a bipolar fuzzy set, the membership degree 0 of an element means that the element is irrelevant to the corresponding property, the membership degree $(0, 1]$ of an element indicates that the element somewhat satisfies the property, and the membership degree $[-1, 0)$ of an element indicates that the element somewhat satisfies the implicit counter-property. Although bipolar fuzzy sets and intuitionistic fuzzy sets look similar to each other, they are essentially different sets. In many domains, it is important to be able to deal with bipolar information. It is noted that positive information represents what is granted to be possible, while negative information represents what is considered to be impossible. This domain has recently motivated new research in several directions. In particular, fuzzy and possibilistic formalisms for bipolar information have been proposed [12], because when we deal with spatial information in image processing or in spatial reasoning applications, this bipolarity also occurs. For instance, when we assess the position of an object in a space, we may have positive information expressed as a set of possible places and negative information expressed as a set of impossible places.

Fuzzy graph theory is finding an increasing number of applications in modeling real time systems where the level of information inherent in the system varies with different levels of precision. Fuzzy models are becoming useful because of their aim in reducing the differences between the traditional numerical models used in engineering and sciences and the symbolic models used in expert systems. Kaufmann's initial definition of a fuzzy graph [16] was based on Zadeh's fuzzy relations [33]. Rosenfeld [26] introduced the fuzzy analogue of several basic graph-theoretic concepts including bridges, cut-nodes, connectedness, trees and cycles. Bhattacharya [7] gave some remarks on fuzzy graphs, and Sunitha and Vijayakumar [29] characterized fuzzy trees. Bhutani and Rosenfeld [8] introduced the concepts of strong arcs, fuzzy end nodes and geodesics in fuzzy graphs, and types of arcs in a fuzzy graph are described in [23]. Akram et al.[1-5] has introduced many concepts, including bipolar fuzzy graphs, regular bipolar fuzzy graphs, bipolar fuzzy hypergraphs and metric aspects of bipolar fuzzy graphs. Samanta et al. [27, 28] discussed Fuzzy k-competition graphs, p-competition fuzzy graphs and $m-$ Step fuzzy competition graphs. In this research article, we introduce certain types of bipolar fuzzy competition graphs including bipolar fuzzy k-competition, bipolar fuzzy p-competition and bipolar fuzzy m-competition. We investigate some properties of these new concepts. We also establish relations between these bipolar fuzzy competition graphs.

2 Preliminaries

We review here some elementary concepts of graphs that are necessary for this paper.
A *directed graph (digraph)* D^* is a graph which consists of non-empty finite set $V(D^*)$ of elements called vertices and a finite set $E(D^*)$ of ordered pairs of distinct vertices called arcs. We will often write $D^* = (V, E)$. For an arc (x, y), x is the tail and y is the head. The *order (size)* of D^* is the number of vertices (arcs) in D^*. The *out-neighbourhood* [15] of a vertex y is the set $N^+(y) = \{x \in V - y : (y, x) \in E\}$. Similarly, the *in-neighbourhood* [15] $N^-(y)$ of a vertex y is the set $\{z \in V - y : (z, y) \in E\}$. The *open-neighbourhood* of a vertex is the union of out-neighbourhood

and in-neighbourhood of the vertex. A *walk* in D^* is an alternating sequence $W = x_1e_1x_2e_2 \ldots x_{k-1}e_kx_k$ of vertices x_i and arcs e_i of D^* such that tail of e_i is x_i and head is x_{i+1} for every $i = 1, 2, \ldots, k-1$. A walk is closed if $x_1 = x_k$. A trail is a walk in which all arcs are distinct. A *path* is a walk in which all vertices are distinct. A path $x_1, x_2, \ldots, x_k$ with $k \geq 3$ is a cycle if $x_1 = x_k$.
For an undirected graph G^*, *open-neighbourhood* [6] $N(x)$ of the vertex x is the set of all vertices adjacent to x in the graph. *Open-neighbourhood graph* [6] $N(G^*)$ of G^* is a graph whose vertex set is same as G and has an edge between two vertices x and y in $N(G^*)$ if and only if $N(x) \cap N(y) \neq \phi$ in G^*. *Closed-neighbourhood* $N[x]$ of x is the set $N(x) \cup \{x\}$. *Closed-neighbourhood graph* $N[G^*]$ of a graph G^* is similarly defined, except having an edge in $N[G^*]$ if and only if $N[x] \cap N[y] \neq \phi$ in G^*.

Definition 2.1. [11] The competition graph $C(D^*)$ of a digraph $D^* = (V, E)$ is an undirected graph $G^* = (V, E)$ which has the same vertex set V and has an edge between distinct two vertices $x, y \in V$ if there exists a vertex $a \in V$ and arcs $(x, a), (y, a) \in E$ in D^*. We say that a graph G^* is a competition graph if there exists a digraph D^* such that $C(D^*) = G^*$.

Definition 2.2. [18] If p is a positive integer, the p-competition graph $C_p(D^*)$ corresponding to the digraph D^* is defined to have a vertex set V with an edge between x and y in V if and only if, for some distinct vertices $a_1, a_2, \ldots, a_p$ in V, $(x, a_1), (y, a_1), (x, a_2), (y, a_2), \ldots, (x, a_p), (y, a_p)$ are arcs in D^*.

If D^* is thought of as a food web whose vertices are the species in some ecosystem, (x, y) is an edge of $C_p(D^*)$ if and only if x and y have at least p common preys. So $C_1(D^*)$ is the competition graph.

Definition 2.3. [32, 33] A *fuzzy subset* μ on a set X is a map $\mu : X \to [0, 1]$. A *fuzzy binary relation* on X is a fuzzy subset μ on $X \times X$. By a fuzzy relation we mean a fuzzy binary relation given by $\mu : X \times X \to [0, 1]$.

Definition 2.4. [26] Directed fuzzy graph (fuzzy digraph) $D = (V, \mu, \nu)$ is a non-empty set V together with a pair of functions

$\mu : V \to [0,1]$ and $\nu : V \times V \to [0,1]$ such that for all $x, y \in V$, $\nu(x,y) \leq \mu(x) \wedge \mu(y)$.

Since ν is well defined, a fuzzy digraph has at most two directed edges (which must have opposite directions) between any two vertices. Here $\nu(x,y)$ is denoted by the membership value of the edge (x,y). The loop at a vertex x is represented by $\nu(x,x) \neq 0$. Here ν need not be symmetric as $\nu(x,y)$ and $\nu(y,x)$ may have different values. The *underlying crisp graph of directed fuzzy graph* is the graph similarly obtained except the directed arcs are replaced by undirected edges.

Definition 2.5. [28] Fuzzy out-neighbourhood of a vertex x of a directed fuzzy graph $D = (\mu, \nu)$ is the fuzzy set $\mathcal{N}^+(x) = (V_x^+, \mu_x^+)$ where $V_x^+ = \{y | \nu(x,y) > 0\}$ and $\mu_x^+ : V_x^+ \to [0,1]$ defined by $\mu_x^+(y) = \nu(y,x)$. Similarly, fuzzy in-neighbourhood of a vertex x of a directed fuzzy graph $D = (\mu, \nu)$ is the fuzzy set $\mathcal{N}^-(x) = (V_x^-, \nu_x^-)$ where $V_x^- = \{y | \nu(y,x) > 0\}$ and $\nu_x^- : V_x^- \to [0,1]$ defined by $\nu_x^-(y) = \nu(y,x)$.

Definition 2.6. [34] Let X be a nonempty set. A *bipolar fuzzy set* B in X is an object having the form

$$B = \{(x,\, \mu_B^P(x),\, \mu_B^N(x)) \,|\, x \in X\}$$

where $\mu_B^P : X \to [0,\, 1]$ and $\mu_B^N : X \to [-1,\, 0]$ are mappings.

We use the positive membership degree $\mu_B^P(x)$ to denote the satisfaction degree of an element x to the property corresponding to a bipolar fuzzy set B, and the negative membership degree $\mu_B^N(x)$ to denote the satisfaction degree of an element x to some implicit counter-property corresponding to a bipolar fuzzy set B. If $\mu_B^P(x) \neq 0$ and $\mu_B^N(x) = 0$, it is the situation that x is regarded as having only positive satisfaction for B. If $\mu_B^P(x) = 0$ and $\mu_B^N(x) \neq 0$, it is the situation that x does not satisfy the property of B but somewhat satisfies the counter property of B. It is possible for an element x to be such that $\mu_B^P(x) \neq 0$ and $\mu_B^N(x) \neq 0$ when the membership function of the property overlaps that of its counter property over some portion of X.

For the sake of simplicity, we shall use the symbol $B = (\mu_B^P, \mu_B^N)$ for the bipolar fuzzy set

$$B = \{(x, \mu_B^P(x), \mu_B^N(x)) \mid x \in X\}.$$

A nice application of bipolar fuzzy concept is a political acceptation (map to $[0,1]$) and opposition (map to [-1,0]).

Definition 2.7. [34] Let X be a nonempty set. Then we call a mapping $A = (\mu_A^P, \mu_A^N) : X \times X \to [0,1] \times [-1,0]$ a *bipolar fuzzy relation* on X such that $\mu_A^P(x,y) \in [0,1]$ and $\mu_A^N(x,y) \in [-1,0]$.

Definition 2.8. [20] The *support* of a bipolar fuzzy set $A = (\mu_A^P, \mu_A^N)$, denoted by supp(A), is defined by supp$(A) = supp^P(A) \cup supp^N(A)$, $supp^P(A) = \{x \mid \mu_A^P(x) > 0\}$, $supp^N(A) = \{x \mid \mu_A^N(x) < 0\}$. We call supp$^P(A)$ as positive support and supp$^N(A)$ as negative support.

Definition 2.9. [20] Let $A = (\mu_A^P, \mu_A^N)$ be a bipolar fuzzy set on X and let $\alpha \in [0,1]$. α-cut A_α of A can be defined as

$$A_\alpha = A_\alpha^P \cup A_\alpha^N, \; A_\alpha^P = \{x \mid \mu_\alpha^P(x) \geq \alpha\}, \; A_\alpha^N = \{x \mid \mu_\alpha^N(x) \leq -\alpha\}.$$

We call A_α^P as positive α-cut and A_α^N as negative α-cut.

Definition 2.10. The *height* of a bipolar fuzzy set $A = (\mu_A^P, \mu_A^N)$ is defined as $h(A) = \max\{\mu_A^P(x) | x \in X\}$. We shall say that bipolar fuzzy set A is *normal*, if there is at least one $x \in X$ such that $\mu_A^P(x)$ $=1$ or $\mu_A^N(x) = -1$.

Notations: Throughout this paper, we denote D^* a crisp digraph, G^* an undirected graph, D a bipolar fuzzy digraph, and G a bipolar fuzzy graph.

3 Bipolar Fuzzy Competition Graphs

In this section, we introduce certain types of bipolar fuzzy competition graphs.

Definition 3.1. Bipolar fuzzy digraph $D = (V, A, B)$ is a non-empty set V together with a pair of functions $A = (\mu_A^P, \mu_A^N) : V \to [0,1] \times [-1,0]$ and $B = (\mu_B^P, \mu_B^N) : V \times V \to [0,1] \times [-1,0]$ such that for all $x, y \in V$,

$$\mu_B^P(x,y) \leq \min(\mu_A^P(x), \mu_A^P(y)) \;\text{ and }\; \mu_B^N(x,y) \geq \max(\mu_A^N(x), \mu_A^N(y)).$$

Notice that $\mu_B^P(xy) > 0$, $\mu_B^N(xy) < 0$ for $(x,y) \in V \times V$, $\mu_B^P(x,y) = \mu_B^N(x,y) = 0$ for $(x,y) \notin V \times V$, and B need not to be symmetric.

Definition 3.2. Bipolar fuzzy out-neighbourhood of a vertex x of a bipolar fuzzy digraph $D = (V, A, B)$ is the bipolar fuzzy set $N^+(x) = (V_x^+, \mu_{A_x}^P, \mu_{A_x}^N)$ where $V_x^+ = \{y | \mu_B^P(x,y) > 0,\ \mu_B^N(x,y) < 0\}$ and $\mu_{A_x}^P : V_x^+ \to [0,1]$, $\mu_{A_x}^N : V_x^+ \to [-1,0]$ defined by $\mu_{A_x}^P(y) = \mu_B^P(x,y)$, $\mu_{A_x}^N(y) = \mu_B^N(x,y)$. Bipolar fuzzy in-neighbourhood of a vertex x of a bipolar fuzzy digraph $D = (A, B)$ is the bipolar fuzzy set $N^-(x) = (V_x^-, \mu_{A_x}^P, \mu_{A_x}^N)$ where $V_x^- = \{y | \mu_B^P(y,x) > 0,\ \mu_B^N(y,x) < 0\}$ and $\mu_{A_x}^P : V_x^- \to [0,1]$, $\mu_{A_x}^N : V_x^- \to [-1,0]$ defined by $\mu_{A_x}^P(y) = \mu_B^P(y,x)$, $\mu_{A_x}^N(y) = \mu_B^N(y,x)$.

Note that $(x, \mu_A^P(x), \mu_A^N(x))$ represents the vertex x with positive membership value $\mu_A^P(x)$ and negative membership value $\mu_A^N(x)$).

Example 3.3. Let D be a bipolar fuzzy digraph as shown in Figure 3.1. Let the vertex set be $\{a,b,c,d\}$ with positive membership values $\mu_A^P(a) = 0.4$, $\mu_A^P(b) = 0.5$, $\mu_A^P(c) = 0.6$, $\mu_A^P(d) = 0.5$, and negative membership values $\mu_A^N(a) = -0.3$, $\mu_A^N(b) = -0.3$, $\mu_A^N(c) = -0.3$, $\mu_A^N(d) = -0.5$. The positive membership values of arcs be $\mu_B^P(b,a) = 0.3$, $\mu_B^P(d,a) = 0.35$, $\mu_B^P(b,c) = 0.5$, and $\mu_B^P(d,c) = 0.4$, and negative membership values of arcs be $\mu_B^N(b,a) = -0.2$, $\mu_B^N(d,a) = -0.3$, $\mu_B^N(b,c) = -0.2$, and $\mu_B^N(d,c) = -0.2$. It is easy to see that $N^+(d) = \{(a, 0.3, -0.3), (c, 0.4, -0.2)\}$. $N^-(c) = \{(b, 0.5, -0.2), (d, 0.4, -0.2)\}$.

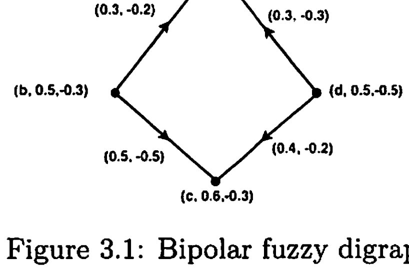

Figure 3.1: Bipolar fuzzy digraph

Definition 3.4. The bipolar fuzzy competition graph $C(D)$ of a bipolar fuzzy digraph D is an undirected bipolar fuzzy graph G which has the same bipolar fuzzy vertex set as in D and has a bipolar fuzzy edge between two vertices x, $y \in V$ in $\mathcal{C}(D)$ if and only if $\mathcal{N}^+(x) \cap \mathcal{N}^+(y)$ is non-empty bipolar fuzzy set in D and the edge positive membership and negative membership values between x and y in $\mathcal{C}(D)$ are $\mu_B^P(x,y) = (\mu_A^P(x) \wedge \mu_A^P(y))h(\mathcal{N}^+(x) \cap \mathcal{N}^+(y))$, $\mu_B^N(x,y) = (\mu_A^N(x) \vee \mu_A^N(y))h(\mathcal{N}^+(x) \cap \mathcal{N}^+(y))$.

Example 3.5. Let D be a bipolar fuzzy digraph as shown in Figure 3.2(a). Let the vertices with positive membership and negative membership values of D be $(a, 0.3, -0.4)$, $(b, 0.6, -0.6)$, $(c, 0.4, -0.5)$, $(d, 0.5, -0.5)$, $(e, 0.4, -0.5)$ with positive membership and negative membership values of arcs be $\mu_B^P(b,a) = 0.2$, $\mu_B^P(c,b) = 0.3$, $\mu_B^P(d,c) = 0.2$, $\mu_B^P(e,d) = 0.2$, $\mu_B^P(e,a) = 0.2$, $\mu_B^P(a,d) = 0.3$, $\mu_B^N(b,a) = -0.2$, $\mu_B^N(c,b) = -0.2$, $\mu_B^N(d,c) = -0.2$, $\mu_B^N(e,d) = -0.1$, $\mu_B^N(e,a) = -0.2$, $\mu_B^N(a,d) = -0.1$. By routine computations, the corresponding bipolar fuzzy competition graph is shown in Figure 3.2(b).

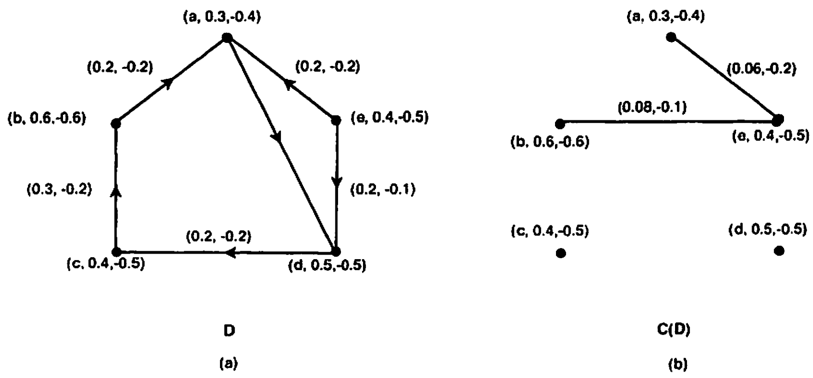

Figure 3.2: Example of bipolar fuzzy competition graph

Definition 3.6. Let k be a non-negative number. The bipolar fuzzy k-competition graph $\mathcal{C}_k(D)$ of a bipolar fuzzy digraph D is an undirected bipolar fuzzy graph G which has the same bipolar fuzzy vertex set as D and has a bipolar fuzzy edge between two vertices x, $y \in V$ in $\mathcal{C}_k(D)$ if and only if $|\mathcal{N}^+(x) \cap \mathcal{N}^+(y)| > k$. The edge positive membership and negative membership values between x and y in $\mathcal{C}_k(D)$ is $\mu_B^P(x,y) = \frac{(k'-k)}{k'}[\mu_A^P(x) \wedge \mu_A^P(y)]h(\mathcal{N}^+(x) \cap \mathcal{N}^+(y))$, $\mu_B^N(x,y) = \frac{(k'-k)}{k'}[\mu_A^N(x) \vee \mu_A^N(y)]h(\mathcal{N}^+(x) \cap \mathcal{N}^+(y))$, where $k' =$

$|\mathcal{N}^+(x) \cap \mathcal{N}^+(y)|$. A bipolar fuzzy k-competition graph is simply bipolar fuzzy competition graph when $k = 0$.

Example 3.7. Consider a bipolar fuzzy digraph D as shown in Figure 3.3(a). Let vertices with positive membership and negative membership values of D be $(x, 0.4, -0.4)$, $(y, 0.6, -0.4)$, $(a, 0.6, -0.4)$, $(b, 0.7, -0.4)$, $(c, 0.8, -0.4)$, $(d, 0.6, -0.4)$ and the positive membership and negative membership values of arcs be $\mu_B^P(x,a) = 0.3$, $\mu_B^P(x,b) = 0.3$, $\mu_B^P(x,c) = 0.3$, $\mu_B^P(x,d) = 0.4$, $\mu_B^P(y,a) = 0.3$, $\mu_B^P(y,b) = 0.5$, $\mu_B^P(y,c) = 0.4$, $\mu_B^P(y,d) = 0.3$, $\mu_B^N(x,a) = -0.2$, $\mu_B^N(x,b) = -0.2$, $\mu_B^N(x,c) = -0.1$, $\mu_B^N(x,d) = -0.2$, $\mu_B^N(y,a) = -0.2$, $\mu_B^N(y,b) = -0.2$, $\mu_B^N(y,c) = -0.2$, $\mu_B^N(y,d) = -0.2$.

Here

$$N^+(x) = \{(a, 0.3, -0.2), (b, 0.3, -0.2), (c, 0.3, -0.1), (d, 0.4, -0.2)\},$$

$$N^+(y) = \{(a, 0.3, -0.2), (b, 0.5, -0.2), (c, 0.4, -0.2), (d, 0.3, -0.2)\},$$

$$N^+(x) \cap N^+(y)$$
$$= \{(a, 0.3, -0.2), (b, 0.3, -0.2), (c, 0.3, -0.1), (d, 0.3, -0.2)\},$$

$$k' = |N^+(x) \cap N^+(y)|$$
$$= \frac{1 + 0.3 + 0.3 + 0.3 + 0.3 - 0.2 - 0.2 - 0.1 - 0.2}{2} = 0.75.$$

For $k = 2$, we have

$$\mu_B^P(x,y) = \frac{(k' - k)}{k'}[\mu_A^P(x) \wedge \mu_A^P(y)]h(\mathcal{N}^+(x) \cap \mathcal{N}^+(y))$$
$$= \frac{(0.75 - 0.2)}{0.75}(0.4)(0.3) = 0.088,$$

$$\mu_B^N(x,y) = \frac{(k' - k)}{k'}[\mu_A^N(x) \vee \mu_A^N(y)]h(\mathcal{N}^+(x) \cap \mathcal{N}^+(y))$$
$$= \frac{(0.75 - 0.2)}{0.75}(-0.1)(0.3) = -0.95.$$

Hence the corresponding bipolar fuzzy 0.2-competition graph is shown in the Figure 3.3(b).

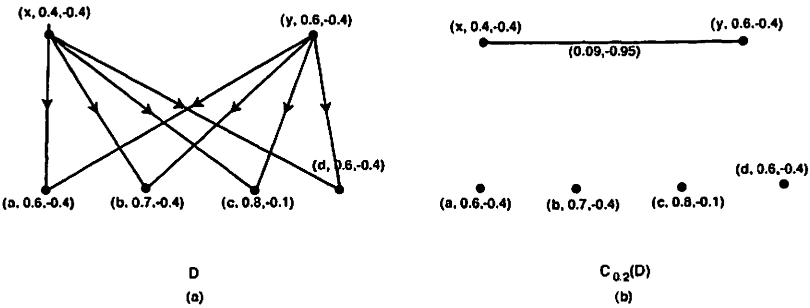

Figure 3.3: Bipolar fuzzy 0.2-competition graph

Theorem 3.8. *Let D be a bipolar fuzzy digraph. If $\mathcal{N}^+(x) \cap \mathcal{N}^+(y)$ contains single element of D, then the edge (x, y) of $C(D)$ is strong if and only if $|\mathcal{N}^+(x) \cap \mathcal{N}^+(y)| > 0.5$.*

Proof. Here D is a bipolar fuzzy digraph. Let $\mathcal{N}^+(x) \cap \mathcal{N}^+(y) = \{(a, m)\}$, where m is the positive membership value and negative membership value of the element a. Here, $|\mathcal{N}^+(x) \cap \mathcal{N}^+(y)| = m = h(\mathcal{N}^+(x) \cap \mathcal{N}^+(y))$. So, $\mu_B^P(x, y) = m \times \mu_A^P(x) \wedge \mu_A^P(y)$, $\mu_B^N(x, y) = m \times \mu_A^N(x) \vee \mu_A^N(y)$. Hence the edge (x, y) in $C(D)$ is strong if and only if $m > 0.5$. □

Remark. If all the edges of a bipolar fuzzy digraph are strong, then all the edges of the corresponding bipolar fuzzy competition graph may not be strong.

Example 3.9. Consider two vertices x, y with $\mu_A^P(x) = 0.3, \mu_A^P(y) = 0.4$, $\mu_A^N(x) = -0.3, \mu_A^N(y) = -0.4$ in a bipolar fuzzy digraph such that the vertices have a common prey z with $\mu_A^P(z) = 0.2$, $\mu_A^N(z) = -0.2$. Let $\mu_B^P(x, z) = 0.2, \mu_B^P(y, z) = 0.1, \mu_B^N(x, z) = -0.1, \mu_B^N(y, z) = -0.1$. Clearly, the edges (x, z) and (y, z) are strong. But positive membership value and negative membership value of the edge (x, y) in corresponding competition graph are $0.3 \times 0.1 = 0.13$ and -0.13, respectively. Routine calculations show that the edge is not strong.

If all the edges are strong of a bipolar fuzzy digraph, then a result can be found from the following theorem.

Theorem 3.10. *If all the edges of a bipolar fuzzy digraph D be strong, then $\frac{\mu_B^P(x,y)}{(\mu_A^P(x) \wedge \mu_A^P(y))^2} > 0.5$ for all edge (x, y) in $C(D)$.*

Proof. Let D be a bipolar fuzzy digraph and every edge of D be strong i.e., $\frac{\mu_B^P(x,y)}{\mu_A^P(x)\wedge\mu_A^P(y)} > 0.5$ for all edge (x,y) in D. Let the corresponding bipolar fuzzy competition graph be $C(D)$.

Let $\mathcal{N}^+(x) \cap \mathcal{N}^+(y)$ be a null set for all $x, y \in V$. Then there exist no edge in $C(D)$ between x and y.

$\mathcal{N}^+(x)\cap\mathcal{N}^+(y)$ is not a null set. Let $\mathcal{N}^+(x)\cap\mathcal{N}^+(y) = \{(a_1,m_1),(a_2,m_2),\ldots,(a_z,m_z)\}$, where $m_i, i = 1,2,\ldots,z$ are the membership values of $a_i, i = 1,2,\ldots,z$, respectively. So $m_i = \min\{\mu_B^P(x,a_i), \mu_B^P(y,a_i)\}, i = 1,2,\ldots,z$. Let $h(\mathcal{N}^+(x)\cap\mathcal{N}^+(y)) = \max\{m_i, i = 1,2,\ldots,z\} = m_{\max}$. $\mu_B^P(x,y) = (\mu_A^P(x)\wedge\mu_A^P(y))h(\mathcal{N}^+(x)\cap\mathcal{N}^+(y)) = m_{\max} \times \mu_A^P(x)\wedge\mu_A^P(y)$. Hence $\frac{\mu_B^P(x,y)}{(\mu_A^P(x)\wedge\mu_A^P(y))^2} = \frac{m_{\max}}{\mu_A^P(x)\wedge\mu_A^P(y)} > 0.5$. □

We have seen that if height of intersection between two out neighbourhoods of two vertices of a bipolar fuzzy digraph is greater than 0.5, the edge between the two vertices in corresponding bipolar fuzzy competition graph is strong. This result is not true in corresponding bipolar fuzzy k-competition graph. A related result is proved below.

Theorem 3.11. *Let D be a bipolar fuzzy digraph. If $h(\mathcal{N}^+(x)\cap\mathcal{N}^+(y)) = 1$ and $|\mathcal{N}^+(x)\cap\mathcal{N}^+(y)| > 2k$, then the edge (x,y) is strong in $C_k(D)$.*

Proof. Let D be a bipolar fuzzy digraph and $C_k(D)$ be the corresponding bipolar fuzzy k-competition graph. Also let, $h(\mathcal{N}^+(x)\cap\mathcal{N}^+(y)) = 1$ and $|\mathcal{N}^+(x)\cap\mathcal{N}^+(y)| > 2k$.

Now, $\mu_B^P(x,y) = \frac{k'-k}{k'}\mu_A^P(x)\wedge\mu_A^P(y)h(\mathcal{N}^+(x)\cap\mathcal{N}^+(y))$, where $k' = |\mathcal{N}^+(x)\cap\mathcal{N}^+(y)|$. So, $\mu_B^P(x,y) = \frac{k'-k}{k'}\mu_A^P(x)\wedge\mu_A^P(y)$. Hence $\frac{\mu_B^P(x,y)}{\mu_A^P(x)\wedge\mu_A^P(y)} = \frac{k'-k}{k'} > 0.5$ as $k' > 2k$. Hence the edge (x,y) is strong. □

Definition 3.12. Let $G = (V, \mu_A^P, \mu_A^N)$ be a bipolar fuzzy digraph. The m-step bipolar fuzzy competition graph of G is denoted by $C_m(G)$ and defined by $C_m(G) = (V, \mu_A^P, \mu_A^N)$ where $B = (\mu_B^P, \mu_B^N) : V\times V \to [0,1]$ such that $\mu_B^P(x,y) = \mu_A^P(x)\wedge\mu_A^P(y)\ h(\mathcal{N}_m^+(x)\cap\mathcal{N}_m^+(y))$, $\mu_B^N(x,y) = \mu_A^N(x)\vee\mu_A^N(y)\ h(\mathcal{N}_m^+(x)\cap\mathcal{N}_m^+(y))$ for all $x, y \in V$. Here $\mu_B^P(x,y)$ represents the edge positive membership value of the edge (x,y), $\mu_B^N(x,y)$ represents the edge negative membership value of the edge (x,y) in m-step bipolar fuzzy competition graph.

If a predator x attacks one prey y, then the linkage is shown by an edge (x, y) in a bipolar fuzzy digraph. But, if the predator needs the help of several mediators $x_1, x_2, \ldots, x_{m-1}$, then the linkage among them is shown by the bipolar fuzzy directed path $P^m_{x,y}$ in a bipolar fuzzy digraph. So m-step prey in a bipolar fuzzy digraph is represented by a vertex which is the m-step out neighbourhood of some vertices. Now, the strength of a bipolar fuzzy competition graphs is defined below.

Definition 3.13. Let $G = (V, \mu^P_A, \mu^N_A)$ be a bipolar fuzzy digraph. Let v be a common vertex of m-step out-neighbourhoods of vertices $x_1, x_2, \ldots, x_k$. Also let $\mu^P(y_1, z_1), \mu^P(y_2, z_2), \ldots, \mu^P(y_k, z_k)$ be the minimum membership values of edges of the paths $P^m_{(x_1,v)}, P^m_{(x_2,v)}, \ldots, P^m_{(x_n,v)}$, respectively, and let $\mu^N(y_1, z_1), \mu^N(y_2, z_2), \ldots, \mu^N(y_k, z_k)$ be the maximum membership values of edges of the paths $P^m_{(x_1,v)}, P^m_{(x_2,v)}, \ldots, P^m_{(x_n,v)}$. The m-step prey $v \in V$ is strong prey if $\mu(y_i, z_i) > 0.5$ for all $i = 1, 2, \ldots, k$.

The strength of the prey v is measured by the mapping $s : V \to [0, 1]$ such that

$$s(v) = \frac{\sum_{i=1}^{k} \mu^P(y_i, z_i)}{k}.$$

Theorem 3.14. *If a prey v of G is strong, then the strength of v, $s(v) > 0.5$.*

Theorem 3.15. *If all preys of G are strong, then all the edges of $C_m(G)$ are strong.*

The following relation is established between m-step bipolar fuzzy competition graph of a bipolar fuzzy digraph and bipolar fuzzy competition graph of m-step bipolar fuzzy digraph.

Theorem 3.16. *If G is a bipolar fuzzy digraph and G_m is the m-step bipolar fuzzy digraph of G, then $C(G_m) = C_m(G)$.*

Theorem 3.17. *Let $G = (V, \mu^P_A, \mu^N_A)$ be directed bipolar fuzzy graph. If $m > |V|$ then $C_m(G)$ has no edge.*

4 Application of bipolar fuzzy competition graphs

Due to the advancement of internet, the people can transfer money from one place to other within a very short time. During transfer of money, there are sources and destinations. Assume that there are three sources (projects) of money such as S_1, S_2, S_3, and five designations (institutions) such as D_1, D_2, D_3, D_4, D_5. The projects, institutions and their relations make a digraph in which projects and institutions are consider as vertices and the relations make directed edges. If two institutions are under a certain project, they are connected by an arc and so on. This kind of graph is called *economic competition graph.* Bipolar fuzzy graph representation is more realistic to represent the competitions. We here introduce bipolar fuzzy economic competition graphs and m-step bipolar fuzzy economic competition graphs.

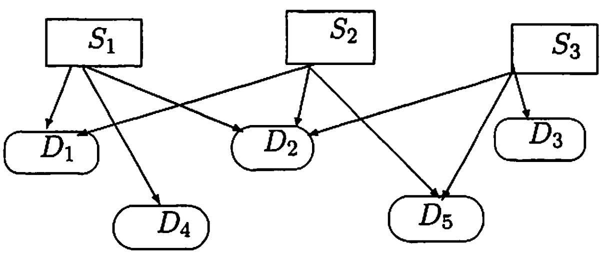

Figure 4.1: Source of Projects

Definition 4.1. The bipolar fuzzy economic competition graph $E(D) = (L, M)$ of a bipolar fuzzy digraph D is an undirected bipolar fuzzy graph $G = (A, B)$ which has the same bipolar fuzzy vertex set as in D and has a bipolar fuzzy edge between two vertices x, $y \in V$ in $E(D)$ if and only if $\mathcal{N}^-(x) \cap \mathcal{N}^-(y)$ is non-empty bipolar fuzzy set in G and the positive membership value of the edge (x, y) in $C(D)$ is $\mu_B^P(x,y) = (\mu_L^P(x) \wedge \mu_L^P(y))\ h(\mathcal{N}^-(x) \cap \mathcal{N}^-(y))$, $\mu_B^P(x,y) = (\mu_L^N(x) \vee \mu_L^N(y))\ h(\mathcal{N}^-(x) \cap \mathcal{N}^-(y))$.

Definition 4.2. Let $D = (A, B)$ be a bipolar fuzzy digraph. The *m-step bipolar fuzzy economic competition graph* is denoted by $E_m(D)$ and is defined by $E_m(D) = (A1, B1)$ where $B1 = (\mu_{B1}^P, \mu_{B1}^N) : V \times V \to [0,1] \times [-1,0]$ such that $\mu_{B1}^P(x,y) = \mu_A^P(x) \wedge \mu_A^P(y)\ h(\mathcal{N}_m^-(x) \cap \mathcal{N}_m^-(y))$, $\mu_{B1}^N(x,y) = \mu_A^N(x) \vee \mu_A^N(y)\ h(\mathcal{N}_m^-(x) \cap \mathcal{N}_m^-(y))$ for all $x, y \in V$. Here $\mu_{B1}^P(x,y)$ and $\mu_{B1}^P(x,y)$ represent the positive and negative membership values of the edge (x, y) of m-step bipolar fuzzy economic competition graph.

We state a nice theorem without proof.

Theorem 4.3. *Bipolar fuzzy competition graphs and bipolar fuzzy economic competition graphs of any complete bipolar fuzzy digraph are same.*

5 Conclusions

Fuzzy digraph theory has numerous applications in modern sciences and technology, especially in the fields of operations research, neural networks, artificial intelligence and decision making. In this paper, m-step competitions are represented by bipolar fuzzy graphs. A formula for strength of preys has been suggested and some results about strong preys have been established. The relation between bipolar fuzzy competition graph and bipolar fuzzy economic competition graph of a complete bipolar fuzzy digraphs without parallel edges has been established. Our study will help to measure the strength of indirect competitions in food web and even in economic markets. These graphs will mark the competitors in competing markets and also identify the subscribers or facilitators from a company.

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