SOME IDENTITIES INVOLVING GENOCCHI POLYNOMIALS AND NUMBERS

DAE SAN KIM AND TAEKYUN KIM

ABSTRACT. In this paper, we derive some identities involving Genocchi polynomials and numbers. These identities follow by evaluating a certain integral in various ways. Also, we express the product of two Genocchi polynomials as a linear combination of Bernoulli polynomials.

1. INTRODUCTION

The Genocchi polynomials are defined by the generating function to be

$$\frac{2t}{e^t+1}e^{xt}=\sum_{n=0}^{\infty}G_n(x)\frac{t^n}{n!},\quad \text{(see [1–10])} \tag{1.1}$$

When $x=0$, $G_n=G_n(0)$are called the Genocchi numbers. From (1.1), we note that

$$\begin{aligned}\sum_{n=0}^{\infty}G_n(x)\frac{t^n}{n!}&=\left(\frac{2t}{e^t+1}\right)e^{xt}=\left(\sum_{l=0}^{\infty}G_l\frac{t^l}{l!}\right)\left(\sum_{m=0}^{\infty}\frac{x^m}{m!}t^m\right)\\&=\sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l}G_lx^{n-l}\right)\frac{t^n}{n!}.\end{aligned} \tag{1.2}$$

Thus, by comparing the coefficients on both sides of (1.2), we get

$$G_n(x)=\sum_{l=0}^{n}\binom{n}{l}G_lx^{n-l}. \tag{1.3}$$

From (1.1), we can also derive the following recurrence relation related to Genocchi numbers:

$$G_0=0,\quad (G+1)^n+G_n=G_n(1)+G_n=2\delta_{1,n}\quad (n\geq 1), \tag{1.4}$$

with the usual convention about replacing G^n by G_n. The first few of them are $0,1,-1,0,\cdots$, and $G_{2k+1}=0$, for $k=1,2,3,\cdots$. It is well known that the Bernoulli polynomials are given by the generating function to be

$$\frac{t}{e^t-1}e^{xt}=\sum_{n=0}^{\infty}B_n(x)\frac{t^n}{n!},\quad \text{(see[4, 12])}. \tag{1.5}$$

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When $x = 0$, $B_n = B_n(0)$ are called the Bernoulli numbers. By (1.1) and (1.5), we easily get

$$\sum_{n=0}^{\infty} G_n \frac{t^n}{n!} = \frac{2t}{e^{2t}-1}e^t - \frac{2t}{e^{2t}-1} = \sum_{n=0}^{\infty} 2^n \left(B_n\left(\frac{1}{2}\right) - B_n\right)\frac{t^n}{n!}. \tag{1.6}$$

Thus, by (1.6), we have

$$G_n = 2^n\left(B_n\left(\frac{1}{2}\right) - B_n\right) = 2\left(1-2^n\right)B_n, \quad (n \geq 0).$$

From (1.1), we can derive the following equation:

$$\sum_{n=0}^{\infty} G_n(x)\frac{t^n}{n!} = -\frac{-2t}{e^{-t}+1}e^{-(1-x)t} = \sum_{n=0}^{\infty}(-1)^{n-1}G_n(1-x)\frac{t^n}{n!}. \tag{1.7}$$

Thus, by comparing the coefficients on both sides of (1.8),

$$G_n(x) = (-1)^{n-1}G_n(1-x), \quad (n \geq 0). \tag{1.8}$$

By (1.3), we see that

$$\frac{d}{dx}G_n(x) = n(G+x)^{n-1} = nG_{n-1}(x), \quad (n \geq 1). \tag{1.9}$$

Thus, from (1.9), we have

$$\int_0^1 G_n(x)\,dx = \frac{1}{n+1}\int_0^1 \frac{d}{dx}G_{n+1}(x)\,dx = \frac{1}{n+1}\left(G_{n+1}(1) - G_{n+1}\right) = -\frac{2G_{n+1}}{n+1}, \quad (n \in \mathbb{N}). \tag{1.10}$$

The gamma and beta functions are defined by the following definite integrals $(\alpha > 0, \beta > 0)$:

$$\Gamma(\alpha) = \int_0^{\infty} e^{-t}t^{\alpha-1}dt, \tag{1.11}$$

and

$$B(\alpha,\beta) = \int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\,dt = \int_0^{\infty}\frac{t^{\alpha-1}}{(1+t)^{\alpha+\beta}}dt, \quad \text{(see[10–21])}. \tag{1.12}$$

Thus, by (1.11) and (1.12), we get

$$\Gamma(\alpha+1)=\alpha\Gamma(\alpha), \quad B(\alpha,\beta)=\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}. \tag{1.13}$$

In this paper, we derive some identities involving Genocchi polynomials and numbers. These identities follow by evaluating a certain integral in various ways. Also, we express the product of two Genocchi polynomials as a linear combination of Bernoulli polynomials.

2. Identities involving Genocchi polynomials and numbers

From (1.3), we note that

$$\begin{aligned} \int_0^1 y^n G_n(x+y)\,dy &= \sum_{l=0}^{n}\binom{n}{l}G_{n-l}(x)\int_0^1 y^{n+l}dy \\ &= \sum_{l=0}^{n}\binom{n}{l}\frac{G_{n-l}(x)}{n+l+1}. \end{aligned} \tag{2.1}$$

By (1.8), we get

(2.2)

$$\begin{aligned} \int_0^1 y^n G_n(x+y)\,dy &= (-1)^{n-1}\int_0^1 y^n G_n(1-(x+y))\,dy \\ &= (-1)^{n-1}\sum_{l=0}^{n} G_{n-l}(-x)\binom{n}{l}\int_0^1 y^n(1-y)^l\,dy \\ &= \sum_{l=0}^{n}\binom{n}{l}(-1)^l G_{n-l}(1+x)\,B(n+1,l+1) \\ &= \sum_{l=0}^{n}\binom{n}{l}(-1)^l\frac{G_{n-l}(1+x)}{n+l+1}\binom{n+l}{l}^{-1}. \end{aligned}$$

Therefore, by (2.1) and (2.2), we obtain the following theorem.

Theorem 1. *For $n \geq 1$, we have*

$$\sum_{l=0}^{n}\frac{\binom{n}{l}G_{n-l}(x)}{n+l+1}=\sum_{l=0}^{n}(-1)^l\frac{G_{n-l}(1+x)}{n+l+1}\frac{\binom{n}{l}}{\binom{n+l}{l}}.$$

In particular, $x=0$,

$$\sum_{l=0}^{n}\frac{\binom{n}{l}G_{n-l}}{n+l+1}=(-1)^{n-1}\sum_{l=0}^{n}\frac{G_{n-l}}{n+l+1}\frac{\binom{n}{l}}{\binom{n+l}{l}}.$$

For $n \in \mathbb{N}$ with $n \geq 3$, from (1.9), we have

$$
\begin{aligned}
(2.3) \quad & \int_0^1 y^n G_n(x+y)\,dy \\
&= \frac{G_n(x+1)}{n+1} - \frac{n}{n+1}\int_0^1 y^{n+1} G_{n-1}(x+y)\,dy \\
&= \frac{G_n(x+1)}{n+1} - \frac{G_{n-1}(x+1)}{n+1}\frac{n}{n+2} \\
&+ (-1)^2 \frac{n(n-1)}{(n+1)(n+2)}\int_0^1 y^{n+2} G_{n-2}(x+y)\,dy \\
&= \frac{G_n(x+1)}{n+1} - \frac{nG_{n-1}(x+1)}{(n+1)(n+2)} + (-1)^2 \frac{n(n-1)G_{n-2}(x+1)}{(n+1)(n+2)(n+3)} \\
&+ \frac{(-1)^3 n(n-1)(n-2)}{(n+1)(n+2)(n+3)}\int_0^1 y^{n+3} G_{n-3}(x+y)\,dy.
\end{aligned}
$$

Continuing this process, we have

$$
\begin{aligned}
(2.4) \quad & \int_0^1 y^n G_n(x+y)\,dy \\
&= \frac{G_n(x+1)}{n+1} + \sum_{l=2}^{n-1} \frac{n(n-1)\cdots(n-l+2)(-1)^{l-1}}{(n+1)(n+2)\cdots(n+l)} G_{n-l+1}(1+x) \\
&+ (-1)^{n-1} \frac{n(n-1)\cdots 2}{(n+1)(n+2)\cdots(2n-1)}\int_0^1 y^{2n-1} G_1(x+y)\,dy \\
&= \frac{G_n(x+1)}{n+1} + \sum_{l=2}^{n-1} \frac{n(n-1)\cdots(n-l+2)}{(n+1)(n+2)\cdots(n+l)}(-1)^{l-1} G_{n-l-1}(1+x) \\
&+ (-1)^{n-1} \frac{n!}{(n+1)(n+2)\cdots 2n} \\
&= \frac{G_n(x+1)}{n+1} + \sum_{l=2}^{n-1} \frac{n(n-1)\cdots(n-l+2)}{(n+1)(n+2)\cdots(n+l)}(-1)^{l-1} G_{n-l+1}(1+x) \\
&+ (-1)^{n-1} \binom{2n}{n}^{-1}.
\end{aligned}
$$

Therefore, by (2.1) and (2.4), we obtain the following theorem.

Theorem 2. *For $n \in \mathbb{N}$ with $n \geq 3$, we have*

$$
\begin{aligned}
& \sum_{l=0}^{n} \frac{\binom{n}{l} G_{n-l}(x)}{n+l+1} \\
&= \frac{G_{n+1}(x+1)}{n+1} + \sum_{l=2}^{n-1} \frac{\binom{n}{l}(-1)^{l-1}}{\binom{n+l}{l}} \frac{G_{n-l+1}(1+x)}{n-l+1} + (-1)^{n-1}\frac{1}{\binom{2n}{n}}.
\end{aligned}
$$

From (1.8), we have $G_n = (-1)^{n-1} G_n(1)$. Taking $x = 0$, from Theorem 2, we obtain the following corollary.

Corollary 3. *For $n \geq 3$, we have*

$$(-1)^{n-1} \sum_{l=0}^{n} \frac{\binom{n}{l} G_{n-l}}{n+l+1} = \frac{-G_{n+1}}{n+1} + \sum_{l=2}^{n-1} \frac{\binom{n}{l}}{\binom{n+l}{l}} \frac{G_{n-l+1}}{n-l+1} + \frac{1}{\binom{2n}{n}}. \tag{2.5}$$

For $n \in \mathbb{N}$, we observe that

$$\begin{aligned}
&\int_0^1 y^n G_n(x+y)\,dy \\
=&\frac{G_{n+1}(x+1)}{n+1} - \frac{n}{n+1}\int_0^1 y^{n-1} G_{n+1}(x+y)\,dy \\
=&\frac{G_{n+1}(x+1)}{n+1} - \frac{n}{n+1}\int_0^1 y^{n-1}(-1)^n G_{n+1}(1-(x+y))\,dy \\
=&\frac{G_{n+1}(x+1)}{n+1} - \frac{n}{n+1}\sum_{l=0}^{n+1}\binom{n+1}{l} G_{n+1-l}(-x)(-1)^n \int_0^1 y^{n-1}(1-y)^l\,dy \\
=&\frac{G_{n+1}(x+1)}{n+1} - \frac{n}{n+1}\sum_{l=0}^{n+1}\binom{n+1}{l} G_{n+1-l}(-x)(-1)^n B(n,l+1) \\
=&\frac{G_{n+1}(x+1)}{n+1} - \frac{1}{n+1}\sum_{l=0}^{n+1}\frac{\binom{n+1}{l}}{\binom{n+l}{l}}(-1)^n G_{n+1-l}(-x) \\
=&\frac{G_{n+1}(x+1)}{n+1} - \frac{1}{n+1}\sum_{l=0}^{n+1}\frac{\binom{n+1}{l}}{\binom{n+l}{l}}(-1)^l G_{n+1-l}(1+x).
\end{aligned} \tag{2.6}$$

Therefore, by (2.2) and (2.6), we obtain the following theorem.

Theorem 4. *For $n \in \mathbb{N}$, we have*

$$\frac{G_{n+1}(x+1)}{n+1} = \frac{1}{n+1}\sum_{l=0}^{n+1}\frac{\binom{n+1}{l}}{\binom{n+l}{l}} G_{n+1-l}(1+x)(-1)^l + \sum_{l=0}^{n}\frac{(-1)^l G_{n-l}(x+1)}{n+l+1}\frac{\binom{n}{l}}{\binom{n+l}{l}}.$$

In particular, for $x = 0$,

$$\frac{(-1)^n G_{n+1}}{n+1} = \frac{1}{n+1}\sum_{l=0}^{n+1}\frac{\binom{n+1}{l}}{\binom{n+l}{l}} G_{n+1-l} - \sum_{l=0}^{n}\frac{G_{n-l}}{n+l+1}\frac{\binom{n}{l}}{\binom{n+l}{l}}.$$

Now, we observe that

$$\text{(2.7)}\qquad \int_0^1 G_n(x)\,G_m(x)\,dx$$
$$=\sum_{l=0}^{n}\binom{n}{l}G_l\,(-1)^{m-1}\sum_{k=0}^{m}\binom{m}{k}G_k\int_0^1 x^{n-l}(1-x)^{m-k}\,dx$$
$$=\sum_{l=0}^{n}\sum_{k=0}^{m}\binom{n}{l}\binom{m}{k}(-1)^{m-1}G_lG_kB(n-l+1,m-k+1)$$
$$=\sum_{l=0}^{n}\sum_{k=0}^{m}\binom{n}{l}\binom{m}{k}(-1)^{m-1}G_lG_k\frac{\Gamma(n-l+1)\Gamma(m-k+1)}{\Gamma(n+m-l-k+2)}$$
$$=\sum_{l=0}^{n}\sum_{k=0}^{m}\frac{\binom{n}{l}\binom{m}{k}}{\binom{n+m-l-k}{n-l}}(-1)^{m-1}\frac{G_lG_k}{n+m-l-k+1}.$$

For $m,n\in\mathbb{N}$ with $m,n\geq 2$, we have

(2.8)

$$\int_0^1 G_m(x)\,G_n(x)\,dx=-\frac{m}{n+1}\int_0^1 G_{n+1}(x)\,G_{m-1}(x)\,dx$$
$$=(-1)^2\frac{m(m-1)}{(n+1)(n+2)}\int_0^1 G_{n+2}(x)\,G_{m-2}(x)\,dx=\cdots$$
$$=(-1)^{m-1}\frac{(m)_{m-1}}{(n+m-1)_{m-1}}\int_0^1 G_{n+m-1}(x)\,G_1(x)\,dx$$
$$=-2\frac{(-1)^{m-1}}{\binom{n+m}{n}}G_{n+m}.$$

Therefore, by (2.7) and (2.8), we obtain the following theorem.

Theorem 5. *For $m,n\in\mathbb{N}$ with $n,m\geq 2$, we have*

$$G_{n+m}=-\frac{1}{2}\binom{n+m}{n}\sum_{l=0}^{n}\sum_{k=0}^{m}\frac{\binom{n}{l}\binom{m}{k}}{\binom{n+m-l-k}{n-l}}\frac{G_lG_k}{n+m-l-k+1}$$
$$=-\frac{1}{2}\binom{n+m}{n}\sum_{l=0}^{n}\sum_{k=0}^{m}\binom{n}{l}\binom{m}{k}\frac{G_{n-l}G_{m-k}}{l+k+1}\frac{1}{\binom{l+k}{k}}.$$

From (1.1), we have

(2.9)

$$\sum_{m,n=0}^{\infty}\left(G_m(x)\frac{G_{n+1}(x)}{n+1}+\frac{G_{m+1}(x)}{m+1}G_n(x)\right)\frac{t^m}{m!}\frac{s^n}{n!}$$
$$=\frac{d}{dx}\left(\frac{4e^{(s+t)x}}{(e^t+1)(e^s+1)}\right)=(s+t)\frac{4}{(e^t+1)(e^s+1)}e^{(s+t)x}$$

$$= \left(\frac{s+t}{e^{s+t}-1} e^{(s+t)x} \right) \left(4 - \frac{4}{e^t+1} - \frac{4}{e^s+1} \right)$$

$$= \left(\sum_{l=0}^{\infty} B_l(x) \frac{(s+t)^l}{l!} \right) \left(4 - 2\sum_{r=0}^{\infty} \frac{G_{r+1}}{r+1} \frac{t^r}{r!} - 2\sum_{r=0}^{\infty} \frac{G_{r+1}}{r+1} \frac{s^r}{r!} \right)$$

$$= \left(\sum_{m,n=0}^{\infty} B_{m+n}(x) \frac{t^m}{m!} \frac{s^n}{n!} \right) \left(-2\sum_{r=0}^{\infty} \frac{G_{2r+2}}{(2r+2)} \frac{1}{(2r+1)!} \left(t^{2r+1} + s^{2r+1} \right) \right)$$

$$= \sum_{m,n=0}^{\infty} \left(-2\sum_{r=0}^{\infty} \frac{G_{2r+2}}{(2r+2)(2r+1)!} \right.$$

$$\left. \times \left(B_{m-2r-1+n}(x) \frac{m!t^m s^n}{(m-2r-1)!n!m!} + \frac{n!B_{m+n-2r-1}(x)}{(n-2r-1)!m!n!} s^n t^m \right) \right)$$

$$= \sum_{m,n=0}^{\infty} \left(-2\sum_{r=0}^{\infty} \frac{G_{2r+2}}{2r+2} B_{m-2r-1+n}(x) \left(\binom{m}{2r+1} + \binom{n}{2r+1} \right) \right) \frac{t^m s^n}{m!n!}.$$

By comparing the coefficients on the both sides of (2.9), we obtain the following thoerem.

Theorem 6. *For $m, n \in \mathbb{N}$, we have*

$$G_m(x) \frac{G_{n+1}(x)}{n+1} + \frac{G_{m+1}(x)}{m+1} G_n(x)$$
$$= -2\sum_{r=0}^{\infty} \frac{G_{2r+2}}{2r+2} B_{m-2r-1+n}(x) \left(\binom{m}{2r+1} + \binom{n}{2r+1} \right).$$

Note that

(2.10)
$$\frac{d}{dx} \left(\frac{G_{m+1}(x)}{m+1} \frac{G_{n+1}(x)}{n+1} \right) = G_m(x) \frac{G_{n+1}(x)}{n+1} + \frac{G_{m+1}(x)}{m+1} G_n(x)$$
$$= -2\sum_{r=0}^{\infty} \frac{G_{2r+2}}{2r+2} B_{m-2r-1+n}(x) \left(\binom{m}{2r+1} + \binom{n}{2r+1} \right).$$

Thus, by (2.10), we get

$$\frac{G_{m+1}(x)}{m+1} \frac{G_{n+1}(x)}{n+1} \tag{2.11}$$
$$= -2\sum_{r=0}^{\infty} \frac{G_{2r+2}}{2r+2} \frac{B_{m+n-2r}(x)}{m+n-2r} \left(\binom{m}{2r+1} + \binom{n}{2r+1} \right) + C,$$

where C is some constant.

By (2.8) and (2.11), we get

$$C = \int_0^1 \frac{G_{m+1}(x)}{m+1} \frac{G_{n+1}(x)}{n+1} dx \tag{2.12}$$

$$
\begin{aligned}
&= \frac{2(-1)^{m+1}}{\binom{n+m+2}{n+1}} G_{n+m+2} \\
&\quad \times \frac{1}{(n+1)(m+1)}.
\end{aligned}
$$

Therefore, by (2.11) and (2.12), we obtain the following theorem.

Theorem 7. *For $m, n \in \mathbb{N}$ with $m + n \geq 2$, we have*

$$
\begin{aligned}
&\frac{G_{m+1}(x)}{m+1} \frac{G_{n+1}(x)}{n+1} \\
&= -2 \sum_{r=0}^{\infty} \frac{G_{2r+2}}{2r+2} \frac{B_{m+n-2r}(x)}{m+n-2r} \left(\binom{m}{2r+1} + \binom{n}{2r+1} \right) \\
&\quad + \frac{2(-1)^{m+1} G_{n+m+2}}{(n+m+2)(n+m+1)\binom{n+m}{n}}.
\end{aligned}
$$

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Department of Mathematics, Sogang University, Seoul 121-742, Republic of Korea
E-mail address: dskim@sogang.ac.kr

Department of Mathematics, Kwangwoon University, Seoul 139-701, Republic of Korea
E-mail address: tkkim@kw.ac.kr