Cyclic 9-coloring of plane graphs with maximum face degree six*

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Abstract

A cyclic coloring is a vertex coloring such that vertices incident with a same face receive different colors. Let G be a plane graph, and let Δ^* be the maximum face degree of G. In 1984, Borodin conjectured that every plane graph admits a cyclic coloring with at most $\lfloor\frac{3\Delta^*}{2}\rfloor$ colors. In this note, we improve a result of Borodin *et al* [On cyclic colorings and their generalizations, Discrete Mathematics 203(1999), 23-40] by showing that every plane graph with $\Delta^* = 6$ can be cyclically colored with 9 colors. This confirms the Cyclic Coloring Conjecture in the case $\Delta^* = 6$.

1 Introduction

We only consider connected and non-empty plane graphs without loops or multiple edges. Undefined concepts and terminologies are all from [3]. A *k-coloring* of a graph G is a mapping $c : V(G) \to \{1, 2, \ldots, k\}$ such that $c(u) \neq c(v)$ whenever $uv \in E(G)$. A *cyclic k-coloring* of an embedded graph is a k-coloring such that any two vertices incident with a common face receive different colors. The *cyclic chromatic number*, denoted by $\chi_c(G)$, of

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a graph G is the smallest integer k such that G admits a cyclic k-coloring. Let $\mathcal{G}$ be a family of embedded graphs. The cyclic chromatic number $\chi_c(\mathcal{G})$ of $\mathcal{G}$ is defined to be $\sup\{\chi_c(G) : G \in \mathcal{G}\}$. Let G be a plane graph. We use $\Delta^*(G)$ (or Δ^* if the graph is clear from context) to denote the *maximum face degree* of G.

For convenience, we use $\mathcal{G}$ to denote the family of plane graphs, and use $\mathcal{G}_k$ to denote the subfamily of plane graphs of which each has maximum face degree at most k.

Motivated by the dual problem of coloring maps such that countries with a common border line or a common border point get different colors, Ore and Plummer [9] introduced the concept of *cyclic chromatic number* in 1969, and proved that $\chi_c(\mathcal{G}) \leq 2 \cdot \Delta^*$. It is easy to verify that $\chi_c(\mathcal{G}_3) \leq 4$ is equivalent to the Four Color Theorem [1, 2]. In 1984, Borodin [5], confirming a conjecture of Ringel [10] on 1-planar graphs, showed that $\chi_c(\mathcal{G}_4) \leq 6$, and proposed a conjecture (which is usually referred to as the *Cyclic Coloring Conjecture*) claiming that $\chi_c(G) \leq \lfloor \frac{3\Delta^*(G)}{2} \rfloor$. This conjecture, if true, is best possible as evidenced by the graph obtained from the 3-prism by replacing the three vertical edges by paths of length $\lfloor \frac{\Delta^*}{2} \rfloor - 1$, $\lfloor \frac{\Delta^*}{2} \rfloor - 1$ and $\lceil \frac{\Delta^*}{2} \rceil - 1$, respectively (see [5]).

In 1992, Borodin [4] improve the upper bound on plane graphs to $\chi_c(\mathcal{G}) \leq 2 \cdot \Delta^* - 3$ whenever $\Delta^* \geq 8$, and proved that $\chi_c(\mathcal{G}_7) \leq 12$, $\chi_c(\mathcal{G}_6) \leq 11$, and $\chi_c(\mathcal{G}_5) \leq 9$. Then, the upper bounds were further improved to $\chi_c(\mathcal{G}) \leq \lceil \frac{5\Delta^*}{3} \rceil$ (Sanders and Zhao [11]), $\chi_c(\mathcal{G}_7) \leq 11$ (Havet, Sereni and Škrekovski [8]), $\chi_c(\mathcal{G}_6) \leq 10$ and $\chi_c(\mathcal{G}_5) \leq 8$ (Borodin, Sanders and Zhao, [7]). In [6], Borodin *et al* introduced a new parameter k^* (which is the maximum number of vertices shared by two faces), proved some new upper bound on $\chi_c(G)$ with respect to Δ^* and k^*. They also conjectured that if both Δ^* and k^* are large enough, then $\chi_c(G) \leq \Delta^* + k^*$, which implies the Cyclic Coloring Conjecture for large Δ^*.

In this note, we prove $\chi_c(\mathcal{G}_6) \leq 9$, that confirms the Cyclic Coloring Conjecture for the case $\Delta^* = 6$.

Theorem 1.1 *Every planar graph with $\Delta^* = 6$ has a cyclic 9-coloring.*

In Section 2, we prove several technical lemmas on reducible configurations. Theorem 1.1 is proved in Section 3 by discharging method.

2 Reducible Configurations

A plane graph G with $\Delta^*(G) \leq \Delta^*$ is said to be a (Δ^*, k)-*minimal* if it has no cyclic k-coloring, and is such a graph with minimum sum of the numbers of vertices and edges. This section contains some lemmas about the structure of a $(6,9)$-minimal graph related to reducible configurations.

Some notations and convention are defined as follows. The *cyclic degree* $cd(x)$ of a vertex x is defined to be the number of vertices that is different from x but lie on the boundary of some faces incident with x. By *identifying* a pair of nonadjacent vertices x and y in a graph G, we mean to replace x and y by a single vertex adjacent to all vertex of $N(x) \cup N(y)$, where $N(x)$ is the neighbor of x in G, and denote the resulting graph by $G/\{x,y\}$. To *contract* an edge xy of a graph G is to delete the edge and then identify its endvertices, and the resulting graph is denoted by G/xy. The *length* of a path or a cycle is the number of its edges. A k-vertex (resp. k^--vertex or k^+-vertex) refers to a vertex of degree k (resp. at most k or at least k). The notations k-face, k^--face and k^+-face are defined similarly. For convenience, we say that two vertices are *cyclically adjacent* if they lie on the boundary of a same face.

Now, we can state and prove our lemmas. Let G be a $(6,9)$-minimal graph. The first two lemmas are from [4].

Lemma 2.1 *[4] No i-face in G has an edge in common with a j-face if $i + j \leq \Delta^* + 2$.*

Lemma 2.2 *[4] There are no separating i-cycles in G where $i \leq \Delta^*$.*

Lemma 2.3 *No $(6,9)$-minimal graph may contain a vertex x with $cd(x) \leq 8$.*

Proof . Suppose that G is a $(6,9)$-minimal graph that contains a vertex x with $cd(x) \leq 8$. Let $e = xy \in E(G)$ and let $H = G/e$. Since $\Delta^*(H) \leq \Delta^*(G) \leq 6$, H has a cyclic 9-coloring, say c, by the minimality of G. This induces a partial cyclic coloring c' of G with only x uncolored. Since $cd(x) \leq 8$, x sees at most eight colors in c', and thus we can color x with a color distinct from all colors used on the vertices cyclically adjacent to it. This yields a cyclic 9-coloring of G, contradicting its $(6,9)$-minimality. □

Below Lemma 2.4 is an easy consequence by the minimality of G. We omit its proof.

Lemma 2.4 *Let G be a $(6,9)$-minimal graph. Then,*

(1) *G have no edge such that $\Delta^*(G - e) = \Delta^*(G)$.*

(2) *G is 2-connected.*

(3) $\delta(G) \geq 3$.

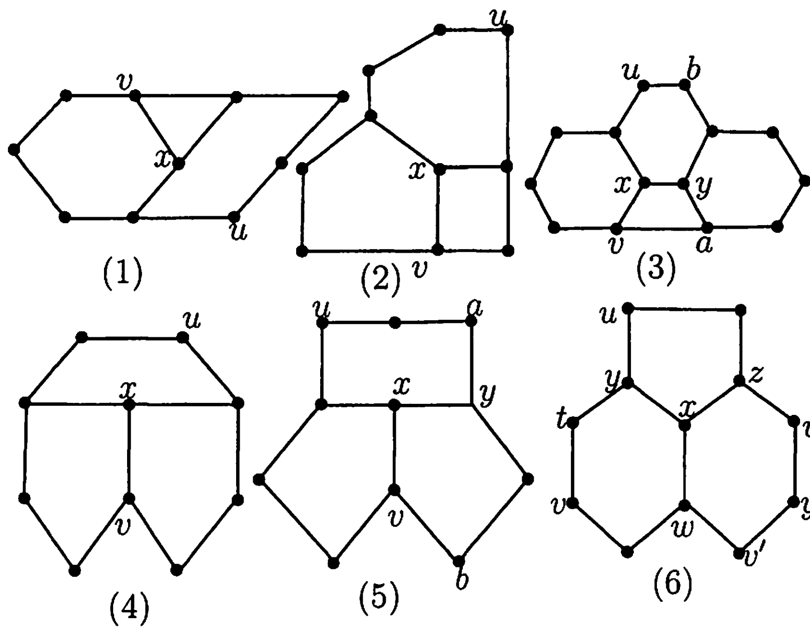

Figure 1.

Now, we show that (6, 9)-minimal graph contains no configurations listed in Figure 1.

Lemma 2.5 *Let G be a $(6, 9)$-minimal graph, and let x be a 3-vertex of G.*

(1) *x is incident with no 3-face.*

(2) *If x is incident with a 4-face, then x is incident with exactly one 4-face and two 6-faces. Furthermore, the neighbors of x incident with 4-face are 4^+-vertices.*

(3) *x cannot be incident with three 5-faces.*

(4) *If x is incident with two 5-faces, then the other face incident with x must be 6-face. Furthermore, the neighbors of x incident with 6-face are 4^+-vertices.*

(5) *If x is incident with a unique 5-face and two 6-faces, then at least one of its neighbors is a 4^+-vertex.*

Proof . By Lemma 2.4(1), G contains no edges such that $\Delta^*(G-e) = \Delta^*(G)$. It is certain that x is incident with at most one 4-face, and if x is incident with a 3-face then the other faces incident with x are both 6-faces. Since $cd(x) \geq 9$ by Lemma 2.3, if x is incident with a 4-face then it must be incident with a 6-face.

Assume to the contrary that one of (1) ~ (5) does not hold.

If x is incident with a 3-face, then it is incident with two 6-faces, and thus we have a configuration as shown in Figure 1(1). If x is incident with three 5-faces, then we have a configuration as shown in Figure 1(4). In both cases, $cd(x) = 9$. Let $H = (G/vx)/\{x, u\}$. It is clear that H is still a plane graph with $\Delta^*(H) \leq 6$, and thus has a cyclic 9-coloring c by the minimality of G. This offers a 9-coloring c' of G with only x uncolored, where $c'(u) = c'(v)$. We need to show that u and v are not cyclically adjacent. Then, c' is a partial cyclic coloring of G, and we can extend it to one of G as $cd(x) = 9$ and $c'(u) = c'(v)$. Assume to its contrary that u and v are cyclically adjacent. Let f be the face incident with both u and v, and let P be the path of length 3 that contains u and vx in Figure 1(1). Since $\Delta^* \leq 6$, P together with the short path on the boundary of f connecting u and v forms a separating i-cycle with $i \leq 6$. This contradiction to Lemma 2.2 proves (1) and (3).

Suppose that x is incident with a 4-face. Then, x is incident with at most one 4-face and must be incident with a 6-face by Lemma 2.4(1). If x is incident with a 5-face, then $cd(x) = 9$, and G has a configuration as shown in Figure 1(2). Let $H = (G/vx)/\{x, u\}$. The same argument as above shows that H has a cyclic 9-coloring which can be extended to one of G, contradicting the choice of G. Therefore, x must be incident with two 6-faces. To prove the second statement of (2), we assume to its contrary that xy is an edge incident with the 4-face such that $d(y) = 3$ (see Figure 1(3)). Now, $cd(x) = cd(y) = 10$. Let $H = (((G/vx)/\{x, u\})/ay)/\{y, b\}$. Similarly to the previous case, we may get a partial cyclic 9-coloring c of G from one of H, where only x and y are not colored, and $c(v) = c(u)$ and $c(a) = c(b)$. Since each of x and y sees at most seven colors with respect to c, we can extend c to a cyclic 9-coloring of G. This completes the proof of (2).

Now, let x be incident with two 5-faces. Then, the other face incident with x is a 6-face by (2) and (3), and thus we have a configuration as shown in Figure 1(5). If the second statement of (4) is not true, we may suppose by symmetry that $d(y) = 3$, and then $cd(x) = 10$ and $cd(y) \leq 11$. Again, by considering the graph $(((G/vx)/\{x, u\})/ay)/\{y, b\}$ as above, we get a partial cyclic 9-coloring c of G, where only x and y are not colored, and $c(v) = c(u)$ and $c(a) = c(b)$. It is easy to check that c can be extended to G by first coloring y and then coloring x. Therefore, (4) holds.

Finally, let x be incident with a unique 5-face and two 6-faces. Assume to its contrary that all the neighbors of x are 3-vertices (see Figure 1(6)). By considering a cyclic 9-coloring of $(((((G/u'z)/\{z,u\})\ /yx)/\{x,y'\})/v'w)/\{w,v\}$, we have a partial coloring c of G where x, w and z are not colored, $c(u) = c(u')$, $c(y) = c(y')$ and $c(v) = c(v')$. We will show that c is a partial cyclic coloring of G and extend it to the whole graph. Since G contains no separating i-cycles for $i \leq 6$ by Lemma 2.2, it is easy to see that neither u and u', nor v and v', are cyclically adjacent in G. If y and y' are cyclically adjacent, then y and y' must be on the boundary of the outer face of G, say f_o, such that both yu and yt are incident with f_o (since $d(y) = 3$), and thus there would be a separating 5-cycle or 6-cycle in G. Therefore, y and y' cannot be acyclically adjacent, and c is indeed a partial cyclic coloring of G. Since $cd(x) = 11, cd(z) \leq 11, cd(w) \leq 12$, we can color w, z and x sequentially to produce a cyclic 9-coloring of G. This completes the proof of (5), and thus all of the lemma. □

3 Proof of Theorem 1.1

To prove Theorem 1.1, we apply a discharging procedure to a minimal (6,9)-graph (it is 2-connected by Lemma 2.4). The *initial charge* $\omega(x)$ for each element x of $V(G) \cup F(G)$ is defined to be $4 - d(x)$. As a consequence of the Euler's formula on plane graphs,

$$\sum_{x \in V \cup F} \omega(x) = 8.$$

We will modify the charge ω to a new one ω' following rules (R_1) and (R_2) below. Since the discharging procedure does not make any change to the total sum, $\sum_{x \in V \cup F} \omega'(x) = \sum_{x \in V \cup F} \omega(x) = 8$. We will deduce a contradiction by showing that $\omega'(x) \leq 0$ for each $x \in V \cup F$.

Let x be a 3-vertex, and let T be a 3-face.

R_1 x sends $\frac{1}{6}$ to each of its incident 5-faces, and $\frac{1}{3}$ to each of its incident 6-face. For each 4^+-vertex y in $N(x)$, x sends further $\frac{1}{6}$, through y, to each 6-face incident with xy.

R_2 T sends $\frac{1}{3}$ to each of its adjacent 6-faces.

We are ready to prove Theorem 1.1.

Proof . Let G be a (6,9)-minimal graph. By Lemma 2.4(3), $\delta(G) \geq 3$.

First we calculate ω' for vertices. Let x be a k-vertex. If $k \geq 4$, then $\omega'(x) = \omega(x) \leq 0$ since R_1 and R_2 make no changes on 4^+-vertices. Suppose that $k = 3$. Then, $\omega(x) = 4 - d(x) = 1$. By Lemma 2.5(1), x is incident with no 3-face. If x is incident with one 4-face, then it is incident with two 6-faces and the neighbors of x incident with 4-face are 4^+-vertices by Lemma 2.5(2), and by R_1, x sends out $\frac{1}{3} \times 2$ to its incident 6-faces and sends further at least $\frac{1}{6} \times 2$, through its 4^+-neighbors, to its incident 6-faces. So, $\omega'(x) \leq 0$. Next, we suppose that x is incident with three 6-faces, then it sends out at least 1 into its incident 6-faces by R_1, thus $\omega'(x) \leq 0$. Finally, we distinguish two cases that x is incident with one or two 5-faces following Lemma 2.5(3)(4)(5). If x is incident with two 5-faces and one 6-face, then the neighbors of x incident with 6-face are both 4^+-vertices, and by R_1, x sends out $\frac{1}{6} \times 2$ to its incident 5-faces and $\frac{1}{3} + \frac{1}{6} \times 2$ to its incident 6-face, respectively. If x is incident with a 5-face and two 6-faces, then at least one neighbor of x incident with 6-face is 4^+-vertex, and x sends out at least $\frac{1}{3} \times 2 + \frac{1}{6} + \frac{1}{6} = 1$ by R_1. Again, $\omega'(x) \leq 0$ in both cases.

Now, we estimate ω' for faces. Let f be an l-face. If $l = 3$, then $\omega(f) = 4 - d(f) = 1$. Since 3-face is only adjacent to 6-face by Lemma 2.1, f sends out $\frac{1}{3} \times 3 = 1$ by R_2, and $\omega'(f) = \omega(f) - 1 = 0$. If $l = 4$, then $\omega'(f) = \omega(f) = 0$ since our discharging rules make no change to the weight of 4-faces. If $l = 5$, then f receives totally at most $\frac{5}{6}$ from its incident 3-vertices by R_1, and so $\omega'(f) \leq \omega(f) + \frac{5}{6} < 0$.

Suppose so that $l = 6$. Then, $\omega(f) = -2$. Let xy be an edge incident with f. If $d(x) = 3$ and $d(y) = 4$, then x sends extra $\frac{1}{6}$ to f through y by R_1. If xy is incident with a 3-face T, then $d(x) \geq 4$ and $d(y) \geq 4$ by Lemma 2.5(1), and T sends $\frac{1}{3}$ to f by R_2. For the simplicity of computation, we may count that $\frac{1}{6}$ (essentially sent out from x to f when $d(x) = 3$ and $d(y) = 4$) as being sent out from y, and count half of that $\frac{1}{3}$ (essentially sent out from T) as being sent out from y also. After such a redistributing procedure, we may suppose that f receives nothing from its adjacent faces, and receives at most $\frac{1}{3}$ from each of its incident vertices. Therefore, $\omega'(f) \leq \omega(f) + 6 \cdot \frac{1}{3} = 0$.

The above arguments yield that $\sum_{x \in V \cup F} \omega'(x) \leq 0$, contradicting $\sum_{x \in V \cup F} \omega(x) = 8$. This completes the proof. □

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