# Uniform cacti with extremal Wiener indices*

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**Abstract.** The Wiener index of a graph is the sum of the distances between all pairs of vertices. In this paper we determine $h$-cacti and $h$-cactus chains with the extremal Wiener indices, respectively.

**Key words:** Wiener index, $h$-cactus, $h$-cactus chain

**MR classification:** O 157.5

## 1 Introduction

The Wiener index was introduced by H. Wiener [8] in 1947. A lot more is done on the Wiener index than what could be mentioned here. The significant applications of the Wiener index in chemistry can be found in [7]. I. Gutman et al. [6] detailed the correlation of Wiener index with certain physicochemical properties of nonpolar organic substances. A. A. Dobrynin et al. [3, 4] have extensively researched the Wiener index in mathematics.

A *cactus* $G$ is a connected graph in which each edge lies on at most one cycle. Therefore, each block in $G$ is either an edge or a cycle. An *h-cactus* is a cactus in which each block is an $h$-cycle. An $h$-cactus chain is an $h$-cactus in which each block contains at most two cut-vertices and each cut-vertex lies in exactly two blocks. Certain invariants of a closely related class of block-cactus graphs have been studied in [2, 9]. T. Došlić and F.

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Måløy [5] considered a type of cactus chain and studied their matching and independence related properties.

In this paper we determine the $h$-cacti and $h$-cactus chains with the extremal Wiener indices, respectively.

# 2 Main results

Suppose that $d_G(u,v)$ is the distance between vertices $u$ and $v$ in a graph $G$, and let $d(v|G) = \sum\limits_{u \in V(G)} d_G(u,v)$. Then $W(G) = \frac{1}{2} \sum\limits_{v \in V(G)} d(v|G)$ is *Wiener index* of $G$. The following lemma is important and will be repeatedly used to obtain our main results.

**Lemma 2.1.** [1] *Let $G$ be a connected graph with a cut-vertex $u$ such that $G_1$ and $G_2$ are two connected subgraphs of $G$ having $u$ as the only common vertex and $G_1 \cup G_2 = G$. Then*

$$W(G_1 \cup G_2) = W(G_1) + W(G_2) + (|V(G_1)| - 1)d(u|G_2) + (|V(G_2)| - 1)d(u|G_1).$$

The number of $h$-cycles in an $h$-cactus is its *length*. Denote by $\mathscr{C}(k)$ the set of all $h$-cacti of length $k$.

**Theorem 2.1.** *If $Y, Y^* \in \mathscr{C}(k)$, then $W(Y) \equiv W(Y^*) mod(h-1)$.*

**Proof.** We proceed by induction on $k$. The case $k = 2$ is obviously true. So suppose $k \geqslant 3$.

Note that any two $Y$ and $Y^*$ of $\mathscr{C}(k)$ can be obtained from two appropriately chosen graphs $X$ and $X^*$ of $\mathscr{C}(k-1)$ by attaching to them two new $h$-cycles $C_h$ and $C_h^*$, respectively, as in Fig. 1.

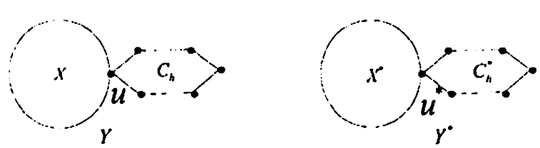


Fig. 1

By Lemma 2.1, we know that

$$W(Y) = W(X) + W(C_h) + (|V(X)| - 1)d(u|C_h) + (|V(C_h)| - 1)d(u|X),$$

$$W(Y^*) = W(X^*)+W(C_h^*)+(|V(X^*)|-1)d(u^*|C_h^*)+(|V(C_h^*)|-1)d(u^*|X^*).$$

Since $W(C_h) = W(C_h^*)$, $|V(X)| = |V(X^*)|$ and $d(u|C_h) = d(u^*|C_h^*)$, we obtain

$$W(Y) - W(Y^*) = (W(X) - W(X^*)) + (h-1)(d(u|X) - d(u^*|X^*)).$$

The result follows from inductive hypothesis that $W(X) \equiv W(X^*) mod(h-1)$. □

*An h-cactus star* is an $h$-cactus that has only one cut-vertex. Denote by $F_k$ the $h$-cactus star of length $k$. Suppose that $F_{k_i}$ is a subgraph of $G_1 \in \mathscr{C}(k)$ whose cut-vertex $u_i$ is on a cycle $C$ ($i = 1, 2$). Two vertices $u$ and $v$ on cycle $C$ are in *t-position* if $d_C(u, v) = t$. If $u_1$ and $u_2$ are in $t$-position on $C$ and $1 \le t \le \lfloor \frac{h}{2} \rfloor$, then we call the process of moving $F_{k_2}$ from $u_2$ to $u_1$ the *flower transformation* of $G_1$, and denote the resulting graph by $G_2$, as in Fig. 2.

**Lemma 2.2.** *Let $G_i$ and $u_i$ ($i = 1, 2$) be defined as above, and $T_0 = G_1 \backslash \{F_{k_1}, F_{k_2}\}$. If $d_{G_2}(u_1|T_0) \le d_{G_1}(u_2|T_0)$, then $W(G_1) > W(G_2)$.*

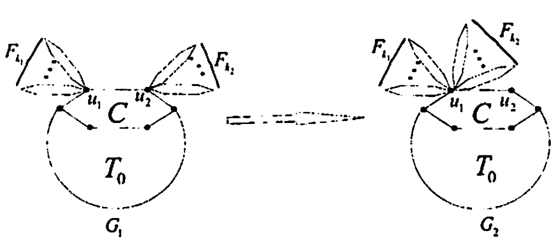


Fig. 2. Flower transformation

**Proof.** By Lemma 2.1, we have

$$\begin{aligned} W(G_1) =& W(T_0 \cup F_{k_1}) + W(F_{k_2}) + (|V(T_0 \cup F_{k_1})| - 1)d_{G_1}(u_2|F_{k_2}) \\ &+ (|V(F_{k_2})| - 1)d_{G_1}(u_2|T_0 \cup F_{k_1}), \end{aligned}$$

$$\begin{aligned} W(G_2) =& W(T_0 \cup F_{k_1}) + W(F_{k_2}) + (|V(T_0 \cup F_{k_1})| - 1)d_{G_2}(u_1|F_{k_2}) \\ &+ (|V(F_{k_2})| - 1)d_{G_2}(u_1|T_0 \cup F_{k_1}). \end{aligned}$$

Since $d_{G_1}(u_2|F_{k_2}) = d_{G_2}(u_1|F_{k_2})$, we have
$W(G_1) - W(G_2) = (|V(F_{k_2})| - 1)(d_{G_1}(u_2|T_0 \cup F_{k_1}) - d_{G_2}(u_1|T_0 \cup F_{k_1}))$.
Note that
$d_{G_1}(u_2|T_0 \cup F_{k_1}) = d_{G_1}(u_2|T_0) + (|V(F_{k_1})| - 1)d_{G_1}(u_2, u_1) + d_{G_1}(u_1|F_{k_1})$,
$d_{G_2}(u_1|T_0 \cup F_{k_1}) = d_{G_2}(u_1|T_0) + d_{G_2}(u_1|F_{k_1})$.

Since $d_{G_1}(u_2|T_0) \geq d_{G_2}(u_1|T_0)$, we have $W(G_1) - W(G_2) = (|V(F_{k_2})| - 1)(d_{G_1}(u_2|T_0) - d_{G_2}(u_1|T_0) + (|V(F_{k_1})| - 1)d_{G_1}(u_2, u_1)) > 0$. □

Suppose that $T_i \in \mathscr{C}(k_i)$ is a subgraph of $G_3 \in \mathscr{C}(k)$ that has common vertex $u_i$ with a cycle $C$ of $G_3$ $(i = 3, 4)$. If $u_3$ and $u_4$ are in $t$-position on $C$ and $1 \leq t \leq \lfloor \frac{h}{2} \rfloor$, then we call the process of moving $T_4$ from $u_4$ to $u_3$ the *inner transformation* of $G_3$, and denote the resulting graph by $G_4$, as in Fig. 3.

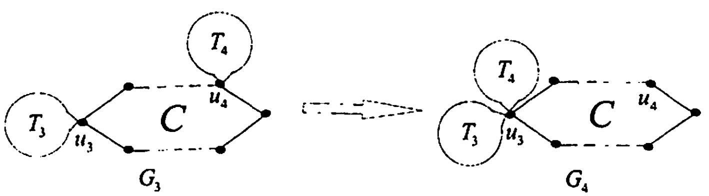


Fig. 3. Inner transformation

**Lemma 2.3.** *Let $G_i$ and $u_i$ $(i = 3, 4)$ be defined as above. Then we have $W(G_3) > W(G_4)$.*

**Proof.** By Lemma 2.1, we have

$$\begin{aligned} W(G_3) =& W(T_3 \cup C) + W(T_4) + (|V(T_3 \cup C)| - 1)d_{G_3}(u_4|T_4) \\ &+ (|V(T_4)| - 1)d_{G_3}(u_4|T_3 \cup C), \end{aligned}$$

$$\begin{aligned} W(G_4) =& W(T_3 \cup C) + W(T_4) + (|V(T_3 \cup C)| - 1)d_{G_4}(u_3|T_4) \\ &+ (|V(T_4)| - 1)d_{G_4}(u_3|T_3 \cup C). \end{aligned}$$

Since $d_{G_3}(u_4|T_4) = d_{G_4}(u_3|T_4)$, we have

$$W(G_3) - W(G_4) = (|V(T_4)| - 1)(d_{G_3}(u_4|T_3 \cup C) - d_{G_4}(u_3|T_3 \cup C)).$$

Note that

$d_{G_3}(u_4|T_3 \cup C) = (|V(T_3)| - 1)d_{G_3}(u_4, u_3) + d_{G_3}(u_3|T_3) + d_{G_3}(u_4|C)$,
$d_{G_4}(u_3|T_3 \cup C) = d_{G_4}(u_3|T_3) + d_{G_4}(u_3|C)$ and $d_{G_3}(u_4|C) = d_{G_4}(u_3|C)$.

Therefore, we have

$$W(G_3) - W(G_4) = (|V(T_4)| - 1)(|V(T_3)| - 1)d_{G_3}(u_4, u_3) > 0. \quad \square$$

We easily observe that a graph $G \in \mathscr{C}(k)$ can be transformed into some $F_k$ through a finite number of steps of flower or inner transformation. Thus, by Lemmas 2.2 and 2.3, we have the following

**Theorem 2.2.** *If $G \in \mathscr{C}(k)$ and $k \geq 3$, then $W(G) \geq W(F_k)$, with equality if and only if $G \cong F_k$.*

A cycle $C$ in an $h$-cactus chain is *internal* if it contains two cut-vertices. An internal cycle $C$ is *para* if the two cut-vertices on $C$ are in $\lfloor \frac{h}{2} \rfloor$-position. Denote by $L_k$ the *para-$h$*-cactus chain of length $k$ whose internal cycles are all para.

Suppose that $L_{k_i}$ is a subgraph of $G_5 \in \mathscr{C}(k)$ that has a common vertex $u_i$ with cycle $C$ of $G_5$ $(i = 5, 6)$ and that $u_7 \in V(L_{k_5})$ is the farthest vertex from $u_5$. If $u_5$ and $u_6$ are in $t$-position $(0 \leq t \leq \lfloor \frac{h}{2} \rfloor)$, then we call the process of moving $L_{k_6}$ from $u_6$ to $u_7$ the *lengthening transformation* of $G_5$, and denote the resulting graph by $G_6$, as in Fig. 4.

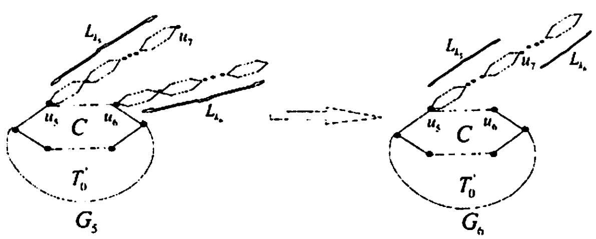

Fig. 4. Lengthening transformation.

**Lemma 2.4.** *Let $G_i$ $(i = 5, 6)$ and $u_j$ $(j = 5, 6, 7)$ be defined as above, and $T_0' = G_5 \backslash \{L_{k_5}, L_{k_6}\}$. If $d_{G_6}(u_5|T_0') \geq d_{G_5}(u_6|T_0')$, then $W(G_5) < W(G_6)$.*

**Proof.** By Lemma 2.1, we have

$$\begin{aligned} W(G_5) =& W(T_0' \cup L_{k_5}) + W(L_{k_6}) + (|V(T_0' \cup L_{k_5})| - 1)d_{G_5}(u_6|L_{k_6}) \\ &+ (|V(L_{k_6})| - 1)d_{G_5}(u_6|T_0' \cup L_{k_5}), \end{aligned}$$

$$\begin{aligned} W(G_6) =& W(T_0' \cup L_{k_5}) + W(L_{k_6}) + (|V(T_0' \cup L_{k_5})| - 1)d_{G_6}(u_7|L_{k_6}) \\ &+ (|V(L_{k_6})| - 1)d_{G_6}(u_7|T_0' \cup L_{k_5}). \end{aligned}$$

Since $d_{G_5}(u_6|L_{k_6}) = d_{G_6}(u_7|L_{k_6})$, we have
$W(G_6) - W(G_5) = (|V(L_{k_6})| - 1)(d_{G_6}(u_7|T_0' \cup L_{k_5}) - d_{G_5}(u_6|T_0' \cup L_{k_5}))$.

Note that

$$\begin{aligned} d_{G_5}(u_6|T_0' \cup L_{k_5}) &= d_{G_5}(u_6|T_0') + d_{G_5}(u_5|L_{k_5}) + (|V(L_{k_5})| - 1)t, \\ d_{G_6}(u_7|T_0' \cup L_{k_5}) &= d_{G_6}(u_5|T_0') + \lfloor \tfrac{h}{2} \rfloor k_8(|V(T_0')| - 1) + d_{G_6}(u_7|L_{k_5}). \end{aligned}$$

Since $d_{G_6}(u_5|T_0') \geq d_{G_5}(u_6|T_0')$, we have

$$\begin{aligned} W(G_6) - W(G_5) &= (|V(L_{k_6})| - 1)(d_{G_6}(u_5|T_0') - d_{G_5}(u_6|T_0') \\ &+ (\lfloor \tfrac{h}{2} \rfloor (k - k_5 - k_6) - t)(h-1)k_5) > 0. \ \square \end{aligned}$$

We easily observe that a graph $G \in \mathscr{C}(k)$ can be transformed into some $L_k$ through a finite number of steps of the lengthening transformation. Thus, by Lemma 2.4, we have the following

**Theorem 2.3.** *If $G \in \mathscr{C}(k)$ and $k \geq 3$, then $W(G) \leq W(L_k)$, with equality if and only if $G \cong L_k$.*

An internal cycle $C$ in an $h$-cactus chain is *ortho* if the two cut-vertices on $C$ are in 1-position. Denote by $H_k$ the *ortho-$h$*-cactus chain of length $k$ whose internal cycles are all ortho. Now we give explicit expression of Wiener indices of $F_k$, $L_k$ and $H_k$.

**Remark 2.1.** *The Wiener index of the $h$-cactus star $F_k$ is given by*

$$W(F_k) = \begin{cases} \frac{h^2(h-1)k^2}{4} - \frac{h^2(h-2)k}{8}, & \text{if } h \text{ is even;} \\ \frac{(h-1)^2(h+1)k^2}{4} - \frac{(h-2)(h-1)(h+1)k}{8}, & \text{if } h \text{ is odd.} \end{cases}$$

**Proof.** Let $C$ be an $h$-cycle in $F_k$. Then $F_k = F_{k-1} \cup C$. Let $x$ be the cut-vertex of $F_k$. By Lemma 2.1, we have

$$W(F_k) = W(C) + W(F_{k-1}) + (h-1)d(x|F_{k-1}) + (h-1)(k-1)d(x|C).$$

If $h$ is even, then $W(C) = \frac{h^3}{8}$, $d(x|C) = \frac{h^2}{4}$ and $d(x|F_{k-1}) = \frac{h^2(k-1)}{4}$, and so

$$W(F_k) = W(F_{k-1}) + \frac{h^3}{8} + \frac{h^2(h-1)(k-1)}{2}.$$

Using iteration method, we obtain $W(F_k) = \frac{h^2(h-1)k^2}{4} - \frac{h^2(h-2)k}{8}$.

Similarly, we can prove that $W(F_k) = \frac{(h-1)^2(h+1)k^2}{4} - \frac{(h-2)(h-1)(h+1)k}{8}$ if $h$ is odd. $\square$

We also have the following Remarks 2.2 and 2.3

**Remark 2.2.** *The Wiener index of the para-h-cactus chain $L_k$ is given by*

$$W(L_k) = \begin{cases} \frac{h(h-1)^2k^3}{12} + \frac{(h^2-h)k^2}{4} - \frac{(h^3-2h^2-2h)k}{24}, & \text{if } h \text{ is even;} \\ \frac{(h-1)^3k^3}{12} + \frac{(h-1)^2k^2}{2} - \frac{(5h^3+6h^2-21h+10)k}{24}, & \text{if } h \text{ is odd.} \end{cases}$$

**Remark 2.3.** The Wiener index of the ortho-$h$-cactus chain $H_k$ is given by

$$W(H_k) = \begin{cases} \frac{h(h-1)k^3}{6} + \frac{h(h-1)(h-2)k^2}{4} + \frac{(5h^3+6h^2-8h)k}{24}, & \text{if } h \text{ is even;} \\ \frac{(h-1)hk^3}{6} + \frac{(h^3+h^2+h+1)k^2}{4} - \frac{(3h^3-14h^2+5h+6)k}{24}, & \text{if } h \text{ is odd.} \end{cases}$$

**Theorem 2.4.** *Suppose that $G \in \mathscr{C}(k)$ is an h-cactus chain. Then $W(H_k) \leq W(G)$, with equality if and only if $G \cong H_k$.*

**Proof.** Suppose that $T_i \in \mathscr{C}(k_i)$ is a subgraph of $h$-cactus chain $G$ that has a common vertex $v_i$ $(i = 1, 2)$ with cycle $C$ so that $G = T_1 \cup C \cup T_2$, as in Fig. 5.

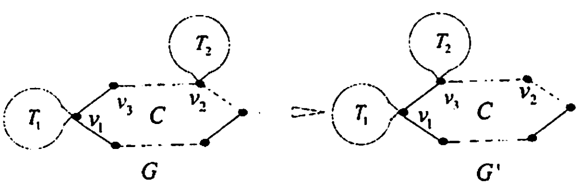


Fig. 5

Suppose that $v_3 \in C$ is adjacent to $v_1$. Then we denote by $G'$ the graph obtained from $G$ by moving $T_2$ from $v_2$ to $v_3$. If $v_1$ and $v_2$ are in $t$-position $(1 \leq t \leq \lfloor \frac{h}{2} \rfloor)$, then, by Lemma 2.1, we have

$$\begin{aligned} W(G) &= W(T_1) + W(C \cup T_2) + (|V(T_1)| - 1)d_G(v_1|C \cup T_2) \\ &+ (|V(T_2)| + h - 2)d_G(v_1|T_1) \\ &= W(T_1) + W(T_2) + (h-1)d_G(v_2|T_2) + (|V(T_2)| - 1)d_G(v_2|C) \\ &+ W(C) + (|V(T_1)| - 1)d_G(v_1|C \cup T_2) + (|V(T_2)| + h - 2)d_G(v_1|T_1) \\ &= W(T_1) + W(C) + W(T_2) + (|V(T_2)| - 1)d_G(v_2|C) \\ &+ (|V(T_1)| - 1)(d_G(v_1|C) - t) + t(|V(T_1)| - 1)|V(T_2)| \\ &+ (|V(T_1)| + h - 2)d_G(v_2|T_2) + (|V(T_2)| + h - 2)d_G(v_1|T_1). \end{aligned}$$

Similarly, we can obtain

$$\begin{aligned} W(G') = & W(T_1) + W(C) + W(T_2) + (|V(T_2)| - 1)d_{G'}(v_3|C) \\ & + (|V(T_1)| - 1)(d_{G'}(v_1|C) - 1) + (|V(T_1)| - 1)|V(T_2)| \\ & + (|V(T_1)| + h - 2)d_{G'}(v_3|T_2) + (|V(T_2)| + h - 2)d_{G'}(v_1|T_1). \end{aligned}$$

Note that $d_G(v_2|C) = d_{G'}(v_3|C)$, $d_G(v_2|T_2) = d_{G'}(v_3|T_2)$ and $d_G(v_1|T_1) = d_{G'}(v_1|T_1)$. Therefore, we have

$$W(G) - W(G') = (t-1)(|V(T_1)| - 1)(|V(T_2)| - 1) \geq 0.$$

This shows that the assertion is true. □

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