Crossing Numbers of Nearly Complete Graphs and Nearly Complete Bipartite Graphs

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Abstract

We determine the crossing numbers (i) of the complete graph K_n with an edge deleted for $n \leq 12$ and (ii) of the complete bipartite graph $K_{m,n}$ with an edge deleted for $m \in \{3,4\}$ and for all natural numbers n, and also for the case $m = n = 5$.

1 Introduction

Let G be a graph whose sets of vertices and edges are denoted by $V(G)$ and $E(G)$ respectively. By a *drawing* of G, denoted $D(G)$, we mean a representation of G in the plane where the vertices of G are represented by distinct points and each edge of G by a simple continuous arc connecting the corresponding pair of points. We further assume that, in such a drawing, (i) no edge passes through any vertex other than its endpoints, (ii) any two edges do not touch each other and they cross each other at most once and (iii) no three edges cross at the same point.

Let $D = D(G)$ be a drawing of a graph G and let $e \in E(G)$. With respect to D, the *responsibility* of e, denoted $r^*_D(e)$ (or just $r^*(e)$ if we do not wish to emphasize the drawing D), is defined to be the number of crossings made by e with the other edges of G. If $v \in V(G)$, the *responsibility* of v, denoted $r_D(v)$ (or just $r(v)$), is defined to be the sum of the responsibilities of all edges incident to v. That is, $r(v) = \sum_{u \in N(v)} r^*(uv)$, where $N(v)$ denotes the neighborhood of v.

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If the drawing $D(G)$ of G achieves the minimum number of crossings, then $D(G)$ is called an *optimal* drawing of G. The *crossing number* of G, denoted $cr(G)$, is the number of crossings in an optimal drawing of G.

A well-known conjecture, called Guy's Conjecture, asserts that the crossing number of the complete graph K_m is $Z(m) = \frac{1}{4}\lfloor\frac{m}{2}\rfloor\lfloor\frac{m-1}{2}\rfloor\lfloor\frac{m-2}{2}\rfloor\lfloor\frac{m-3}{2}\rfloor$. Guy's Conjecture is known to be true for $m \leq 12$ (see [7]).

As for the complete bipartite graph $K_{m,n}$, it has been conjectured that $cr(K_{m,n}) = Z(m,n)$ where $Z(m,n) = \lfloor\frac{m}{2}\rfloor\lfloor\frac{m-1}{2}\rfloor\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor$. This conjecture, called Zarankiewicz's Conjecture, is known to be true for $m \leq 6$ and all n, (see [5]) and also for $m \in \{7, 8\}$ with $n \leq 10$ (see [10]).

In the present paper, we attempt to determine the crossing numbers of graphs which are very close to being complete graphs or complete bipartite graphs. These are graphs that are obtained by deleting an edge from each of K_n and $K_{m,n}$.

Incidentally, we note the following. Much work been done on showing the existence of an edge whose deletion does not reduce the crossing number too much. See [2], [3], [6], [8] and [9]. Here, we look for an edge whose deletion reduces the crossing number by the most. This might be a generally interesting problem. Of course, in this case, every edge of K_n or $K_{m,n}$ reduces the crossing number by the same amount. But then this could be a starting point for a more general problem.

2 Complete graphs

We begin with the following lemma. Let $Z^*(m) = Z(m) - \binom{\lfloor\frac{m-1}{2}\rfloor}{2}$.

Lemma 1 *For any edge e in K_n, $cr(K_n - e) \leq Z^*(n)$ where $n \geq 5$.*

PROOF: Consider the case $n = 2m$ first.

Take a cylinder and place m vertices $x_1, x_2, \ldots, x_m$ on the circular edge of the top surface (in a clock-wise manner) and join any pair of vertices (so that the edges appear on the top face). Do the same to the bottom face with another m vertices $y_1, y_2, \ldots, y_m$ so that x_i is vertically above y_i for all $i = 1, 2, \ldots, m$.

Now for each $i = 1, 2, \ldots m$, join x_i to $y_i, y_{i+1}, \ldots, y_m, y_1, \ldots, y_{i-1}$ where each vertex y_j appears to be on the bottom left position of x_i. In so doing, all edges x_iy_j appear to be on the "left-hand side" of x_iy_i (see Figure 1 for the case $m = 5$). This gives a drawing D of K_{2m} having $Z(2m)$ crossings. Moreover, $r^*_D(x_my_m) = \binom{m-1}{2}$.

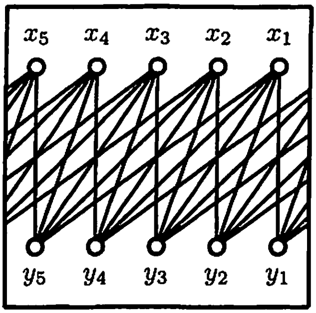

Figure 1: Part of the drawing of K_{10} viewed from the side of the cylinder.

For $n = 2m + 1$, to the drawing D, we add the vertex y_{m+1} on the bottom face of the cylinder so that it lies in between y_1 and y_m (and join it to all the $y_i's$). Join y_{m+1} to all the $x_j's$ which appear to be on the top right position of y_{m+1}. This gives a drawing D' of K_{2m+1} with $Z(2m+1)$ crossings. Moreover $r^*_{D'}(x_m y_m) = \binom{m}{2}$.

This completes the proof. □

Lemma 2 *Let $u_1u_2 \in E(K_n)$ and let D be an optimal drawing of $K_n - u_1u_2$ on the plane with c crossings. Then either $r_D(u_1) + r_D(u_2) \leq 2(c - Z(n-1))$ or else $cr(K_{n-1}) < Z(n-1)$.*

PROOF: Assume that $cr(K_{n-1}) = Z(n-1)$.

Note that $K_n^- - u_1$ is K_{n-1}. If $r_D(u_1) > c - Z(n-1)$, then $cr(K_n^- - u_1) = c - r_D(u_1) < c - (c - Z(n-1)) = Z(n-1)$ which contradicts the assumption. Hence $r_D(u_1) \leq c - Z(n-1)$ and similarly $r_D(u_2) \leq c - Z(n-1)$. But this implies that $r_D(u_1) + r_D(u_2) \leq 2(c - Z(n-1))$. □

Theorem 1 *Suppose $cr(K_m) = Z(m)$ for $m \in \{n-1, n\}$. Then for any edge e in K_n,*

$$\left\lceil \frac{n-2}{n}\left(Z(n) - \left\lfloor \frac{n-3}{2} \right\rfloor\right) + \frac{2}{n} Z(n-1) \right\rceil \leq cr(K_n - e) \leq Z^*(n)$$

for any natural number $n \geq 5$.

PROOF: It follows easily from Lemma 1 that $cr(K_n - e) \leq Z^*(n)$.

Suppose $e = u_1u_2$. Let $K_n^- = K_n - e$ and suppose $cr(K_n^-) = c$. Let D be an optimal drawing of K_n^- in the plane with c crossings. Then $K_n^- - u_1$

is K_{n-1}. By Lemma 2, we have

$$r_D(u_1) + r_D(u_2) \le 2(c - Z(n-1)). \quad (1)$$

Let $S = V(K_n) - \{u_1, u_2\}$ and let $v \in S$. We shall show that

$$Z(n) - c - \left\lfloor \frac{n-3}{2} \right\rfloor \le r_D^*(u_1v) + r_D^*(vu_2). \quad (2)$$

Now, from the optimal drawing D of K_n^-, we can obtain a drawing D' of K_n by adding the edge u_1u_2 using the so called embedding method. Recall that this can be done by drawing the edge u_1u_2 so that it is "as close as possible" to the path u_1vu_2 as is shown (by the dotted line) in Figure 2. (See [3] for more detail for the embedding method.)

Clearly, the responsibility of u_1u_2 in D' comes from the number of crossings made by (i) u_1u_2 and some edges incident to v, and by (ii) u_1u_2 with all edges which cross with u_1v and vu_2 (see Figure 2).

To minimize the responsibility of u_1u_2 in case (i), we draw u_1u_2 so that it crosses with at most $\lfloor \frac{n-3}{2} \rfloor$ edges (that are incident with v).

Note that in case (ii), the number of crossings is $r_D^*(u_1v) + r_D^*(vu_2)$. If $r_D^*(u_1v)+r_D^*(vu_2) < Z(n)-c-\lfloor \frac{n-3}{2} \rfloor$, then a drawing of K_n with less than $Z(n)$ crossings is produced, a contradiction. This proves the inequality in (2).

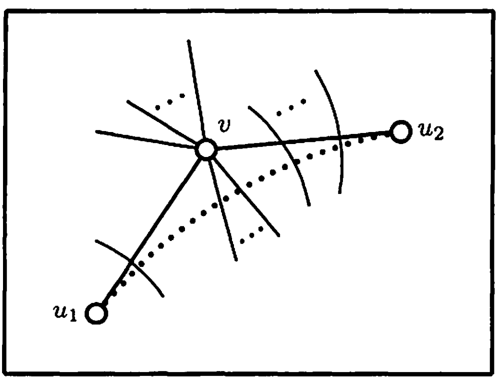

Figure 2: Adding a new edge u_1u_2 close to the path u_1vu_2.

Summing the inequality in (2) over all vertices v in S we have

$$\sum_{v\in S}(Z(n) - c - \left\lfloor \frac{n-3}{2} \right\rfloor) \le \sum_{v\in S}(r_D^*(u_1v) + r_D^*(vu_2)).$$

Since $r_D(u) = \sum_{w \in N(u)} r^*_D(uw)$, we have $\sum_{v \in S} r^*_D(u_1 v) = r_D(u_1)$ and $\sum_{v \in S} r^*_D(vu_2) = r_D(u_2)$. Therefore it follows from the inequality in (1) that

$$(n-2)\left(Z(n) - c - \left\lfloor \frac{n-3}{2} \right\rfloor\right) \leq 2(c - Z(n-1))$$

and this implies

$$c \geq \left\lceil \frac{n-2}{n}\left(Z(n) - \left\lfloor \frac{n-3}{2} \right\rfloor\right) + \frac{2}{n} Z(n-1) \right\rceil$$

and the proof is complete. □

Remark 1 *We note that the gap between the lower and upper bounds in Theorem 1 is a linear function of the form cn for some positive number $c \leq 1/8$.*

Theorem 2 *Suppose $n \geq 3$ is a natural number such that $cr(K_{2n-1}) = Z(2n-1)$. Then, for any edge e of K_{2n}, $cr(K_{2n} - e) = Z^*(2n)$.*

PROOF: Suppose $cr(K_{2n} - e) = c < Z^*(2n) = Z(2n) - \binom{n-1}{2}$. Let D be an optimal drawing of $K_{2n} - e$ in the plane with c crossings. Suppose $e = u_1 u_2$ and let $A = r_D(u_1) + r_D(u_2)$. Then, by Lemma 2, we have

$$A \leq 2(c - Z(2n-1)). \tag{3}$$

Let $S = V(K_{2n}) - \{u_1, u_2\}$. We claim that there is a vertex $v \in S$ such that

$$r_D(v) \geq c - Z(2n-1) + 1 + r^*_D(u_1 v) + r^*_D(vu_2). \tag{4}$$

To see this, assume on the contrary that there is no such vertex in S. Then

$$r_D(v) \leq c - Z(2n-1) + r^*_D(u_1 v) + r^*_D(vu_2) \tag{5}$$

for any vertex $v \in S$. By summing up the inequality in (5) over all vertices in S, we have

$$\sum_{v \in S} r_D(v) \leq \sum_{v \in S} (c - Z(2n-1)) + \sum_{v \in S} (r^*_D(u_1 v) + r^*_D(vu_2)).$$

That is,

$$4c - A \leq (2n-2)(c - Z(2n-1)) + A$$

which yields

$$2c \leq (n-1)(c - Z(2n-1)) + A$$

and hence

$$2c \leq (n-1)(c - Z(2n-1)) + 2(c - Z(2n-1))$$

by the inequality in (3). Therefore

$$c \geq \frac{n+1}{n-1} Z(2n-1).$$

It is routine to check that $\frac{n+1}{n-1} Z(2n-1) = Z^*(2n)$ and this contradicts the assumption that $c < Z^*(2n)$.

Now, let $v \in S$ be a vertex that satisfies the inequality in (4). Now add the edge $u_1 u_2$ to the drawing D (again using the embedding method) so that it is "as close as possible" to the path $u_1 v u_2$ (see Figure 2) and produce a drawing D' of K_{2n} with

$$c + r_D^*(u_1 v) + r_D^*(v u_2) + d$$

crossings, where d is the number of crossings made by $u_1 u_2$ with some edges incident to v in D.

Note that, in this drawing D' of K_{2n}, the responsibility of v is given by $r_{D'}(v) = r_D(v) + d$.

If we delete the vertex v from the drawing D' of K_{2n}, we get a drawing of K_{2n-1} with

$$c + r_D^*(u_1 v) + r_D^*(v u_2) + d - (r_D(v) + d) = c'$$

crossings. But then

$$\begin{aligned} c' &= c + r_D^*(u_1 v) + r_D^*(v u_2) - r_D(v) \\ &\leq c + r_D^*(u_1 v) + r_D^*(v u_2) - \\ &\quad (c - Z(2n-1) + 1 + r_D^*(u_1 v) + r_D^*(v u_2)) \qquad \text{by (4)} \\ &= Z(2n-1) - 1, \end{aligned}$$

a contradiction since $cr(K_{2n-1}) = Z(2n-1)$. Hence $cr(K_{2n} - e) \geq Z^*(2n)$. Combining this with Lemma 1, we have $cr(K_{2n}) = Z^*(2n)$. □

Remark 2 *Note that Theorem 2 gives a precise formula only for graphs with even number of vertices while Theorem 1 provides a lower bound for graphs with odd number of vertices.*

Corollary 1 *For any edge e in K_n, $cr(K_n - e) = Z^*(n)$ for each natural number $n \leq 12$.*

PROOF: If $n \leq 11$ and $n \neq 10, 12$, the result follows from Theorem 1 since the upper bound equals the lower bound. For $n = 10, 12$, the result follows from Theorem 2. □

Conjecture 1 *Let e be an edge in K_m. Then $cr(K_m - e) = Z^*(m)$.*

3 Complete bipartite graphs

We now turn our attention to complete bipartite graphs. In what follows, let the vertices of $K_{m,n}$ be denoted by $a_1, \ldots, a_m, b_1, \ldots, b_n$ where a_i is adjacent to b_j for every $1 \leq i \leq m$ and $1 \leq j \leq n$. Further, unless otherwise stated, we shall assume that $e = a_1 b_1$.

Let $Z^*(m,n) = Z(m,n) - \lfloor \frac{m-1}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor$.

Lemma 3 *For any edge e in $K_{m,n}$, $cr(K_{m,n} - e) \leq Z^*(m,n)$.*

PROOF: For ease of reference, we recall the following drawing of $K_{m,n}$ having $Z(m,n)$ crossings due to Zarankiewicz [11].

Place the vertices of $K_{m,n}$ at the coordinators $(i, 0)$ and $(0, j)$ where $-\lfloor \frac{m}{2} \rfloor \leq i \leq \lceil \frac{m}{2} \rceil$, $-\lfloor \frac{n}{2} \rfloor \leq i \leq \lceil \frac{n}{2} \rceil$ and $i, j \neq 0$. Then join $(i, 0)$ to $(j, 0)$ with straight line segment. Then one can verify that the responsibility of the edge $(0,1)(\lceil \frac{m}{2} \rceil, 0)$ is $\lfloor \frac{m-1}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor$. This proves the lemma. □

Theorem 3 *Suppose e is an edge in $K_{m,n}$. Then*

$$\left\lceil \frac{(n-1)cr(K_{m,n-1}-e)+cr(K_{m,n-1})}{n-2} \right\rceil \leq cr(K_{m,n} - e) \leq Z^*(m,n).$$

PROOF: The upper bound follows from Lemma 3.

Consider a drawing D of $K_{m,n} - e$. If we delete a vertex b_j from D, we obtain either a drawing of $K_{m,n-1}$ when $j = 1$ or else a drawing of $K_{m,n-1} - e$ when $j \neq 1$ (there are $n-1$ such j). Since any crossing $a_i b_j, a_k b_l$ occurs in $n - 2$ of these drawings (where neither b_j nor b_k is deleted), we have

$$(n-2)cr(K_{m,n} - e) \geq (n-1)cr(K_{m,n-1} - e) + cr(K_{m,n-1})$$

and this proves the lower bound. □

Corollary 2 *Suppose $n \geq 3$ is a natural number. Let e be an edge in $K_{m,n}$. Then*
(i) $cr(K_{3,n} - e) = Z^*(3, n)$ and (ii) $cr(K_{4,n} - e) = Z^*(4, n)$.

PROOF: (i) Since $K_{3,n} - e$ contains a subgraph $K_{3,n-1}$, we have $cr(K_{3,n} - e) \geq cr(K_{3,n-1}) = Z(3, n-1) = Z^*(3, n)$ and the proof then follows from Theorem 3.

(ii) Evidently,

$$\left\lceil \frac{(n-1)Z^*(4, n-1) + Z(4, n-1)}{n-2} \right\rceil = Z^*(4, n)$$

if $n \geq 4$. Since $cr(K_{4,3} - e) = cr(K_{3,4} - e) = Z^*(3, 4) = Z^*(4, 3)$ by (i), the result follows from Theorem 3 and by induction on n. □

Remark 3 Alternatively, we may prove Theorem 3 in the following way. Suppose $cr(K_{m,n} - e) = c$ where $e = a_1 b_1$. Let D be a drawing of $K_{m,n} - e$ with c crossings. Then

$$r_D(b_1) \leq c - cr(K_{m,n-1})$$

otherwise $(K_{m,n} - e) - b_1$ (which is isomorphic to $K_{m,n-1}$) has less than $cr(K_{m,n-1})$ crossings. Likewise,

$$r_D(b_i) \leq c - cr(K_{m,n-1} - e)$$

for any $i = 2, \ldots, n$. As such, we have

$$2c = \sum_{i=1}^{n} r_D(b_i) \leq c - cr(K_{m,n-1}) + (n-1)(c - cr(K_{m,n-1} - e))$$

and this implies the lower bound in Theorem 3. □

Proposition 1 $cr(K_{5,5} - e) = Z^*(5, 5)$ *for any edge e in $K_{5,5}$.*

PROOF: By Theorem 3 and Corollary 2, we have $11 \leq cr(K_{5,5} - e) \leq 12$.

If $cr(K_{5,5} - e) \neq Z^*(5, 5)$, then $cr(K_{5,5} - e) = 11$. Let D be a drawing of $K_{5,5} - e$ with 11 crossings. From the proof of Remark 3, we have

$$r_D(a_1),\ r_D(b_1) \leq 11 - cr(K_{5,4}) = 3 \quad (6)$$
$$r_D(a_i),\ r_D(b_i) \leq 11 - cr(K_{5,4} - e) = 5 \quad (7)$$

for $2 \leq i \leq 5$. As such, $22 = 2cr(K_{5,5}-e) = \sum_{i=1}^{5} r_D(a_i) = \sum_{i=1}^{5} r_D(b_i) \leq 23$ which means that $r_D(a_1) \in \{2,3\}$. This implies that there is an edge a_1b_j such that $r^*_D(a_1b_j) = 0$.

We claim that $r^*_D(b_ja_i) + r^*_D(a_ib_1) \geq 2$ for every $i \in \{2,\dots,5\}$.

To see this, assume that $r^*_D(b_ja_i)+r^*_D(a_ib_1) < 2$ for some $i \in \{2,\dots,5\}$. Now we can obtain a drawing D' of $K_{5,5}$ by joining a new edge a_1b_1 so that it is as close to the path $a_1b_ja_ib_1$ as possible. Further, this new edge can be drawn in such a way that it crosses at most twice with edges of the form b_ja_r (where $r \neq 1$) and at most once with edges of the form a_ib_k (where $k \neq 1, j$). But this implies that the number of crossings in D' is at most $4 + 11$ which is impossible (since $cr(K_{5,5}) = 16$).

It follows from the claim that $\sum_{i\neq 1}(r^*_D(b_ja_i) + r^*_D(a_ib_1)) \geq 8$. That is, $r_D(b_j) + r_D(b_1) \geq 8$. This together with (6) and (7) imply that $r_D(b_j) = 5$ and $r_D(b_1) = 3$. Consequently, $r^*_D(b_ja_i) + r^*_D(a_ib_1) = 2$ for all $i \geq 2$. This means that we can obtain a drawing D' of $K_{5,5}$ by joining a new edge a_1b_1 so that it is as close to the path $a_1b_ja_ib_1$ as possible for some $i \geq 2$ and that it is routine to check that the responsibility of a_1b_1 in D' is at most 4. However this is a contradiction (since $cr(K_{5,5}) = 16$). This completes the proof. □

Conjecture 2 *Let e be an edge in $K_{m,n}$. Then $cr(K_{m,n} - e) = Z^*(m,n)$.*

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References

[1] L. Beineke and R. Wilson, The early history of the brick factory problem, *Math. Intellingencer* **32** (2010) 42–48.

[2] J. Černý, J. Kynčl and G. Tóth, Improvement on the decay of crossing numbers, *Lecture Notes in Computer Science* **4875** Springer-Verlag (2008) 25–30.

[3] J. Fox and C.D. Tóth, On the decay of crossing numbers, *J. Combin. Theory Ser. B* **98** (2008) 33–42.

[4] F. Harary and A. Hill, On the number of crossings in a complete graph, *Proc. Edinb. Math. Soc. (II)* **13** (1962-63) 333–338.

[5] D.J. Kleitman, The crossing number of $K_{5,n}$, *J. Combin. Theory* **9** (1971) 315–323.

[6] J. Pach and G. Tóth, Thirteen problems on crossing numbers, *Geombinatorics* **9** (2000) 194–207.

[7] S. Pan and R.B. Richter, The crossing number of K_{11} is 100, *J. Graph Theory* **56** (2007) 128–134.

[8] R.B. Richter and C. Thomassen, Minimal graphs with crossing number at least k, *J. Combin. Theory Ser. B* **58** (1993) 217–224.

[9] G. Salazar, On the crossing number result of Richter and Thomassen, *J. Combin. Theory Ser. B* **79** (2000) 98–99.

[10] D.R. Woodall, Cyclic-order graphs and Zarankiewicz's crossing-number conjecture, *J. Graph Theory* **17** (1993) 657–671.

[11] K. Zarankiewicz, On a problem of P. Turán concerning graphs, *Fund. Math.* **41** (1954) 137–145.