Automorphism groups of a family of Cayley graphs of the alternating groups *

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Abstract

Let A_n be the alternating group of degree n with $n \geq 5$. Set $S = \{(1ij), (1ji) \mid 2 \leq i, j \leq n, i \neq j\}$. In this paper, it is shown that the full automorphism group of the Cayley graph $\mathrm{Cay}(A_n, S)$ is the semi-product $R(A_n) \rtimes \mathrm{Aut}(A_n, S)$, where $R(A_n)$ is the right regular representation of A_n and $\mathrm{Aut}(A_n, S) = \{\phi \in \mathrm{Aut}(A_n) \mid S^{\phi} = S\} \cong S_{n-1}$.

Key words: Cayley graph; alternating group; automorphism group; independent set.

AMS Classifications: 05C25, 05C69

1 Introduction

For a graph Γ, we denote its vertex set, edge set and full automorphism group respectively by $V(\Gamma), E(\Gamma)$ and $\mathrm{Aut}(\Gamma)$. Let G be a finite group with identity element e and let Ω be a finite set. An *action* of G on Ω is defined as a mapping $\Omega \times G \to \Omega$, $(\alpha, g) \mapsto \alpha^g$ such that $\alpha^e = \alpha$ and $(\alpha^g)^h = \alpha^{gh}$ for $\alpha \in \Omega$ and $g, h \in G$. The subgroup $K = \{g \in G \mid \alpha^g = \alpha$, for any $\alpha \in \Omega\}$ of G is called the *kernel* of G acting on Ω. If $K = \{e\}$, then the action of G on Ω is called *faithful*.

Let G be a finite group with identity element e and let S be a subset of G not containing e with $S = S^{-1}$. The *Cayley graph* $\mathrm{Cay}(G, S)$ of G with respect to S is defined as the graph with vertex set G and edge set $\{\{g, sg\} \mid g \in G, s \in S\}$. Let us set $A = \mathrm{Aut}(\mathrm{Cay}(G, S))$, then $A = R(G)A_e$, where $R(G)$ is the right regular representation $R(G)$ of G, i.e., the action of G on itself by right multiplication, and

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A_e is the stabilizer of the vertex e in A. Furthermore, $\mathrm{Aut}(G, S) = \{\phi \in \mathrm{Aut}(G) \mid S^\phi = S\}$ is a subgroup of A_e. Let $N_A(R(G))$ be the normalizer of $R(G)$ in A. By Godsil [6], $N_A(R(G)) = R(G) \rtimes \mathrm{Aut}(G, S)$.

Let S_n and A_n denote the symmetric group and the alternating group of degree n, respectively. In the past few years the problem of determining the full automorphism groups of Cayley graphs has received considerable attention. This is mainly due to the fact that Cayley graphs, especially of S_n and A_n, are widely used as models for interconnection networks [8, 9]. A major consideration in constructing interconnection networks is the symmetry, which is mainly characterized by their full automorphism groups. Therefore, one important problem is to determine the full automorphism groups of Cayley graphs of S_n and A_n. This problem has been studied extensively by a number of researchers. For example, for any minimal generating set S of transpositions of S_n, Feng [4] showed that the full automorphism group of $\mathrm{Cay}(S_n, S)$ is the semi-product $R(S_n) \rtimes \mathrm{Aut}(S_n, S)$, which generalized the results of Godsil and Royle [7, Theorem 3.10.4] and Zhang and Huang [10]. Furthermore, Ganesan [5] showed that if S is a generating set of transpositions of S_n such that the girth of the transposition graph of S is at least 5, then the same result as [4] holds. Zhou [11] completely determined the full automorphism group of $\mathrm{Cay}(A_n, T)$, where $T = \{(12i), (1i2) \mid 3 \le i \le n\}$. In [4, 5, 7, 10, 11], one common technique to determine the full automorphism groups of Cayley graphs is to prove the uniqueness of cycles of certain length passing through several given vertices. Another technique to determine the full automorphism groups of Cayley graphs can be found in [1, 2]. In [1], Deng and Zhang determined the full automorphism group of the derangement graph Γ_n by proving the faithfulness of the actions of $\mathrm{Aut}(\Gamma_n)$ on the set Ω of all maximum-size independent sets of Γ_n and some particular subset of the power set of Ω. Similarly, Deng and Zhang [2] determined the full automorphism group of the pancake graph P_n by proving the faithfulness of the action of $\mathrm{Aut}(P_n)$ on the set of all efficient dominating sets of P_n.

In this paper, combining the above two techniques, we demonstrate an approach to obtain the full automorphism groups of Cayley graphs by proving the following main result:

Theorem 1.1 *Let* $A\Gamma_n = \mathrm{Cay}(A_n, S)$, *where* $S = \{(1ij), (1ji) \mid 2 \le i, j \le n, i \ne j\}$. *Then* $\mathrm{Aut}(A\Gamma_n) = R(A_n) \rtimes \mathrm{Aut}(A_n, S)$ *for* $n \ge 5$. *Furthermore,* $\mathrm{Aut}(A_n, S) = \{c(\sigma) \mid 1^\sigma = 1, \sigma \in S_n\} \cong S_{n-1}$, *where* $c(\sigma)$ *is the automorphism of* A_n *induced by the conjugacy of* σ.

The rest of this paper is organized as follows. In Section 2, we give an equivalent condition of efficient dominating set in a Cayley graph, and thereby characterize all the independent sets of size $\frac{(n-1)!}{2}$ of $A\Gamma_n$. In Section 3, we present the proof of Theorem 1.1.

2 The independent sets

Let G be a finite group with multiplicative notation. We define the product of two subsets M, N of G by $MN = \{mn \mid m \in M, n \in N\}$. If each $x \in MN$ has a unique representation in the form $x = mn$ with $m \in M$ and $n \in N$, then the product MN is called *direct*, denoted by $M \times N$.

Lemma 2.1 *Let G be a finite group and let M, N be subsets of G. Then $G = M \times N$ if and only if $M^{-1}M \cap NN^{-1} = \{e\}$ and $|G| = |M||N|$.*

Proof. ($\Longrightarrow$) Let $G = M \times N$. Suppose to the contrary that $M^{-1}M \cap NN^{-1} \neq \{e\}$. Then there exist distinct elements m_1 and m_2 in M and n_1 and n_2 in N such that $m_1^{-1}m_2 = n_1n_2^{-1}$, and thus $m_1n_1 = m_2n_2$, which contradicts $G = M \times N$. Hence $M^{-1}M \cap NN^{-1} = \{e\}$. Suppose that $M^{-1}M \cap NN^{-1} = \{e\}$ and $|G| \neq |M||N|$. Clearly $|G| > |M||N| = |MN|$, thus some element of G is not in MN, which also contradicts $G = M \times N$.
($\Longleftarrow$) Suppose that $M^{-1}M \cap NN^{-1} = \{e\}$ and $|G| = |M||N|$. Assume that $m_1n_1 = m_2n_2$ with $m_1, m_2 \in M$ and $n_1, n_2 \in N$. Then $m_2^{-1}m_1 = n_2n_1^{-1} \in M^{-1}M \cap NN^{-1} = \{e\}$, which implies $m_1 = m_2$ and $n_1 = n_2$. Thus the product MN is direct, and so $|MN| = |M||N|$. Since $|G| = |M||N|$, we have $|G| = |MN|$. Hence $G = M \times N$. ■

A subset D of vertices in a graph is called an *efficient dominating set* if D is an independent set and each vertex not in D is adjacent to exactly one vertex in D.

Lemma 2.2 *For any Cayley graph* $\mathrm{Cay}(G, S)$, *D is an efficient dominating set of* $\mathrm{Cay}(G, S)$ *if and only if $S^{-1}S \cap DD^{-1} = \{e\}$ and $|D| = \frac{|G|}{|S|+1}$.*

Proof. It follows from the definitions of a Cayley graph and an efficient dominating set that D is an efficient dominating set of $\mathrm{Cay}(G, S)$ $\Leftrightarrow$ $G \setminus D = S \times D$ $\Leftrightarrow$ $S^{-1}S \cap DD^{-1} = \{e\}$, $|G \setminus D| = |S||D|$ (by Lemma 2.1) $\Leftrightarrow$ $S^{-1}S \cap DD^{-1} = \{e\}$, $|D| = \frac{|G|}{|S|+1}$. The assertion holds.■

Proposition 2.3 [3]. *Let ST_n denote the star graph* $\mathrm{Cay}(S_n, C_n)$, *where $C_n = \{(1i) \mid 2 \leq i \leq n\}$. Then all the efficient dominating sets of ST_n are $D_k = \{\sigma \in S_n \mid 1^\sigma = k\}$, $k = 1, 2, \cdots, n$.*

Theorem 2.4 *All the independent sets of size $\frac{(n-1)!}{2}$ of $A\Gamma_n$ ($n \geq 3$) are $B_k = \{\sigma \in A_n \mid 1^\sigma = k\}$, $k = 1, 2, \cdots, n$.*

Proof. First for any $u, v \in B_k$, we have $1^{uv^{-1}} = k^{v^{-1}} = 1 \Rightarrow uv^{-1} \notin S$ (where $S = \{(1ij), (1ji) \mid 2 \leq i, j \leq n,\ i \neq j\}$) $\Rightarrow u$ is not adjacent to v in $A\Gamma_n$. Thus B_k is an independent set. Note that $|B_k| = \frac{(n-1)!}{2}$. Hence B_k ($k = 1, 2, \cdots, n$) are the independent sets of size $\frac{(n-1)!}{2}$.

Next we claim that B_k ($k = 1, 2, \cdots, n$) are the only independent sets of size $\frac{(n-1)!}{2}$ of $A\Gamma_n$. Let I_1 be an independent set of size $\frac{(n-1)!}{2}$. Then $I_1 I_1^{-1} \cap S = \emptyset$. Let (ij) be a transposition in S_n and set $I = I_1 \cup I_1(ij)$ ($\subseteq S_n$). Then $II^{-1} = (I_1 \cup I_1(ij))(I_1^{-1} \cup (ij)I_1^{-1}) = I_1 I_1^{-1} \cup I_1(ij)I_1^{-1} \cup I_1(ij)I_1^{-1} \cup I_1(ij)(ij)I_1^{-1} = I_1 I_1^{-1} \cup I_1(ij)I_1^{-1}$. Since $I_1(ij)I_1^{-1} \subseteq S_n \setminus A_n$ and $S \subseteq A_n$, we have $I_1(ij)I_1^{-1} \cap S = \emptyset$, and thus $II^{-1} \cap S = \emptyset$. Set $ST_n = \mathrm{Cay}(S_n, C_n)$, where $C_n = \{(1i) \mid 2 \le i \le n\}$. Since $C_n^{-1} C_n = S \cup \{e\}$, we have $C_n^{-1} C_n \cap II^{-1} = \{e\}$. Note that $|I| = (n-1)! = \frac{|S_n|}{|C_n|+1}$. By Lemma 2.2, I is an efficient dominating set of ST_n. By Proposition 2.3, we have $I = D_k$, and thus $I_1 = B_k$ for some k. The assertion holds. ■

3 Proof of Theorem 1.1

Lemma 3.1 *The action of A_e on S is faithful.*

Proof. Let A_e^* be the kernel of A_e acting on S. We shall show that A_e^* fixes all neighbors of s for any $s \in S$. Let $s = (1ij) \in S$ and set $E = \{1, 2, \cdots, n\} \setminus \{1, i, j\}$. Then the neighborhood of s is $N(s) = \{e, (1ji), (1kj), (jki), (1j)(ik), (1k)(ij), (1klij) \mid k, l \in E, k \neq l\}$. Clearly, A_e^* fixes e, $(1ji)$ and $(1kj)$, which all belong to $\{e\} \cup S$. In order to show that A_e^* fixes other neighbors of s, we first prove the following four claims.

Claim 1. $C_1 = (e, (1ij), (1ji), (1ki))$, $C_2 = (e, (1ij), (jki), (1ki))$, $C_3 = (e, (1ij), (1j)(ik), (1ki))$ are the only 4-cycles in $A\Gamma_n$ passing through $e, (1ij)$ and $(1ki)$.

One can easily check that for any $x \in N(s)$, x is adjacent to $(1ki)$ if and only if $x \in \{e, (1ji), (jki), (1j)(ik)\}$. Hence Claim 1 holds.

Claim 2. $C_4 = (e, (1ij), (1j)(ik), (1ik))$, $C_5 = (e, (1ij), (1k)(ij), (1ik))$ are the only 4-cycles in $A\Gamma_n$ passing through $e, (1ij)$ and $(1ik)$.

One can easily check that for any $x \in N(s)$, x is adjacent to $(1ik)$ if and only if $x \in \{e, (1j)(ik), (1k)(ij)\}$. Hence Claim 2 holds.

Claim 3. $C_6 = (e, (1ij), (1klij), (1ijkl), (1kl))$ is the unique 5-cycle in $A\Gamma_n$ having the form $(e, (1ij), x, y, (1kl))$ with $x \in B_k$, $y \in B_i$.

First note that $N((1ij)) \cap B_k = \{(1kj), (1k)(ij), (1kpij) \mid p \neq 1, i, j, k\}$ and $N((1kl)) \cap B_i = \{(1il), (1i)(kl), (1iqkl) \mid q \neq 1, k, l, i\}$. Then it is easy to check that for any $x \in N((1ij)) \cap B_k$, $y \in N((1kl)) \cap B_i$, x is adjacent to y if and only if $x = (1klij)$ and $y = (1ijkl)$. Hence Claim 3 holds.

Claim 4. A_e^* fixes B_i setwise for any $i \in \{1, 2, \cdots, n\}$.

Since any $\sigma \in A_e^*$ must permute the independent sets of size $\frac{(n-1)!}{2}$ of $A\Gamma_n$, by Theorem 2.4, A_e^* naturally acts on $\{B_1, B_2, \cdots, B_n\}$. Clearly, $(\{e\} \cup S) \cap B_i \neq \emptyset$, $B_i \cap B_j = \emptyset$ for any $i, j \in \{1, 2, \cdots, n\}$ with $i \neq j$. Since A_e^* fixes each element of $\{e\} \cup S$, A_e^* fixes each B_i setwise.

Next we show that A_e^* fixes each element of $\{(jki), (1j)(ik), (1k)(ij), (1klij) \mid k, l \in E, k \neq l\}$. By Claim 1, any $\sigma \in A_e^*$ permutes C_1, C_2, C_3, and thus permutes $(1ji), (jki), (1j)(ik)$. Clearly, A_e^* fixes $(1ji)$ because $(1ji) \in S$. Since $(jki) \in$

B_1, $(1j)(ik) \in B_j$, by Claim 4, $A_e^\star$ fixes (jki) and $(1j)(ik)$. By Claim 2, any $\sigma \in A_e^\star$ permutes C_4, C_5, and thus permutes $(1j)(ik)$, $(1k)(ij)$. Since $A_e^\star$ fixes $(1j)(ik)$, $A_e^\star$ also fixes $(1k)(ij)$. By Claims 3 and 4, $A_e^\star$ fixes $(1klij)$.

In conclusion, $A_e^\star$ fixes all neighbors of s for any $s \in S$. By the connectedness of $A\Gamma_n$, $A_e^\star$ fixes all vertices of $A\Gamma_n$, and thus $A_e^\star = 1$, that is, the action of A_e on S is faithful. ■

Lemma 3.2 *Let $S_k = S \cap B_k$ and $\Omega = \{S_2, S_3, \cdots, S_n\}$. Then A_e induces an action on Ω and this action is faithful. In particular, $|A_e| \leq (n-1)!$.*

Proof. Clearly any $\sigma \in A_e$ must permute the independent sets of size $\frac{(n-1)!}{2}$ of $A\Gamma_n$. By Theorem 2.4, A_e naturally acts on $\{B_1, B_2, \cdots, B_n\}$. Note that A_e fixes e, $e \in B_1$ and $e \notin B_i$, $i = 2, 3 \cdots, n$. It follows that A_e fixes B_1 setwise. Thus A_e acts on $\{B_2, B_3, \cdots, B_n\}$. Since A_e also acts on S and $S_k = S \cap B_k$, A_e induces an action on $\Omega = \{S_2, S_3, \cdots, S_n\}$. In order to show that the action of A_e on Ω is faithful, we first prove the following two claims.

Claim 1. For any $i \neq k$, each vertex in S_i is adjacent to exactly one vertex in S_k.

In fact, for any $(1ij) \in S_i$, we have $N((1ij)) \cap S = \{((1ji), (1kj) \mid k \neq 1, i, j\} \Rightarrow |N((1ij)) \cap S_k| = 1$. Hence Claim 1 holds.

Claim 2. For any $2 \leq i \neq j \neq k \leq n$ there is a unique 6-cycle whose vertices belong to $S_i \cup S_j \cup S_k$. We denote such a 6-cycle by $C_{i,j,k}$.

In fact, since S_i, S_j, S_k are independent sets, then by Claim 1 and its proof, one can easily check that $C_{i,j,k} = ((1ij), (1ji), (1ki), (1ik), (1jk), (1kj))$ is such a unique 6-cycle.

Next we shall show that the action of A_e on Ω is faithful. Assume that $\phi \in A_e$ such that $S_k^\phi = S_k$ for each $k \in \{2, 3, \cdots, n\}$. Then by Claim 2, ϕ setwise stabilizes each $C_{i,j,k}$. By the proof of Claim 2 , for any $(1ij) \in S$, we have $\{(1ij), (1ji)\} = C_{i,j,k} \cap C_{i,j,l}$ for any $k \neq l$ (since $n \geq 5$). It follows that ϕ setwise stabilizes $\{(1ij), (1ji)\}$. Since $(1ij) \in S_i$ and $(1ji) \in S_j$, ϕ fixes $(1ij)$. In consideration of arbitrariness of $(1ij)$ in S, ϕ fixes all vertices in S. By Lemma 3.1, we have $\phi = 1$, which implies that the action of A_e on Ω is faithful. Thus $|A_e| \leq (n-1)!$. ■

Proof of Theorem 1.1: For $\sigma \in S_n$, let $c(\sigma)$ denote the automorphism of A_n induced by the conjugacy of σ. Clearly, $\mathrm{Aut}(A_n, S) = \{\phi \in \mathrm{Aut}(A_n) \mid S^\phi = S\} \supseteq \{c(\sigma) \mid 1^\sigma = 1, \sigma \in S_n\} \cong S_{n-1}$. Hence $(n-1)! = |\{c(\sigma) \mid 1^\sigma = 1, \sigma \in S_n\}| \leq |\mathrm{Aut}(A_n, S)| \leq |A_e| \leq (n-1)!$ (by Lemma 3.2), and thus $A_e = \mathrm{Aut}(A_n, S) = \{c(\sigma) \mid 1^\sigma = 1, \sigma \in S_n\}$. So $\mathrm{Aut}(A\Gamma_n) = R(A_n) \rtimes \mathrm{Aut}(A_n, S)$.

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