FINDING A BIPLANAR IMBEDDING OF $C_n \times C_n \times C_l \times P_m$

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ABSTRACT. Determining the biplanar crossing number of the graph $C_n \times C_n \times C_n \times P_n$ was a problem proposed in a paper by Czabarka, Sýkora, Székely, and Vrťo [2]. We find as a corollary to the main theorem of this paper that the biplanar crossing number of the aforementioned graph is zero. This result follows from the decomposition of $C_n \times C_n \times C_l \times P_m$ into one copy of $C_{n^2} \times P_{lm}$, $l-2$ copies of $C_{n^2} \times P_m$, and a copy of $C_{n^2} \times P_{2m}$.

1. INTRODUCTION

In today's society, economical circuit design has become a topic of interest. With wire crossings causing short circuits, it is desirable for chip designers to look to graph theory for a solution. The notion of a crossing number leads to the minimizing of crossings in a circuit design. More formally, the crossing number of a graph G, denoted by $\nu(G)$, is the minimum number of crossings in a drawing of a graph G in the plane [9]. Designers use this information and place crossing wires on different layers of a printed circuit board. Drilling holes through a printed circuit board, commonly referred to as vias, allow designers to escape the possibility of a wire crossing [6]. Yet a problem with this method is that too many vias increase the area, resulting in a greater chance for a faulty chip.

Another problem with using vias is the costly nature of the wear and breaking on the mechanical bits used in drilling small vias. Therefore chip designers have used another method that once again has a direct correlation with graph theory. Designers embed their circuits onto two or more circuit boards, trying to avoid crossings at all costs. If no crossings occur on any of the circuit boards, then the circuit design directly relates to the thickness of a graph. When properly defined, the thickness of a graph G, denoted by $\theta(G)$, is the minimum number of planar graphs whose union is G [10]. On the other hand, if the circuit designers use vias on the different circuit boards, the designers are dealing with the k-planar crossing number of a graph. In other words, they are looking at the minimum of $\nu(G_1) + \nu(G_2) + \cdots + \nu(G_k)$, where the minimum is taken over all edge disjoint subgraphs $G_1, G_2, \cdots, G_k$ of G such that $G = G_1 \cup G_2 \cup \cdots \cup G_k$ [9].

For the purposes of this paper, we will concern ourselves with the case when $k = 2$, which commonly is referred to as the biplanar crossing number. If the biplanar crossing number happens to be zero, then we refer to such a drawing as a biplanar imbedding. One easily finds that a graph has a biplanar imbedding if and only if the graph has thickness less than or equal to two.

2. Pursuit of a Problem

When looking for reasons as to why finding a biplanar imbedding of $C_n \times C_n \times C_l \times P_m$ is of interest in the mathematical community, we must first consider some of the results in the field. In [2], the authors mentioned how the biplanar crossing number of $C_k \times C_l \times C_m$ is zero. Also in that paper, we found the thickness of $C_k \times C_l \times C_m \times C_n$ to be greater than or equal to three. This implies that the biplanar crossing number of the aforementioned graph must be greater than or equal to one. Serving as a middle ground between these two problems, Czabarka, Sýkora, Székely, and Vrťo asked for the biplanar crossing number of $C_n \times C_n \times C_n \times P_n$. In this paper, we will answer that question and generalize the result to help find the biplanar crossing number for the more general case of $C_k \times C_l \times C_m \times P_n$.

2.1. **The First Plane.** Before beginning to find a planar graph to be imbedded on the first plane of our biplanar imbedding, we must develop a useful notation to help keep track of all the edges that are coming from the proper vertices. We will again use the idea behind the notation used in [5].

Remark 2.1.1. We begin by giving each vertex in $C_n \times C_n \times C_l \times P_m$ the label $v_{\alpha,\beta,\gamma,\delta}$, where $0 \leq \alpha \leq n-1$, $0 \leq \beta \leq n-1$, $0 \leq \gamma \leq l-1$, and $0 \leq \delta \leq m-1$. Notice that we can think of $\alpha, \beta, \gamma, \delta$ respectively corresponding to the vertices $v_\alpha, v_\beta, v_\gamma, v_\delta$ in the graphs C_n, C_n, C_l, P_m. With this in mind, one notices that the edges of $C_n \times C_n \times C_l \times P_m$ are of the form $v_{\alpha,\beta,\gamma,\delta} - v_{\alpha+1,\beta,\gamma,\delta}$, $v_{\alpha,\beta,\gamma,\delta} - v_{\alpha,\beta+1,\gamma,\delta}$, $v_{\alpha,\beta,\gamma,\delta} - v_{\alpha,\beta,\gamma+1,\delta}$, and $v_{\alpha,\beta,\gamma,\delta} - v_{\alpha,\beta,\gamma,\delta+1}$ where we consider $(\alpha+1) \bmod n$, $(\beta+1) \bmod n$, $(\gamma+1) \bmod l$ and $1 \leq (\delta+1) \leq m-2$.

Now we look to lay the groundwork for our biplanar imbedding.

Lemma 2.1.2. *The graph $C_{n^2} \times P_{lm}$ is a subgraph of $C_n \times C_n \times C_l \times P_m$.*

Proof. We will find our desired subgraph as an imbedding in $\mathbb{R}^2$. We begin by placing the vertices of $C_n \times C_n \times C_l \times P_m$ in a format resembling a grid. When considering the vertex $v_{\alpha,\beta,\gamma,\delta}$, α and β will help determine the y-coordinate of our vertex in $\mathbb{R}^2$, whereas γ and δ determine the x-coordinate. We will start with determining the y-coordinate of each vertex.

In order to lay the groundwork for our proof, we determine where to place vertices of the form $v_{\alpha,\beta,0,0}$. Of these vertices, we first find which ones are best suited for the coordinate $(0, n\beta)$. This requires some familiarity with modular arithmetic. We place the vertex $v_{\alpha,\beta,0,0}$ at $(0, n\beta)$ if and only if $\alpha+\beta \equiv 0 \bmod n$. Our next step will be to find where to place all the vertices such that $\alpha + \beta \not\equiv 0 \bmod n$. This requires us to consider the permutation $(0, 1, \cdots, \alpha-1, \alpha, \alpha+1, \cdots, n-2, n-1)$.

Suppose that $\alpha + \beta \not\equiv 0 \mod n$ for the vertex $v_{\alpha,\beta,0,0}$, but the vertex $v_{i,\beta,0,0}$ has the desired result of $i+\beta \equiv 0 \mod n$. We will coin the term *distance between* i *and* α to represent the corresponding number of places between i and α when rotating the permutation $(0, 1, \cdots, \alpha-1, \alpha, \alpha+1, \cdots, n-2, n-1)$ so that i is the leading term. Suppose the distance between i and α is j. Then we will place the vertex $v_{\alpha,\beta,0,0}$ at the coordinate $(0, n\beta+j)$. A cause for concern might be the case where $j \geq n$, but once considering the distance in a permutation one finds the maximum value j can attain is $n-1$.

One can easily find the y-coordinate of $v_{\alpha,\beta,\gamma,\delta}$ by looking at the corresponding y-coordinate of $v_{\alpha,\beta,0,0}$. Hence we should move forward towards finding the x-coordinate of $v_{\alpha,\beta,\gamma,\delta}$. In a similar manner to finding the y-coordinate, we will first consider vertices of the form $v_{0,0,\gamma,\delta}$. The two cases to consider here are when δ is odd or even. In the case where δ is even, we find the corresponding coordinate for $v_{0,0,\gamma,\delta}$ is $(l\delta+\gamma, 0)$. Likewise, when δ is odd we have $v_{0,0,\gamma,\delta}$ is located at $(l\delta+l-\gamma-1, 0)$. Hence one can find the exact coordinate of $v_{\alpha,\beta,\gamma,\delta}$ by considering the y-coordinate of $v_{\alpha,\beta,0,0}$ and the x-coordinate of $v_{0,0,\gamma,\delta}$.

Now we must look for edges to form in our graph in the first plane of our biplanar imbedding. When considering the edges mentioned in Remark 2.1.1, one easily finds that we can form edges between the coordinates (i,j) and $(i,j+1)$ where $0 \leq i \leq lm-1$ and $0 \leq j \leq n^2-2$. Similarly, one also notices that we can add edges from (a,b) to $(a+1,b)$ with $0 \leq a \leq lm-2$ and $0 \leq b \leq n^2-1$. Thus far we have formed a graph that resembles a grid, but there are more edges waiting to be added.

With our subgraph in sight, we now look to add edges from $(i,0)$ to (i,n^2-1). Before doing this though we must show that such an edge exists in the original graph of $C_n \times C_n \times C_l \times P_m$. Once again we must consider the permutation $(0, 1, \cdots, \alpha-1, \alpha, \alpha+1, \cdots, n-2, n-1)$. Recall that the y-coordinate of $v_{\alpha,\beta,\gamma,\delta}$ is at $n\beta$ if and only if $\alpha + \beta \equiv 0 \; mod \; n$ with the maximum value β can obtain being $n-1$. Hence the vertex $v_{1,n-1,\gamma,\delta}$ has a y-coordinate of $n^2 - n$. Therefore 1 is the corresponding lead term in the permutation $(0, 1, \cdots, \alpha - 1, \alpha, \alpha+1, \cdots, n-2, n-1)$, implying 1 and 0 must have a distance of $n-1$ in this instance. Thus, $v_{0,n-1,\gamma,\delta}$ must have a y-coordinate of $(n^2-n)+(n-1)$ which equals n^2-1. Recalling from Remark 2.1.1, we find an edge between $v_{0,0,\gamma,\delta}$ and $v_{0,n-1,\gamma,\delta}$. Hence we have our desired edge from $(i,0)$ to (i,n^2-1), as well as our desired subgraph, $C_{n^2} \times P_{lm}$.

□

In order to develop a better understanding for Lemma 2.1.2, we should familiarize ourselves with the subgraph we have just constructed. As a result, we will consider the following example.

Example 2.1.3. *Consider the graph $C_4 \times C_4 \times C_5 \times P_3$. Lemma 2.1.2 claims that we should find a copy $C_{16} \times P_{15}$ as a subgraph which could be imbedded in*

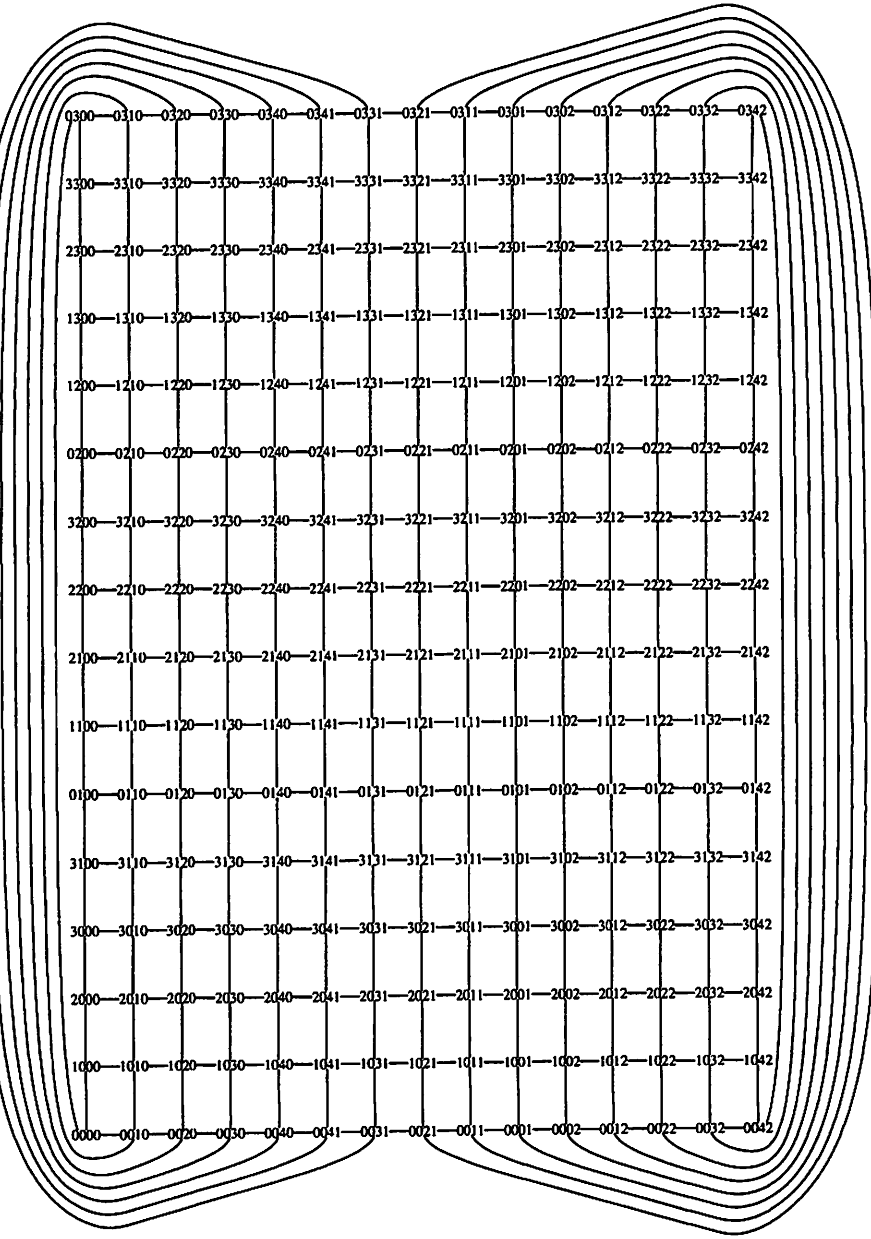

FIGURE 1. The subgraph $C_{16} \times P_{15}$ of $C_4 \times C_4 \times C_5 \times P_3$ as constructed in Lemma 2.1.2

the first plane of our biplanar imbedding of $C_4 \times C_4 \times C_5 \times P_3$. Figure 1 helps display the desired subgraph described in the proof of Lemma 2.1.2.

Before moving to the next section of this paper, it is important to discuss the edges formed in Lemma 2.1.2.

Remark 2.1.4. As illustrated in Example 2.1.3, we can consider the edges formed parallel to the y-axis along with those edges parallel to the x-axis. We begin by looking at the edges formed parallel to the y-axis in Lemma 2.1.2. In this case, we have constructed all of the edges of the form $v_{\alpha,\beta,\gamma,\delta} - v_{\alpha+1,\beta,\gamma,\delta}$, with the only exception being when $(\alpha + \beta) \equiv -1 \; mod \; n$. Such an exception yields edges of the form $v_{\alpha,\beta,\gamma,\delta} - v_{\alpha,\beta+1,\gamma,\delta}$.

Now we can consider the edges parallel to the x-axis. The majority of these edges are of the form $v_{\alpha,\beta,\gamma,\delta} - v_{\alpha,\beta,\gamma+1,\delta}$, with the only exception occurring when $\gamma = l-1$. This leads us to the two types of edges of the form $v_{\alpha,\beta,\gamma,\delta} - v_{\alpha,\beta,\gamma,\delta+1}$ constructed in Lemma 2.1.2. If δ is even, then we find $\gamma = l - 1$. Similarly, if δ is odd, then we have $\gamma = 0$. This complete list of all the edges formed in Lemma 2.1.2 will become useful in our upcoming lemmata.

2.2. **The Second Plane.** In order to produce a biplanar imbedding of $C_n \times C_n \times C_l \times P_m$ we will use the subgraph produced in the proof of Lemma 2.1.2 to obtain the resulting graphs on the second plane. From this point forward we will refer to the graph produced in the proof of Lemma 2.1.2 as G_1. Without further ado, we look to producing the remaining graphs in our biplanar imbedding.

Lemma 2.2.1. *The vertices labeled $v_{\alpha,\beta,i,\delta}$ with $1 \le i \le l-2$ produce $l-2$ disjoint copies of $C_{n^2} \times P_m$ in $C_n \times C_n \times C_l \times P_m - E(G_1)$.*

Proof. Consider the vertices $v_{\alpha,\beta,i,\delta}$ with $1 \le i \le l-2$ in G_1. Notice all such vertices have degree 4 in G_1, implying that the aforementioned vertices have either degree 3 or 4 in $C_n \times C_n \times C_l \times P_m - E(G_1)$. We now look to find one copy of $C_{n^2} \times P_m$.

In order to produce one copy of $C_{n^2} \times P_m$ in $C_n \times C_n \times C_l \times P_m - E(G_1)$, we begin by finding a cycle from the vertices labeled $v_{\alpha,\beta,i,\delta}$ regardless of the value for i and δ. Our desired cycle begins with constructing a series of paths. Once again we look to modular arithmetic for our desired paths. Each path begins at the vertex $v_{\alpha,k,i,\delta}$ where $\alpha + k \equiv 0 \bmod n$ and $0 \le k \le n-1$. With the second coordinate of our vertex always considered modulo n, we can create the path $v_{\alpha,k,i,\delta} - v_{\alpha,k+1,i,\delta} - \cdots - v_{\alpha,k+n-1,i,\delta}$, which Remark 2.1.4 states is missing from G_1. Since we looked at the second coordinate modulo n, one immediately finds that the vertex $v_{\alpha,k+n-1,i,\delta}$ coincides with $v_{\alpha,k-1,1,\delta}$.

Continuing from this point, we notice from Remark 2.1.4 that the edge between $v_{\alpha,k-1,i,\delta}$ and $v_{\alpha+1,k-1,i,\delta}$ is missing from G_1. Thus we can create the path $v_{\alpha,k,i,\delta} - v_{\alpha,k+1,i,\delta} - \cdots - v_{\alpha,k-1,i,\delta} - v_{\alpha+1,k-1,i,\delta} - v_{\alpha+1,k,i,\delta} - \cdots - v_{\alpha+1,k-2,i,\delta}$ in $C_n \times C_n \times C_l \times P_m - E(G_1)$. When beginning our path at $v_{0,0,i,\delta}$, we immediately obtain the path $v_{0,0,i,\delta} - \cdots - v_{0,n-1,i,\delta} - v_{1,n-1,i,\delta} - \cdots - v_{1,n-2,i,\delta} -$

$v_{2,n-2,i,\delta} - \cdots - v_{n-1,1,i,\delta} - \cdots - v_{n-1,0,i,\delta}$ on n^2 vertices. By realizing that $v_{0,0,i,\delta}$ and $v_{n-1,0,i,\delta}$ do not have an edge in G_1, we obtain the desired cycle C_{n^2}.

In order to finish our proof, we must find a path from the vertices $v_{\alpha,\beta,i,\delta}$ regardless of the values given for α and β. The desired path of length m is given by $v_{\alpha,\beta,i,0} - v_{\alpha,\beta,i,1} - \cdots - v_{\alpha,\beta,i,m-1}$. Notice that such a path can be constructed for each value of α and β, implying that we have constructed a copy of $C_{n^2} \times P_m$. Since a copy exists for each value of i, there exists $l-2$ disjoint copies $C_{n^2} \times P_m$ in $C_n \times C_n \times C_l \times P_m - E(G_1)$. □

Before beginning the final piece of our biplanar imbedding, we shall consider an example for a complete understanding of the proof of Lemma 2.2.1. In the upcoming example, we once again consider the graph constructed in Example 2.1.3.

Example 2.2.2. *Let us once again consider the graph $C_4 \times C_4 \times C_5 \times P_3$ along with its associated subgraph, $C_{16} \times P_{15}$. Figure 2 helps to demonstrate how Lemma 2.2.1 constructs C_{16} from the figure constructed in Example 2.1.3.*

The sequence of paths mentioned in the proof of Lemma 2.2.1 is obtained in Figure 2 with "×" indicating where each path begins. In the three columns of Figure 2 where shapes appear, the vertices surrounded by the objects "○" and "□" give two examples of the path $v_{\alpha,k,i,\delta} - v_{\alpha,k+1,i,\delta} - \cdots - v_{\alpha,k+n-1,i,\delta}$ in Lemma 2.2.1, while the vertices surrounded by "△" demonstrate an example of the path $v_{\alpha,k-1,i,\delta} - v_{\alpha+1,k-1,i,\delta}$ in Lemma 2.2.1.

For Figure 2, the vertices enclosed by "○" help demonstrate the path $v_{0,0,1,\delta} - v_{0,1,1,\delta} - v_{0,2,1,\delta} - v_{0,3,1,\delta}$ where $0 \leq \delta \leq 2$. Meanwhile the vertices surrounded by "□" help indicate the path $v_{1,3,1,\delta} - v_{1,0,1,\delta} - v_{1,1,1,\delta} - v_{1,2,1,\delta}$ missing from $C_{16} \times P_{15}$. Finally, we find the vertices with the attached shape "△" describe the path $v_{2,1,1,\delta} - v_{3,1,1,\delta}$. These paths help become a part of the C_{16} described in Lemma 2.2.1.

Figure 2 also allows us to picture the desired path P_3 to be obtained from our subgraph $C_{16} \times P_{15}$. Any row consisting of three of the same shapes help indicate our desired path P_3. Hence Figure 2 gives us ten examples of P_3. Combining this result with the one from the previous paragraph allows us to see exactly how $C_{16} \times P_3$ is constructed from the graph $C_{16} \times P_{15}$ given in Example 2.1.3.

After seeing the construction of $C_{n^2} \times P_m$ in the form of an example, we are ready to move towards finishing our desired biplanar imbedding.

Lemma 2.2.3. *The vertices labeled $v_{\alpha,\beta,j,\delta}$ with $j \in \{0, l-1\}$ produce a copy of $C_{n^2} \times P_{2m}$ in $C_n \times C_n \times C_l \times P_m - E(G_1)$.*

Proof. Recall from the proof of Lemma 2.2.1 that we constructed a cycle of length n^2 for arbitrary values i and δ. The same construction can be applied to find C_{n^2} regardless of our values of j and δ. Thus, we need only look for a path of length $2m$ where α and β are arbitrary. In order to complete our task we must consider the two cases when m is odd or even.

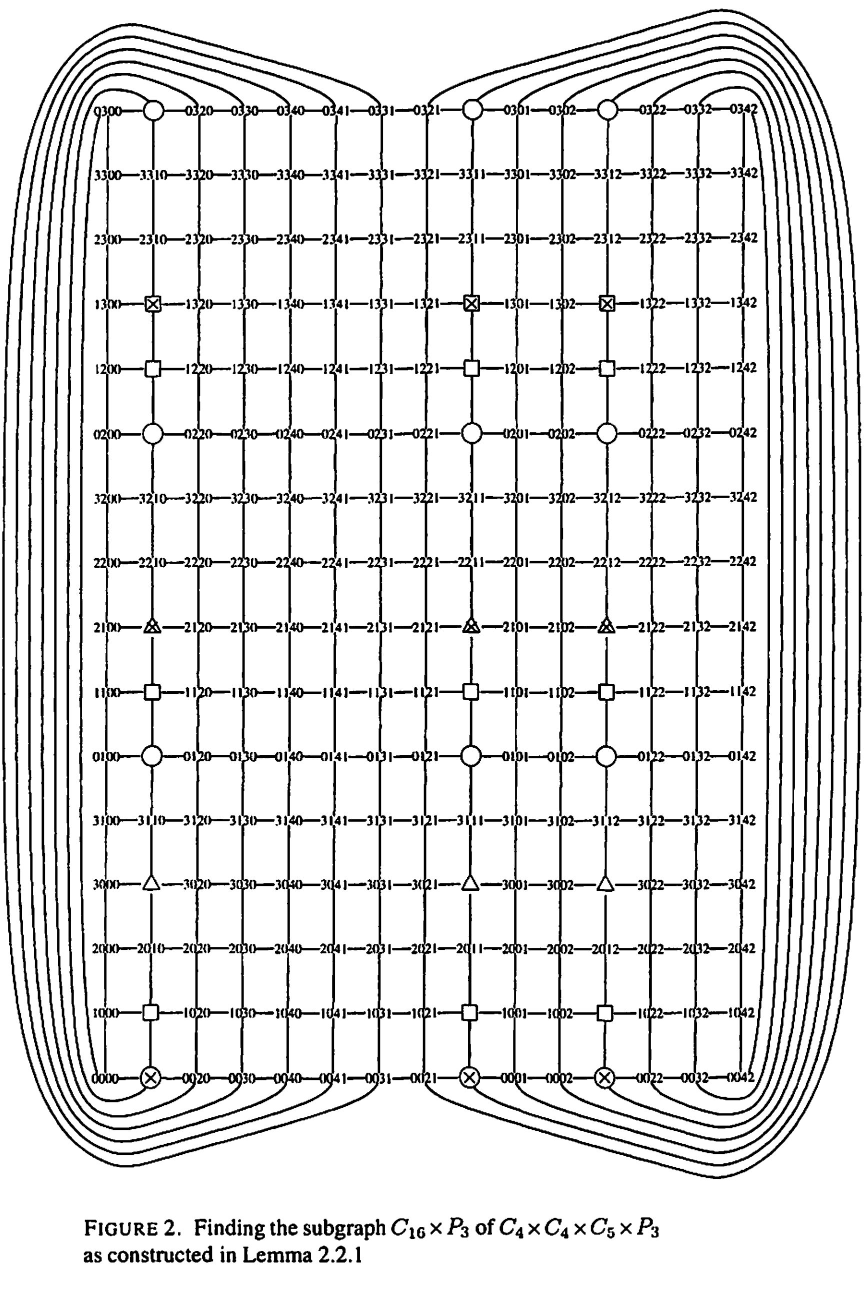

FIGURE 2. Finding the subgraph $C_{16} \times P_3$ of $C_4 \times C_4 \times C_5 \times P_3$ as constructed in Lemma 2.2.1

In the case where m is odd, we can construct the path $v_{\alpha,\beta,l-1,0} - v_{\alpha,\beta,0,0} - v_{\alpha,\beta,0,1} - v_{\alpha,\beta,l-1,1} - v_{\alpha,\beta,l-1,2} - \cdots - v_{\alpha,\beta,l-1,m-1} - v_{\alpha,\beta,0,m-1}$. When carefully considering the aforementioned path, we find that all of the given edges are missing from G_1, as we desired. Moving to the case where m is even, we find the edges needed in $C_n \times C_n \times C_l \times P_m - E(G_1)$ come in the form of the path $v_{\alpha,\beta,l-1,0} - v_{\alpha,\beta,0,0} - v_{\alpha,\beta,0,1} - v_{\alpha,\beta,l-1,1} - \cdots - v_{\alpha,\beta,0,m-1} - v_{\alpha,\beta,l-1,m-1}$.

Notice in either case, we have constructed a path of length $2m$. Piecing this information together with our cycle of length n^2, we have constructed a copy of $C_{n^2} \times P_{2m}$ in $C_n \times C_n \times C_l \times P_m - E(G_1)$.

□

Although we have not completely touched on the subject, the idea behind these proofs can be easily extended to find a biplanar imbedding of $C_n \times C_{kn} \times C_l \times P_m$. Unfortunately, the resulting decomposition of $C_n \times C_{kn} \times C_l \times P_m$ does not fit in the scope of this paper. Hence we move forward with the results obtained in the previous lemmata.

2.3. **The Results.** Considering the proofs of Lemma 2.1.2, Lemma 2.2.1, and Lemma 2.2.3, we find all edges are accounted for in the graph $C_n \times C_n \times C_l \times P_m$. This leads us to the next stream of results.

Theorem 2.3.1. *The graph $C_n \times C_n \times C_l \times P_m$ can be decomposed into one copy of $C_{n^2} \times P_{lm}$, $l-2$ copies of $C_{n^2} \times P_m$, and a copy of $C_{n^2} \times P_{2m}$.*

Theorem 2.3.2. *There exists a biplanar imbedding of $C_n \times C_n \times C_l \times P_m$.*

Proof. Just place the copy of $C_{n^2} \times P_{lm}$ obtained in Lemma 2.1.2 on the first plane, while putting the $l-2$ copies of $C_{n^2} \times P_m$ from Lemma 2.2.1 and copy of $C_{n^2} \times P_{2m}$ in Lemma 2.2.3 on the second plane.

□

Theorem 2.3.3. *The biplanar crossing number of $C_n \times C_n \times C_l \times P_m$ is zero.*

Last, but certainly not least, we give the answer to the motivating question behind this paper.

Corollary 2.3.4. *The biplanar crossing number of $C_n \times C_n \times C_n \times P_n$ is zero.*

2.4. **Open Problem.** Using the results from this paper along with those obtained in [5], we have made significant progress towards finding the biplanar crossing number of $C_k \times C_l \times C_m \times P_n$. The significance of such a result is that it would tell us the middle ground between the biplanar graph $C_k \times C_l \times C_m$ and the thickness-three graph $C_k \times C_l \times C_m \times C_n$.

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