Extending matchings in planar odd graphs *

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Abstract

Let n and k be two non-negative integers. A graph G with $1 \leq n \leq |V(G)| - 2$ is said to be n-factor-critical if any n vertices of G are deleted then the resultant graph has a perfect matching. An odd graph G with $2k \leq |V(G)| - 3$ is said to be near k-extendable if G has a k-matching and any k-matching of G can be extended to a near perfect matching of G. Lou and Yu [Australsa. J. Combin. 29 (2004) 127-133] showed that any 5-connected planar odd graph is 3-factor-critical. In

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this paper, as an improvement of Lou and Yu's result, we prove that any 4-connected planar odd graph is 3-factor-critical and also near 2-extendable. Furthermore, we prove that all 5-connected planar odd graphs are near 3-extendable.

Keywords: k-extendable graphs, near k-extendable graphs, n-factor-critical graphs.

1 Introduction

All graphs we considered in this paper are simple and finite. For the terminology and notation not defined in this paper, the reader is referred to [2, 7].

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. G is said to be an *odd (even) graph* if it has an odd (even) number of vertices. Let S be a subset of $V(G)$. Denote by $c_o(G-S)$ the number of odd components of $G-S$. A *matching* M of G is a subset of $E(G)$ such that any two edges of M have no vertices in common. A matching of size k is called a *k-matching*. Let d be a non-negative integer. A matching M of G is called a *defect-d-matching* if it exactly covers $|V(G)|-d$ vertices in G. In particular, a matching M of G is also called a *perfect matching* (resp. a *near perfect matching*) of G if it covers all vertices (resp. $|V(G)|-1$ vertices) in G. Berge [1] proved the following result.

Theorem 1.1 *[1] Let G be a graph and let d be an integer such that $0 \le d \le |V(G)|$ and $|V(G)| \equiv d \pmod 2$. Then G has a* *defect-d-matching if and only if for any subset S of $V(G)$*

$$c_o(G-S) \leq |S| + d.$$

Let n be a non-negative integer. A graph G is called n-*factor-critical* for $0 \leq n \leq |V(G)| - 2$ if the subgraph $G - S$ has a perfect matching for any subset S of $V(G)$ with $|S| = n$. This concept was introduced by Favaron [3] and Yu [12], independently, which is a generalization of the notions of the *factor-critical graphs* and the *bicritical graphs* (the case of $n = 1$ and $n = 2$) [7]. Yu [12] proved the following theorem.

Theorem 1.2 *[12] Let G be a graph of order p and n an integer such that $0 \leq n \leq p-2$ and $n \equiv p$ (mod 2). Then G is n-factor-critical if and only if for each subset $S \subset V(G)$ with $|S| \geq n$, $c_o(G-S) \leq |S| - n$.*

Let G be an even graph and k a non-negative integer with $2k \leq |V(G)| - 2$. G is said to be k-*extendable* if G has a k-matching and any k-matching of G can be extended to a perfect matching of G. The concept of k-extendable graphs was introduced by Plummer [8] in 1980.

In general, Liu and Yu [4] at the first time introduced (n, k, d)-graphs. For a graph G, and three non-negative integers n, k and d such that $n + 2k + d \leq |V(G)| - 2$ and, $|V(G)|$ and $n + d$ have the same parity, if when deleting any n vertices from G the remaining subgraph of G contains a k-matching and each k-matching of the subgraph can be extended to a defect-d-matching of the subgraph, then G is called a (n, k, d)-*graph*. It is easy to see that k-extendable graphs (resp. n-factor-critical graphs) are just the $(0, k, 0)$-graphs (resp. $(n, 0, 0)$-graphs).

As a generalization of k-extendable graphs to odd graphs, an odd graph G with $|V(G)| \geq 2k+3$ is said to be *near k-extendable* [13] if G has a k-matching and any k-matching of G can be extended to a near perfect matching of G. Clearly, a near k-extendable graph is just a $(0, k, 1)$-graph. Zhai and Guo [13] improved the characterization of (n, k, d)-graphs in [4], and consequently obtained a characterization of near k-extendable graphs. Furthermore, a characterization of near k-extendable bipartite graphs and the relations between near k-extendable graphs and n-factor-critical graphs are also investigated in [13].

Theorem 1.3 *[13] A graph G is near k-extendable if and only if the following conditions hold:*
(1) For any $S \subseteq V(G)$, then $c_o(G-S) \leq |S| + |V(G)| - 2k$,
(2) For any $S \subseteq V(G)$ such that $|S| \geq 2k$ and $G[S]$ contains a k-matching, then $c_o(G-S) \leq |S| - 2k + 1$.

Theorem 1.4 *[13] Any $(2k-1)$-factor-critical graph with order $p \geq 2k+3$ is near k-extendable.*

An further topic is the extendability of planar graphs. Lou [5] and Plummer [9] independently proved that all 5-connected planar even graphs are 2-extendable. Lou and Yu [6] proved that any 5-connected planar odd graph is 3-factor-critical.

Theorem 1.5 *[5, 9] Any 5-connected planar even graph is 2-extendable.*

Theorem 1.6 *[6] Any 5-connected planar odd graph is 3-factor-critical.*

In this paper, as an improvement of Theorem 1.6, we prove that any 4-connected planar odd graph is 3-factor-critical, and so, by Theorem 1.4, any 4-connected planar odd graph is also near 2-extendable. Furthermore, we prove that all 5-connected planar odd graphs are near 3-extendable.

2 Extending matchings of planar odd graphs

By Euler's Formula of planar graph, Douglas [2] gave the following result.

Theorem 2.1 *[2] If G is a simple planar graph with at least three vertices, then $|E(G)| \leq 3|V(G)| - 6$. If also G is triangle-free, then $|E(G)| \leq 2|V(G)| - 4$.*

A graph G is called *hamiltonian connected* if, for any two vertices, there is a hamiltonian path joining them.

Tutte [11] firstly proved the following theorem in 1956. Later, Thmassen [10] obtained a simple proof of it in 1983.

Theorem 2.2 *[10, 11] Any 4-connected planar graph is hamiltonian connected.*

By the above theorem, the following lemma is obvious.

Lemma 2.3 *Any 4-connected planar odd graph is factor-critical.*

We now give an improvement of Theorem 1.6 as follows.

Theorem 2.4 *Any 4-connected planar odd graph is 3-factor-critical.*

Proof. Let G be a 4-connected planar odd graph. To the contrary, suppose that G is not 3-factor-critical, then, by Theorem 1.2, there exists a subset $S \subset V(G)$ with $|S| \geq 3$ such that $c_o(G-S) > |S|-3$, by parity, $c_o(G-S) \geq |S|-1$. By Lemma 2.3, G is also factor-critical, and so $c_o(G-S) \leq |S|-1$ by Theorem 1.2. Combining the above two inequalities, we have $c_o(G-S) = |S|-1 \geq 2$. Then S is a vertex cut of G. Since G is 4-connected, $|S| \geq 4$. Let B be the graph obtained from G by deleting all edges of $G[S]$ and all even components of $G-S$, and contracting each odd component to a single vertex, respectively. Clearly, B is bipartite and every vertex of $B-S$ is adjacent to at least four distinct vertices in S. Since $|B-S| = c_o(G-S) = |S|-1$, we have $|E(B)| \geq 4(|S|-1) = 4|S|-4$. On the other hand, B is a triangle-free and simple planar graph, by Theorem 2.1, $|E(B)| \leq 2|V(B)|-4 = 2(|S|+|S|-1)-4 = 4|S|-6$, contradicting the above inequality.

Now it follows that any 4-connected planar odd graph is 3-factor-critical. □

By Theorem 1.4 and Theorem 2.4, we have the following corollary.

Corollary 2.5 *Any 4-connected planar odd graph is near 2-extendable.*

Furthermore, for 5-connected planar odd graphs, we obtain the following result.

Theorem 2.6 *Any 5-connected planar odd graph is near 3-extendable*

Proof. Let G be a 5-connected planar odd graph, then G is near 2-extendable by Corollary 2.5. Suppose that G is not near 3-extendable. Then there exist three independent edges e_1, e_2 and e_3 which do not lie in any near perfect matching of G. Let $G' = G - V(e_1, e_2, e_3)$, then G' has no near perfect matching. Thus, by Theorem 1.1, there exists a subset $S' \subset V(G')$ such that $c_o(G' - S') > |S'| + 1$, by parity, $c_o(G' - S') \geq |S'| + 3$. Let $S = S' \cup V(e_1, e_2, e_3)$. Then $|S| \geq 6$ and $c_o(G - S) = c_o(G' - S') \geq |S'| + 3 = |S| - 3$, that is $c_o(G - S) \geq |S| - 3$. Since G is near 2-extendable, by Theorem 1.3, $c_o(G - S) \leq |S| - 3$. Hence, we have $c_o(G - S) = |S| - 3$. Let B be the graph obtained from G by deleting all edges of $G[S]$ and all even components of $G - S$, and contracting each odd component to a single vertex, respectively. Clearly, B is bipartite and every vertex of $B - S$ is adjacent to at least five distinct vertices in S. Since $|B - S| = c_o(G - S) = |S| - 3$, we have $|E(B)| \geq 5(|S| - 3) = 5|S| - 15$. On the other hand, B is a triangle-free and simple planar graph, by Theorem 2.1, $|E(B)| \leq 2|V(B)| - 4 = 2(|S| + |S| - 3) - 4 = 4|S| - 10 = 5|S| - 15 - (|S| - 5)$, contradicting the above inequality.

Therefore, any 5-connected planar odd graph is near 3-extendable. □

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