Characterizing $K_{r+1} - C_k$-graphic Sequences *

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Abstract

For given a graph H, a graphic sequence $\pi = (d_1, d_2, \cdots, d_n)$ is said to be potentially H-graphic if there exists a realization of π containing H as a subgraph. Let $K_{r+1} - C_k$ be the graph obtained from K_{r+1} by removing the k edges of a k-cycle. In this paper, we first characterize potentially $A_{r+1} - C_k(3 \leq k \leq r+1)$-graphic sequences which is analogous to Yin et.al characterization [19] using a system of inequalities. Then we obtain a sufficient and necessary condition for a graphic sequence π to have a realization containing $K_{r+1} - C_k$ as an induced subgraph.

Key words: graph; degree sequence; potentially $A_{r+1} - C_k$-graphic sequences; potentially $K_{r+1} - C_k$-graphic sequences

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1 Introduction

We consider finite simple graphs. Any undefined notation follows that of Bondy and Murty [1]. The set of all non-increasing nonnegative integer sequence $\pi = (d_1, d_2, \cdots, d_n)$ is denoted by NS_n. A sequence $\pi \in NS_n$ is said to be graphic if it is the degree sequence of a simple graph G of order n; such a graph G is called a realization of π. The set of all graphic sequences

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in NS_n is denoted by GS_n. A graphic sequence π is potentially H-graphic if there is a realization of π containing H as a subgraph. Let C_k and P_k denote a cycle on k vertices and a path on $k+1$ vertices, respectively. Let $G-H$ denote the graph obtained from G by removing the edges set $E(H)$ where H is a subgraph of G. In the degree sequence, r^t means r repeats t times, that is, in the realization of the sequence there are t vertices of degree r.

In the research of degree sequences, an important question is to characterize the potentially G-graphic sequences without zero terms, where G is a simple graph. *Erdös* and Gallai[3] gave a characterization for π to be graphic. Rao [14] and Kézdy and Lehel [10] independently gave a characterization for a sequence π to be potentially K_{r+1}-graphic. Lai and Hu in [12] proposed the following question: Characterizing $K_{r+1}-H$-graphic sequences for the graph $H \subseteq K_{r+1}$ and $H \neq K_{r+1}$. For $K_{r+1}-H = C_k$, Luo [13] characterized the potentially C_k-graphic sequences for each $k = 3, 4, 5$. Chen [2] characterized the potentially C_6-graphic sequences. If $\pi = (d_1, d_2, \cdots, d_n) \in GS_n$ has a realization G with the vertex set $V(G) = \{v_1, v_2, \cdots, v_n\}$ such that $d_G(v_i) = d_i$ for $1 \leq i \leq n$ and $v_1v_2\cdots v_rv_1$ is a cycle of length r in G, then π is said to be potentially C_r''-graphic. Recently, Yin[16] characterized the potentially C_r''-graphic sequences. Let $r \geq 3$ and $S(r)$ be the set of all circular arrangements of $1, 2, \cdots, r$. Let $\alpha = i_1i_2\cdots i_r \in S(r)$ and $\pi = (d_1, d_2, \cdots, d_n)$ be a graphic sequence with $n \geq r$. If π has a realization G with vertex set $V(G) = \{1, 2, \cdots, n\}$ such that $d_G(i) = d_i$ for $1 \leq i \leq n$ and $i_1i_2\cdots i_ri_1$ is a cycle of length r in G, then π is said to be potentially C_r^α-graphic. Yin and Wang[17] characterized the potentially C_r^α-graphic sequences for each $\alpha \in S(r)$. An extremal problem on potentially C_k-graphic sequences was investigated by Lai [11].

For the case $H = C_k (k \geq 3)$, Yin et.al[18-19] characterized the potentially $K_6 - C_3$ and $K_{r+1} - C_3$-graphic sequences. Hu and Lai[6-9] characterized the potentially $K_5 - C_3$, $K_5 - C_4$, $K_6 - C_4$ and $K_6 - C_6$-graphic sequences. Xu and Lai[15] characterized the potentially $K_6 - C_5$-graphic sequences. In this paper, we first characterize potentially $A_{r+1} - C_k (3 \leq k \leq r+1)$-graphic sequences which is analogous to Yin et.al characterization [19] using a system of inequalities. Then we obtain a sufficient and necessary condition for a graphic sequence π to have a realization containing $K_{r+1} - C_k$ as an induced subgraph.

2 Preparations

The following definitions will be useful for us to prove our main theorems.

If $\pi = (d_1, d_2, \cdots, d_n) \in GS_n$ has a realization G with the vertex set $V(G) = \{v_1, v_2, \cdots, v_n\}$ such that $d_G(v_i) = d_i$ for $1 \le i \le n$ and $G[\{v_1, v_2, \cdots, v_{r+1}\}] = K_{r+1} - C_k (3 \le k \le r+1)$ such that $d_{K_{r+1}-C_k}(v_i) = r$ for $1 \le i \le r+1-k$ and $d_{K_{r+1}-C_k}(v_i) = r-2$ for $r+2-k \le i \le r+1$, then π is said to be potentially $A_{r+1} - C_k$-graphic. For given graphs $H_1, H_2, \cdots, H_k$, π is said to be potentially $\{H_1, H_2, \cdots, H_k\}$-graphic if there is a realization of π containing one of $H_1, H_2, \cdots, H_k$ as a subgraph. Let H be a simple graph. We say that H satisfies the odd-cycle condition if between any two disjoint odd cycles there is an edge. We need the following results.

Theorem 2.1 [5] If $\pi = (d_1, d_2, \cdots, d_n)$ is a graphic sequence with a realization G containing H as a subgraph, then there exists a realization G' of π containing H as a subgraph so that the vertices of H have the largest degrees of π.

Theorem 2.2 [4] Assume that $H = (V(H), E(H))$ satisfies the odd-cycle condition, where $V(H) = \{v_1, v_2, \cdots, v_n\}$. There exists a subgraph $G \subseteq H$such that every vertex v_i has degree d_i, if and only if

(1) $\sum\limits_{i=1}^{n} d_i$ is even,

(2) for every $A, B \subseteq V(H)$ such that $A \cap B = \emptyset$, we have

$$\sum_{v_i \in A} d_i \le | \{(v_i, v_j) : v_i v_j \in E(H), v_i \in A, v_j \in V(H) \setminus B\} | + \sum_{v_i \in B} d_i$$

Theorem 2.3 [3] Let $\pi = (d_1, d_2, \cdots, d_n) \in NS_n$, where $\sum\limits_{i=1}^{n} d_i$ is even. Then π is graphic if and only if

$$\sum_{i=1}^{t} d_i \le t(t-1) + \sum_{i=t+1}^{n} min\{t, d_i\}$$

for each t, $1 \le t \le n$.

Lemma 2.4 [19] If $\pi = (d_1, d_2, \cdots, d_n)$ has a realization containing H as an induced subgraph so that the vertices of H have the largest degrees of π, then there exists a realization G of π with the vertex set $V(G) = \{v_1, v_2, \cdots, v_n\}$ such that $d_G(v_i) = d_i$ for $1 \le i \le n$, $G[\{v_1, v_2, \cdots, v_{|V(H)|}\}] = H$ and $d_H(v_1) \ge d_H(v_2) \ge \cdots \ge d_H(v_{|V(H)|})$.

3 Main Theorems

Theorem 3.1 Let $n \geq r+1$, $3 \leq k \leq r+1$, $\pi = (d_1, d_2, \cdots, d_n) \in GS_n$ with $d_{r+1-k} \geq r$ and $d_{r+1} \geq r-2$. Then π is potentially $A_{r+1} - C_k$-graphic if and only if

$$\sum_{i=1}^{p}(d_i - r) + \sum_{i=r+2-k}^{r+1-k+q}(d_i - r + 2) + \sum_{i=r+2}^{s+r+1} d_i \leq s(s-1) + 2(p+q)s$$
$$+ \sum_{i=p+1}^{r+1-k} min\{s, d_i - r\}$$
$$+ \sum_{i=r+2-k+q}^{r+1} min\{s, d_i - r + 2\}$$
$$+ \sum_{i=s+r+2}^{n} min\{p+q+s, d_i\}$$

for any p, q and s, $0 \leq p \leq r+1-k$, $0 \leq q \leq k$ and $0 \leq s \leq n-r-1$.

Proof: First we prove the necessity. Assume that $\pi = (d_1, d_2, \cdots, d_n) \in GS_n$ satisfies the conditions of Theorem 3.1, and, G is a realization of π with the vertex set $V(G) = \{v_1, v_2, \cdots, v_n\}$ such that $d_G(v_i) = d_i$ for $i = 1, 2, \cdots, n$ and $G[\{v_1, v_2, \cdots, v_{r+1}\}] = K_{r+1} - C_k$ so that $v_{r+2-k}v_{r+3-k}$, $v_{r+3-k}v_{r+4-k}, \cdots, v_r v_{r+1}$ and $v_{r+1}v_{r+2-k} \notin E(G)$. For $0 \leq p \leq r+1-k$, $0 \leq q \leq k$ and $0 \leq s \leq n-r-1$, denote $P = \{v_i \mid 1 \leq i \leq p\}$, $P' = \{v_i \mid p+1 \leq i \leq r+1-k\}$, $Q = \{v_i \mid r+2-k \leq i \leq r+1-k+q\}$, $Q' = \{v_i \mid r+2-k+q \leq i \leq r+1\}$, $S = \{v_i \mid r+2 \leq i \leq s+r+1\}$, $S' = \{v_i \mid s+r+2 \leq i \leq n\}$. The removal of the edges induced by $\{v_1, v_2, \cdots, v_{r+1}\}$ results in a graph G' in which all degrees in $\{v_1, v_2, \cdots, v_{r+1-k}\}$ are reduced by r and all degrees in $\{v_{r+2-k}, \cdots, v_{r+1}\}$ are reduced by $r-2$. Hence,

$$m = \sum_{i=1}^{p}(d_i - r) + \sum_{i=r+2-k}^{r+1-k+q}(d_i - r + 2) + \sum_{i=r+2}^{s+r+1} d_i - (s(s-1) + 2(p+q)s)$$

is the minimum number of edges of G' with exactly one endvertex in $P \cup Q \cup S$ and

$$M = \sum_{i=p+1}^{r+1-k} min\{s, d_i - r\} + \sum_{i=r+2-k+q}^{r+1} min\{s, d_i - r + 2\} + \sum_{i=s+r+2}^{n} min\{p+q+s, d_i\}$$

is the maximum number of edges of G' with exactly one endvertex in $P' \cup Q' \cup S'$. Thus, $m \le M$ is true.

Now we prove the sufficiency. Let $n \ge r+1$ and $\pi = (d_1, d_2, \cdots, d_n) \in GS_n$ with $d_{r+1-k} \ge r$, $d_{r+1} \ge r-2$. Let $\pi' = (d'_1, d'_2, \cdots, d'_{r+1-k}, d'_{r+2-k}, \cdots, d'_{r+1}, d'_{r+2}, \cdots, d'_n)$ where $d'_i = d_i - r$ for $1 \le i \le r+1-k$, $d'_i = d_i - r + 2$ for $r+2-k \le i \le r+1$ and $d'_i = d_i$ for $r+2 \le i \le n$. Let H be the graph obtained from K_n with the vertex set $V(K_n) = \{v_1, v_2, \cdots, v_n\}$ by deleting all edges between v_i and v_j for any $i, j \in \{1, \cdots, r+1\}$. It is easy to see that π is potentially $A_{r+1} - C_k$-graphic if and only if H has a subgraph G with the degree sequence π' such that every vertex v_i has degree d'_i. Notice that H satisfies the odd-cycle condition, we may use Theorem 2.2.

Let $T_1 = \{v_1, v_2, \cdots, v_{r+1-k}\}$, $T_2 = \{v_{r+2-k}, \cdots, v_{r+1}\}$ and $A, B \subseteq V(H)$ such that $A \cap B = \emptyset$. Let $A_1 = A \cap T_1$, $A_2 = A \cap T_2$, $A_3 = A \backslash \{T_1 \cup T_2\}$, $B_1 = B \cap T_1$, $B_2 = B \cap T_2$, $B_3 = B \backslash \{T_1 \cup T_2\}$ and set $p = |A_1|$, $q = |A_2|$, $s = |A_3|$, $b_1 = |B_1|$, $b_2 = |B_2|$, $b_3 = |B_3|$. For simplicity, we denote

$$K(p,q,s) = \sum_{i=1}^{p}(d_i - r) + \sum_{i=r+2-k}^{r+1-k+q}(d_i - r + 2) + \sum_{i=r+2}^{s+r+1} d_i$$

$$\begin{aligned} L(p,q,s) &= s(s-1) + 2(p+q)s + \sum_{i=p+1}^{r+1-k} min\{s, d_i - r\} \\ &+ \sum_{i=r+2-k+q}^{r+1} \min\{s, d_i - r + 2\} \\ &+ \sum_{i=s+r+2}^{n} min\{p+q+s, d_i\} \end{aligned}$$

$$K'(A,B) = \sum_{v_i \in A} d'_i = \sum_{v_i \in A_1}(d_i - r) + \sum_{v_i \in A_2}(d_i - r + 2) + \sum_{v_i \in A_3} d_i$$

$$\begin{aligned} L'(A,B) &= |\{(v_i, v_j) : v_i v_j \in E(H), v_i \in A, v_j \in V(H)\backslash B\}| + \sum_{v_i \in B} d'_i \\ &= |\{(v_i, v_j) : v_i v_j \in E(H), v_i \in A, v_j \in V(H) \setminus B\}| \\ &+ \sum_{v_i \in B_1}(d_i - r) + \sum_{v_i \in B_2}(d_i - r + 2) + \sum_{v_i \in B_3} d_i \end{aligned}$$

Clearly, $K'(A,B) \le K(p,q,s)$ and

$|\{(v_i, v_j) : v_i v_j \in E(H), v_i \in A, v_j \in V(H)\backslash B\}|$

$$= 2(p+q)s+s(s-1)+s(r+1-k-p-b_1)+s(k-q-b_2)+(p+q+s)(n-(r+1)-s-b_3)$$

$$= 2(p+q)s+s(s-1)+\sum_{i=p+1}^{r+1-k-b_1} s+\sum_{i=r+2-k+q}^{r+1-b_2} s+\sum_{i=s+r+2}^{n-b_3} (p+q+s)$$

Thus,

$$L'(A,B) = 2(p+q)s+s(s-1)+\sum_{i=p+1}^{r+1-k-b_1} s+\sum_{i=r+2-k+q}^{r+1-b_2} s+\sum_{i=s+r+2}^{n-b_3} (p+q+s)$$
$$+\sum_{v_i\in B_1} (d_i-r)+\sum_{v_i\in B_2} (d_i-r+2)+\sum_{v_i\in B_3} d_i$$
$$\geq 2(p+q)s+s(s-1)+\sum_{i=p+1}^{r+1-k-b_1} s+\sum_{i=r+2-k+q}^{r+1-b_2} s+\sum_{i=s+r+2}^{n-b_3} (p+q+s)$$
$$+\sum_{i=r+2-k-b_1}^{r+1-k} (d_i-r)+\sum_{i=r+2-b_2}^{r+1} (d_i-r+2)+\sum_{i=n+1-b_3}^{n} d_i$$
$$\geq 2(p+q)s+s(q-1)+\sum_{i=p+1}^{r+1-k} min\{s,d_i-r\}+\sum_{i=r+2-k+q}^{r+1} \min\{s,d_i-r+2\}$$
$$+\sum_{i=s+r+2}^{n} min\{p+q+s,d_i\}$$
$$= L(p,q,s)$$

Since $K(p,q,s) \leq L(p,q,s)$, we have $K'(A,B) \leq L'(A,B)$. By Theorem 2.2, H has a subgraph G with the degree sequence π' such that every vertex v_i has degree d'_i.

This completes the proof.

Theorem 3.2 Let $n \geq r+1$, $3 \leq k \leq r+1$, $\pi = (d_1, d_2, \cdots, d_n) \in GS_n$ with $d_{r+1-k} \geq r$ and $d_{r+1} \geq r-2$. If π is not potentially $\{K_{r+1}, K_{r+1} - H(H \subseteq C_k \text{ and } H \neq C_k)\}$-graphic, then π is potentially $K_{r+1}-C_k$-graphic if and only if π is potentially $A_{r+1} - C_k$-graphic.

Proof: Clearly, we only need to show that if π is potentially $K_{r+1}-C_k$-graphic, then π is potentially $A_{r+1} - C_k$-graphic. This is the immediate consequence of Theorem 2.1 and Lemma 2.4.

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