A kind of conditional vertex connectivity of recursive circulants*

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Abstract A vertex subset F is a R_k-vertex-cut of a connected graph G if $G-F$ is disconnected and every vertex in $G-F$ has at least k good neighbors in $G-F$. The cardinality of the minimum R_k-vertex-cut of G is the R_k-connectivity of G, denoted by $\kappa^k(G)$. This parameter measures a kind of conditional fault tolerance of networks. In this paper, we determine R_1-connectivity and R_2-connectivity of recursive circulant graphs $G(2^m, 2)$.

Keywords: Interconnection networks; Cayley graphs; Conditional connectivity; Recursive circulants

1 Introduction

For graph-theoretical terminologies and notations not given here, we follow Bondy [1]. In a network, traditional connectivity is an important measure since it can correctly reflect the fault tolerance of network systems with few processors. However, it always underestimates the resilience of large networks. There is a discrepancy because the occurrence of events which would disrupt a large network after a few processor or link failures

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is highly unlikely. To overcome such shortcoming, Latifi et al [7] defined R_h-connectivity [5] as follows. A vertex subset F is a R_k-vertex-cut of a connected graph G if $G-F$ is disconnected and every vertex in $G-F$ has at least k good neighbors in $G-F$. The cardinality of the minimum R_k-vertex-cut of G is the R_k-connectivity of G, denoted by $\kappa^k(G)$. Park and Chwa [11] proposed an interconnection structure for multi-computer networks as follows.

Definition 1. *The* recursive circulant $G(N,d)$ *has vertex set* $V=\{0,1,\ldots,N-1\}$*, and edge set* $E=\{(v,w)\ v\in V, w\in V|$ *there exists* i, $0\le i\le \lceil\log_d N\rceil-1$, *such that* $v+d^i\equiv w(\mod N)\}$.

Definition 2. *For a group* X*, let* S *be a subset of* X *such that* $1_X\notin S$ *and* $S^{-1}=S$*, the* Cayley graph $Cay(X,S)$ *is a graph with vertex set* X *and edge set* $\{(g,sg)|\ g\in X, s\in S\}$.

The hypercube is a well known model for computer networks which has been attracted many attentions in the past four decades, for example [2, 3] and [6, 7, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19]. An *n-dimensional hypercube* is an undirected graph $Q_n=(V,E)$ with $|V|=2^n$ and $|E|=n2^{n-1}$. Each vertex can be represented by an n-bit binary string. There is an edge between two vertices whenever their binary string representation differs in exactly one bit position.

Combining Definition 1 with Definition 2, it can be seen that $G(N,d)\cong Cay(Z_N,\{\pm d^0,\pm d^1,\ldots,\pm d^{\lceil\log_d N\rceil}\})$. Thus, $G(N,d)$ is vertex symmetric. By the definition of the hypercube, we have $G(2^m,2)$ is a supergraph of Q_m, see [8, 18] for the studies of embedding a graph in recursive circulants. Three simple examples of recursive circulants are given in Fig.1.

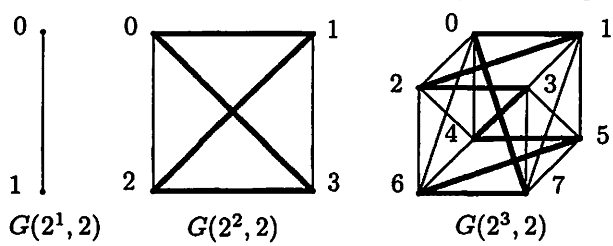

Fig.1. Examples of recursive circulants

In particular, $G(2^m,2)$ has has many advantages over the m-dimensional hypercube, see Table.1. It can been seen that if a hypercube and recursive circulant $G(2^m, 2)$ have the same number of vertices, the recursive circulant $G(2^m, 2)$ has higher connectivity which increases the fault tolerance, smaller diameter which reduces the transmission delay, etc., see [7, 11] for the details.

Graph	Dimension	Vertices	Degree	Diameter	Connectivity
Q_m	m	2^m	m	m	m
$G(2^m, 2)$	m	2^m	$2m-1$	$\lfloor \frac{m}{2} \rfloor$	$2m-1$

Table.1

There are many results on the R_h-connectivity of hypercubes, see [7, 9, 13, 15] for examples. Park and Chwa [11] shown that the connectivity of $G(2^m, 2)$ $\kappa(G(2^m, 2))$ equals its regular degree $2m-1$. In this note, we determine the R_h-connectivity of $G(2^m, 2)$ for $h = 1, 2$.

2 preliminaries

For the graphs $G(cd^m, d)$, we always assume $m \geq 1$. Park and Chwa shown in [11] that recursive circulant $G(N, d)$ has a recursive structure when $N = cd^m, 1 \leq c < d$ as follows.

Property 1. *Let V_i be a subset of vertices in $G(cd^m, d)$ such that $V_i = \{v \mid v \equiv i(\mod d)\}, m \geq 1, 0 \leq i < d$. The subgraph of $G(cd^m, d)$ induced by each V_i is isomorphic to $G(cd^{m-1}, d)$.*

$G(cd^m, d), m \geq 1$, can be defined recursively on d copies of $G(cd^{m-1}, d)$ as follows. Let $G_i(V_i, E_i), 0 \leq i < d$, be a copy of $G(cd^{m-1}, d)$. We assume that $V_i = \{v_0^i, v_1^i, \ldots, v_{cd^{m-1}-1}^i\}$ and relabel v_j^i by $jd + i$. Thus $V(G(cd^m, d)) = \bigcup_{0 \leq i < d} V_i$, and $E(G(cd^m, d)) = \bigcup_{0 \leq i < d} E_i \bigcup X$, where $X = \{(v, w) \mid v - w \equiv \pm 1(\mod cd^m)\}$. Furthermore, the edges of X form a hamiltonian cycle of $G(cd^{m-1}, d)$, see [11] for the details. The construction of $G(2^3, 2)$ on two copies of $G(2^2, 2)$ is shown in Fig.1.

For an edge $e = (x, y)$ of $G(2^m, 2)$, if $x - y \equiv \pm 2^i(\mod 2^m)\}$, we say e has label i. For $G(2^m, 2)$, let $V_i = \{v \mid v \equiv i(\mod 2)\}, i = 0, 1$. By the Property 1, $G(2^m, 2)[V_i]$ is isomorphic to $G(2^{m-1}, 2)$. For notational convenience, we use X to denote the hamiltonian cycle of $G(2^m, 2)$ consisting of the edge set $\{(x, y) \mid x - y \equiv \pm 1(\mod 2^m)\}$, use $G_i(2^m, 2)$ to denote $G(2^m,2)[V_i]$ and assume that M_1 and M_2 are two perfect matching of $G(2^m,2)$ such that $X = M_1 \bigcup M_2$. Let S be a vertex set of $G_i(2^m,2)$. We use $N_G(x)$ to denote the neighborhood of the vertex x in G, $N_G(S)$ to denote the set $\bigcup_{x\in S} N_G(x) \setminus S$, $N_{M_i}(S)$ to denote the set of the vertex adjacent to a vertex of S by some edge of M_i. If no confusion, the subscripts of $N_G(x)$ and $N_G(S)$ are always omitted in the following.

Lemma 2.1. *Let H be a triangle in $G(2^m,2)$. Then there are two edges of H having the same label l, and another edge has label $l+1$.*

Proof. Assume that $V(H) = \{x,y,z\}$ with $x<y<z$, and $(x,y),(y,z)$ and (x,z) have label i,j and k, respectively. Since H is a triangle, we have

$$x + 2^i + 2^j \equiv x + 2^k (\text{mod} 2^m)$$

Notice that i,j,k are less than m, we have

$$\begin{cases} i = j \\ k = i+1 = j+1. \end{cases}$$

Thus, there are two edges of H having the same label l, and another edge has label $l+1$. □

Let (x,y) be an edge of $G(2^m,2)$ with label i. Without loss of generality we can assume $y = x+2^i$, then we can see that all the common neighbors of x and y are $y+2^i, x-2^i, x+2^{i-1}$ if $1 \le i \le m-2$. By an argument similar to above and Lemma 2.1, we have the following corollary.

Corollary 2.2. *If $(x,y) \in E(G(2^m,2))$ has label i, then*

$$|N(x)\bigcap N(y)| = \begin{cases} 2, & i=0 \text{ or } i=m-1, \\ 3, & 1 \le i \le m-2. \end{cases}$$

Lemma 2.3. *Suppose x,y are two distinct nonadjacent vertices of $G(2^m,2)$. Then $|N(x)\bigcap N(y)| \le 4$.*

Proof. Let x,y be two distinct nonadjacent vertices of $G(2^m,2)$. Note that $G(2^m,2))$ is vertex symmetry, without loss of generality we can assume $x=0$. Assume $y = 2^i+2^j$. By the definition of $G(2^m,2))$, we can see that all the common neighbors of x and y are $2^i, 2^j$ and $-2^k, -2^l$ if k,l exist, where k,l are two integers such that $-2^k - 2^l \equiv 2^i + 2^j(\bmod 2^m)$. Note that $i,j,k,l \le m-1$, we have $|N(x)\bigcap N(y)| \le 4$. □

In particular, it is not difficult to see that $|N(x)\bigcap N(y)| = 4$ if and only if $x - y \equiv \pm(2^{m-2} + 2^{m-3})(\mod 2^m)$. For example, the common neighborhood of 0 and $2^{m-2} + 2^{m-3}$ is $\{2^{m-3}, 2^{m-2}, -2^{m-1}, -2^{m-3}\}$.

Observation 1. *Let $(x,y) \in E(G(2^4,2))$. Then $N(\{x,y\})$ is a R_1-vertex-cut of $G(2^4,2)$ (see Fig.2).*

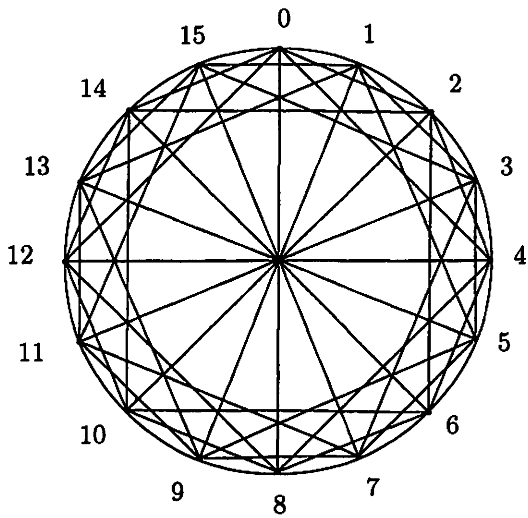

*Fig.*2 $G(2^4,2)$

Lemma 2.4. *Let $(x,y) \in E(G(2^m,2))$. Then $N(\{x,y\})$ is a R_1-vertex-cut of $G(2^m,2)$. Furthermore, $\kappa^1(G(2^m,2)) \le 4m-7$.*

Proof. By induction on m. Let $(x,y) \in E(G(2^m,2))$ with label i. If $m=4$, then the result holds by Observation 1. Suppose $m \ge 5$ in the following. We shall show that $V(G(2^m,2)) - N(\{x,y\})$ contains no isolated vertex.

Case 1. $i = 0$.

In this case, (x,y) is an edge between $G_0(2^m,2)$ and $G_1(2^m,2)$. Without loss of generality, we assume that $x \in V_0, y \in V_1$. It is sufficient to show that each vertex $z \in V(G(2^m,2)) - N(\{x,y\})\bigcup\{x,y\}$ has a neighbor in $V(G(2^m,2)) - N(\{x,y\})\bigcup\{x,y\}$. Without loss of generality, we assume that $z \in V(G_0(2^m,2))$. By Lemma 2.3, x and z have at most 4 common neighbors, and note that $|N_{G_0(2^m,2)}(z)| = 2(m-1)-1 = 2m-3 \ge 7$ for $m \ge 5$, thus there exists a neighbor of z in $V(G_0(2^m,2)) - N(\{x,y\})\bigcup\{x,y\}$.

Case 2. $1 \le i \le m-1$.

In this case, we have either $(x,y) \in E(G_0(2^m,2))$ or $(x,y) \in E(G_1(2^m,2))$. Without loss of generality, we assume that $(x,y) \in E(G_0(2^m,2))$. Note that $|N(\{x,y\}) \bigcap V(G_1(2^m,2)| \le 4$ and $\kappa(G_1(2^m,2)) = 2(m-1)-1 = 2m-3 \ge 7$ for $m \ge 5$. Hence $G_1(2^m,2) - N(\{x,y\})$ is connected, that is, there exists no isolated vertex in $G_1(2^m,2) - N(\{x,y\})$. By induction hypothesis, $N_{G_0(2^m,2)}(\{x,y\})$ is a R_1-vertex-cut of $G_0(2^m,2)$, that is, there exists no isolated vertex in $G_0(2^m,2) - N(\{x,y\})$. Thus, $N(\{x,y\})$ is a R_1-vertex-cut of $G(2^m,2)$.

Note that if $(x,y) \in E(G(2^m,2))$ has label $i, 1 \le i \le m-2$, then $|N(x) \bigcap N(y)| = 3$ by Corollary 2.2. Thus, $\kappa^1(G(2^m,2)) \le |N(x)| + |N(y)| - |\{x,y\}| - |N(x) \bigcap N(y)| = 2(2m-1) - 2 - 3 = 4m-7$. □

By an argument similar to above, we have the following results.

Observation 2. *For any triangle H of $G(2^5,2)$, $N(H)$ is a R_2-vertex-cut of $G(2^5,2)$.*

Lemma 2.5. *For any triangle H of $G(2^m,2), m \ge 5$, $N(H)$ is a R_2-vertex-cut of $G(2^m,2)$. Furthermore, $\kappa^2(G(2^m,2)) \le 6m-15$.*

3 Main result

Theorem 3.1. $\kappa^1(G(2^m,2)) = 4m-7$ *for* $m \ge 4$.

Proof. By Lemma 2.4, $\kappa^1(G(2^m,2)) \le 4m-7$. Now we show that $\kappa^1(G(2^m, 2)) \ge 4m-7$.

Suppose by the way of contradiction that $\kappa^1(G(2^m,2)) \le 4m-8$. Let F be a minimum R_1-vertex-cut, then $|F| \le 4m-8$. Denote by F_i the vertex set $F \bigcap V(G_i(2^m,2))$ in the following arguments. Note that $\kappa(G_0(2^m,2)) = \kappa(G_0(2^m,2)) = 2(m-1)-1$, we have at least one of $G_0(2^m,2) - F_0$ and $G_1(2^m,2) - F_1$ is connected.

Case 1. $G_0(2^m,2) - F_0$ and $G_1(2^m,2) - F_1$ are both connected.

We claim that $G_0(2^m,2) - F_0$ is connected to $G_1(2^m,2) - F_1$. Note that $|N_{M_1}(G_0(2^m,2)) \bigcap V(G_1(2^m,2))| = 2^{m-1}$, $|F| = 4m-8$ and $2^{m-1} \ge 4m-8$ for $m \ge 4$. If $m \ge 5$, then the above inequality is strict, that is, there exists $x \in G_0(2^m,2) - F_0, y \in G_1(2^m,2) - F_1$ such that $(x,y) \in M_1$.

That is, $G_0(2^m,2)-F_0$ is connected to $G_1(2^m,2)-F_1$. On the other hand, if $m = 4$, then the above inequality becomes equality, but clearly there exists $x \in G_0(2^m,2) - F_0, y \in G_1(2^m,2) - F_1$ such that $(x,y) \in M_2$. That is, $G_0(2^m,2) - F_0$ is connects to $G_1(2^m,2) - F_1$.

Case 2. One of $G_0(2^m,2) - F_0$ and $G_1(2^m,2) - F_1$ is disconnected.

Without loss of generality, we assume that $G_0(2^m,2)-F_0$ is disconnected and has r components, $H_1, H_2, \ldots H_r, r \geq 2$. Since F is a R_1-vertex-cut, there exists a nontrivial component H_j of $G_0(2^m,2) - F_0$ such that it is disconnected to $G_1(2^m,2)-F_1$. Take $(x,y) \in E(H_j)$ and assume that (x,y) has label i. Let $S_1 = (N_{G_0(2^m,2)}(\{x,y\})) \bigcap F$, $S_2 = (N_{G_0(2^m,2)}(\{x,y\})) \setminus F$ and $S_3 = N_{G_1(2^m,2)}(x) \bigcup N_{G_1(2^m,2)}(y)$, see Fig.3. Clearly, $S_3 \in F_1$. By the definition of $G(2^m,2)$, it is not difficult to see that S_2 has at least $|S_2|$ neighbors in $G_1(2^m,2)$ which are not in S_3. Thus $|F| \geq |S_1|+|S_2|+|S_3| \geq 4m-7 > 4m-8$, a contradiction. □

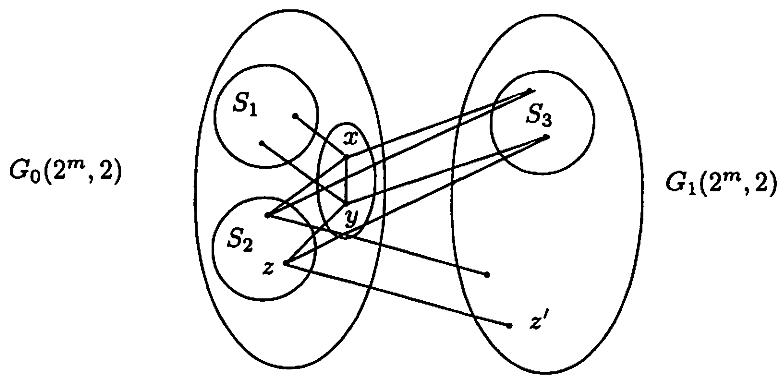

Fig.3

Theorem 3.2. $\kappa^2(G(2^m,2)) = 6m - 15$ *for* $m \geq 5$.

Proof. By Lemma 2.5, $\kappa^2(G(2^m,2)) \leq 6m-15$. Now we show that $\kappa^2(G(2^m, 2)) \geq 6m - 15$. Suppose by the way of contradiction that $\kappa^2(G(2^m,2)) \leq$

$6m-16$. Let F be a minimum R_1-vertex-cut, then $|F| \leq 6m-16$. Denote by F_i the vertex set $F \bigcap V(G_i(2^m, 2))$ in the following argument. We denote the number of all isolated vertices contained in $G_0(2^m, 2) - F_0$ and $G_1(2^m, 2) - F_1$ by k. Clearly, $k < 3$.

Case 1. $k = 0$.

If $G_0(2^m, 2) - F_0$ and $G_1(2^m, 2) - F_1$ are connected. This case can be proved by a similar argument of Case 1 of Theorem 2.4.

If one of $G_0(2^m, 2) - F_0$ and $G_1(2^m, 2) - F_1$ is disconnected. Without loss of generality, we assume that $G_0(2^m, 2) - F_0$ is disconnected and has r components, $H_1, H_2, \ldots H_r$. Note that F is a R_2-vertex-cut, then there exists a component H_i with $\delta(H_i) \geq 2$ such that H_i is disconnected to $G_1(2^m, 2) - F_1$. Take a path P_3 of H_i with three vertices, similar to Case 2 of Theorem 3.1, we can complete the proof of this case.

If $G_0(2^m, 2) - F_0$ and $G_1(2^m, 2) - F_1$ are disconnected. Since there exists no isolated vertex in $G_0(2^m, 2) - F_0$ and $G_1(2^m, 2) - F_1$, F_0, F_1 are both R_1-vertex-cuts of $G_0(2^m, 2) - F_0$ and $G_1(2^m, 2) - F_1$, respectively. Note that $|F_0| \geq \kappa^1(G_0(2^m, 2))$, $|F_1| \geq \kappa^1(G_1(2^m, 2))$, and $\kappa^1(G_0(2^m, 2)) = \kappa^1(G_1(2^m, 2)) = 4(m-1) - 7 = 4m - 11$ we have $|F| = |F_0| + |F_1| \geq 8m - 22 > 6m - 16$ for $m \geq 5$, a contradiction.

Case 2. $k = 1$.

Without loss of generality, We assume that $G_0(2^m, 2) - F_0$ contains an isolated vertex.

If $G_1(2^m, 2) - F_1$ is disconnected. Note that F_0 is a vertex cut of $G_0(2^m, 2)$ and F_1 is a R_1-vertex-cut of $G_1(2^m, 2)$, we have $|F| = |F_0| + |F_1| \geq 2(m-1) - 1 + 4(m-1) - 7 = 6m - 14 > 6m - 16$, a contradiction.

If $G_1(2^m, 2) - F_1$ is connected. Thus, there exists a component H of $G_0(2^m, 2) - F_0$ with $\delta(H) \geq 2$ such that H is disconnected to $G_1(2^m, 2) - F_1$. Take a path P_3 in H, similar to Case 2 of Theorem 3.1, we can complete the proof of this case.

Case 3. $k = 2$.

Let x, y be the two isolated vertices of $G(2^m, 2) - F$. If x, y are both in $G_0(2^m, 2)$ or $G_1(2^m, 2)$. We can prove this case by an argument similar to case 2.

With loss of generality, we next assume $x \in G_0(2^m,2)$ and $y \in G_1(2^m,2)$. Let $F' = F\bigcup\{x\}$. Denote by F_i' the vertex set $F'\bigcap V(G_i(2^m,2))$. If $G_0(2^m,2) - F_0'$ is disconnected, then F_0' is a R_1-vertex-cut of $G_0(2^m,2)$. Hence, $|F| = |F'|-1 = |F_0'|+|F_1'|-1 \geq 4(m-1)-7+2(m-1)-1-1 = 6m-15 > 6m-16$, a contradiction. If $G_0(2^m,2) - F_0'$ is connected. Similarly, let $F'' = F\bigcup\{y\}$, we have $G_1(2^m,2) - F_1''$ is connected. Assume that $G_0(2^m,2) - F_0' = B_0$ and $G_1(2^m,2) - F_1'' = B_1$. Note that $|N_{G_1(2^m,2)}(B_0))| \geq |B_0|+1$, $|N_{G_0(2^m,2)}(B_1))| \geq |B_1|+1$, if B_0 is disconnected to B_1, we have

$$\begin{aligned} |F| &= |F_0|+|F_1| \geq |N_{G_0(2^m,2)}(B_1))|-1+|N_{G_1(2^m,2)}(B_0))|-1 \\ &\geq |B_1|+|B_0| \geq 2^m-|F|-2 = 2^m-2-|F| \end{aligned}$$

Hence, $2|F| \geq 2^m - 2$, this is impossible for $m \geq 5$, a contradiction. □

References

[1] J.A. Bondy and U.S.R. Murty, Graph Theory with Applications. Macmillan, New York, 1976.

[2] M. Bae and B. Bose, Edge Disjoint Hamiltonian Cycles in k-Ary n-Cubes and Hypercubes, IEEE.Trans.on Computers, 52 (2003) 1271-1284.

[3] A,H.Esfahanian Generalized Measure of Fault Tolerance with Application to N-cube Networks, IEEE Trans. Comput., 38 (1989) 1586-1591.

[4] K.Day and A.Tripathi, A comparative study of topological properties of hypercubes and star graphs, IEEE Trans. Comput. 5(1) (1994) 31-38.

[5] F. Haray, Conditional connectivity. Networks, 13 (1983) 346-357.

[6] P. Lai, J. Tan, C. Chang, and L. Hsu, Conditional Diagnosability Measures for Large Multiprocessor Systems, IEEE Tans. Comput., 54 (2005) 165-175.

[7] S.Latifi, M.Hegde and M.Naraghi-Pour, Conditional connectivity measures for large multiprocessor systems, IEEE.Trans.on Computers, 43 (1994) 218-222.

[8] C. Micheneau, Disjoint Hamiltonian cycles in recursive circulant graphs, Information processing Letters 61 (1997) 259-264.

[9] A. D. Oh and H. Choi, Generalized Measures of Fault Tolerance in n-Cube Networks, IEEE Transactions on Parallel and Distributed Systems, 4 (1993) 702-703.

[10] Y.Saad and M.H.Schultz, Topological properties of hypercubes, IEEE Trans. on Compters, 37(7) (1988) 867-872.

[11] J.H. Park and K.Y. Chwa, Recursive circulants and their embeddings among hypercubes. Theoretical Computer Science, 244 (2000), 35-62.

[12] J. Wu and G. Guo, Fault Tolerance Measures for m-Ary n-Dimensional Hypercubes Based on Forbidden Faulty Sets, IEEE Trans. Comput., 47 (1998) 888-893.

[13] J.M.Xu, Q.Zhu, X.M.Hou and T.Zhou, On restricted connectivity and extra connectivity of hypercubes and foled hypercubes, J. of Shanghai Jiaotong Univ., 10(2)(2005) 203-207.

[14] J.M.Xu, On conditional edge-connectivity of graphs, Acta Math. Appli. Sinica, 16B 4 (2000), 414-419.

[15] W. Yang and J. Meng, Generalized Measures of Fault Tolerance in Hypercube Networks, submitted to Applied Mathematics Letters.

[16] W.Yang and J.Meng, Extraconnectivity of Hypercubes, Applied Mathematics Letters, 22 (2009) 887-891.

[17] X.Yang, J. Cao, M. Megson and J. Luo, Minimum neighborhood in a generalized cube, Information Processing Letters, 97 (2006) 88-93.

[18] X. Yang, D.J. Evans and G.M. Megson, Maximum induced subgraph of a recursive circulant, Information Processing Letters 95 (2005) 293 - 298.

[19] Q.Zhu and J.M.Xu, On restricted edge connectivity and extra edge connectivity of hypercubes and foled hypercubes, J. University of Science and Technology of China, 36(3) (2006) 246-253.