# Special minimum cuts in directed graphs
NOTE

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**Abstract**

The purpose of this paper is to solve the odd minimum $S$-cut, the odd minimum $\overline{T}$-cut and the odd minimum $(S, T)$-cut problems in directed graphs using triple families. We also provide here two properties of triple families.

## 1 Introduction

Let $\vec{G} = (V, \vec{E})$ be a directed graph with at least two vertices. Usually, a cut of the graph is defined as a bipartition of its vertex set $V$ into $C \subset V$ and its complement. (Sometimes, in the definition of cut instead of the bipartition the edges joining $C$ and $V - C$ are considered.)

For simplicity, in this paper the cut of the graph is a subset $C$ of the nodes, the value of the cut $f(C)$ is the number (or total capacity) of the edges leaving $C$.

Note that the cut value function $f$ is submodular over the ground set $V$, i.e. all subsets $X, Y \subseteq V$ satisfy $f(X) + f(Y) \geq f(X \cap Y) + f(X \cup Y)$.

Grötschel et al. ([3]) define triple families as a generalization of families of odd (cardinality) sets as follows. A family $\mathcal{G}$ of subsets of a ground set $V$ forms a *triple family over* $V$ if for all $X \subseteq V$ and $Y \subseteq V$ whenever three of the four sets $X$, $Y$, $X \cap Y$ and $X \cup Y$ are not in the triple family, then so is the fourth.

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## 2 Properties of triple families

Many problems in combinatorial optimization can be solved by minimizing a submodular function over a triple family. Before considering some applications here, let us examine triple families.

**Theorem 2.1.** *If $\mathcal{G}$ is a triple family over $V$ and $C_i \notin \mathcal{G}$, $i \in \{1, \ldots, k\}$ such that $\bigcup\{C_j : j \in L\} \notin \mathcal{G}$ for each nonempty subset $L \subseteq \{1, \ldots, k\}$ then we have $\bigcap\{C_i : i = 1, \ldots, k\} \notin \mathcal{G}$.*

*Proof.* By induction on $k$. For $k = 2$ Theorem 2.1 follows from the definition of triple families. Let us consider $C_i \notin \mathcal{G}$, $i \in \{1, \ldots, k+1\}$ such that $\bigcup\{C_j : j \in L\} \notin \mathcal{G}$ for each nonempty $L \subseteq \{1, \ldots, k+1\}$. Supposing indirectly that $\bigcap\{C_i : i = 1, \ldots, k+1\} \in \mathcal{G}$ and using the definition of triple families and the induction hypothesis we obtain $(C_1 \bigcap \cdots \bigcap C_k) \bigcup C_{k+1} \in \mathcal{G}$, i.e. $(C_1 \bigcup C_{k+1}) \bigcap \cdots \bigcap (C_k \bigcup C_{k+1}) \in \mathcal{G}$, but this set is non-member by induction, contradiction. □

Similarly we can prove

**Theorem 2.2.** *If $\mathcal{G}$ is a triple family over $V$ and $C_i \notin \mathcal{G}$, $i \in \{1, \ldots, k\}$ such that $\bigcap\{C_j : j \in L\} \notin \mathcal{G}$ for each nonempty subset $L \subseteq \{1, \ldots, k\}$ then we have $\bigcup\{C_i : i = 1, \ldots, k\} \notin \mathcal{G}$.*

## 3 Special minimum cuts

Let $S$ and $T$ be two disjoint subsets of $V$ different from $\emptyset$ and $V$. We consider the following special minimum cut problems in the directed graph $\vec{G} = (V, \vec{E})$ with at least two nodes:

- The odd (even) minimum $S$-cut problem asks for cut $C$ such that $S \subseteq C$, $|C|$ is odd (even) with $f(C)$ minimum.
- The odd (even) minimum $\overline{T}$-cut problem asks for a cut $C$ such that $T \bigcap C = \emptyset$, $|C|$ is odd (even) with $f(C)$ minimum.
- The odd (even) minimum $(S, T)$-cut problem asks for a cut $C$ such that $S \subseteq C$, $T \bigcap C = \emptyset$, $|C|$ is odd (even) with $f(C)$ minimum.

The third problem is a generalization of the directed odd or even minimum $(s, t)$-cut problem [2]. Notice that if $\mathcal{G}$ is an arbitrary triple family over $V$, ($|V| \geq 2$), then for two arbitrarily fixed nonempty disjoint subsets $S \subset V$ and $T \subset V$ (different from $V$), families of sets $\mathcal{G} \cap \{X \subseteq V : S \subseteq X\}$,

$\mathcal{G}\cap\{X \subset V: X\bigcap T = \emptyset\}$ and $\mathcal{G}\cap\{X \subset V: S \subseteq X, X\bigcap T = \emptyset\}$ are not triple families over $V$. Thus the above mentioned three problems do not ask for minimum value cut in triple families. They ask for the minimum value $S$-cut, $\overline{T}$-cut and ($S$, $T$)-cut in triple families.

**Theorem 3.1.** *Let $\mathcal{G}\subseteq 2^V$ be a triple family, $S$ and $T$ be two disjoint subsets of the ground set $V$ different from $\emptyset$ and $V$. Let us denote $\mathcal{G}_1 := \{X - S: X \in \mathcal{G}, S \subseteq X\}$, $\mathcal{G}_2 := \{X: X \in \mathcal{G}, T\bigcap X = \emptyset\}$ and $\mathcal{G}_3 := \{X - S: X \in \mathcal{G}, S \subseteq X, T\bigcap X = \emptyset\}$, furthermore $V_1 := V - S$, $V_2 := V - T$ and $V_3 := V - (S\bigcup T)$. Then for all $i \in \{1,2,3\}$, $\mathcal{G}_i$ forms a triple family over $V_i$.*

*Proof.* For $i = 3$ by definition of $\mathcal{G}_3$, a subset $A$ of $V_3$ is not in $\mathcal{G}_3$ iff $A\bigcup S$ is not in $\mathcal{G}$. Let $A$ and $B$ be two arbitrarily fixed subsets of $V_3$. Suppose that three of the four sets $A$, $B$, $A\bigcap B$, $A\bigcup B$ are not in $\mathcal{G}_3$, this means that three of the four sets $A\bigcup S$, $B\bigcup S$, $(A\bigcap B)\bigcup S$, $(A\bigcup B)\bigcup S$ are not the triple family $\mathcal{G}$, hence so is the fourth. If we leave out $S$ from the fourth set we obtain that the fourth set from $A$, $B$, $A\bigcap B$, $A\bigcup B$ (which is also a subset of $V_3$) is not in $\mathcal{G}_3$. The proof for $i = 1$ and 2 is similar. □

Let us consider the triple family $\mathcal{G}_i$ over $V_i$ from Theorem 3.1. We can use our algorithm for minimizing submodular functions over triple families from [1], which may return $\emptyset$ or $V_i$ with $O\left(|V|^2 \cdot |\vec{E}| + |V| \cdot M(|V|, |\vec{E}|)\right)$ running time, where $M(|V|, |\vec{E}|)$ denotes the time of a ($u$, $v$)-minimum cut computation ([1], Section 4.2).

Our algorithm from [1] uses the Cheng-Hu flow-equivalent tree and a specific uncrossing procedure that we call parity uncrossing, and means a factor $O(n)$ improvement over the running time of the previous most efficient algorithm of Goemans and Ramakrishnan for triple families [4].

In case of $i = 1$ (i.e. the odd (even) minimum $S$-cut problem) if $Y_0$ is the output of the algorithm from [1], i.e. $Y_0 \subseteq V - S$, $Y_0 \in \mathcal{G}_1$ with $f(Y_0)$ minimum, then $C := Y_0\bigcup S$ is an $f$-minimizer over $\mathcal{G}$ such that $S \subseteq C$.

In case of $i = 2$ (i.e. the odd (even) minimum $\overline{T}$-cut problem) if $Y_0$ is the output, namely $Y_0 \subseteq V - T$, $Y_0 \in \mathcal{G}_2$ with $f(Y_0)$ minimum, then $C := Y_0$ is an $f$-minimizer over $\mathcal{G}$ such that $C\bigcap T = \emptyset$.

In the remaining case of $i = 3$ (the odd (even) minimum ($S$, $T$)-cut problem) if $Y_0$ is the output of our algorithm from [1], which means that $Y_0 \subseteq V - (S\bigcup T)$, $Y_0 \in \mathcal{G}_3$ with $f(Y_0)$ minimum, then $C := Y_0\bigcup S$ is an $f$-minimizer over $\mathcal{G}$ such that $S \subseteq C$ and $C\bigcap T = \emptyset$.

# References


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