Path Extendability and Degree Sum in Graphs[1]

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Abstract: In this paper, we study the relations between degree sum and extending paths in graphs. The following result is proved. Let G be a graph of order n, if $d(u) + d(v) \geq n + k$ for each pair of nonadjacent vertices u, v in $V(G)$, then every path P of G with $\frac{n}{k+2} + 2 \leq |P| < n$ is extendable. The bound $\frac{n}{k+2} + 2$ is sharp.
Key words: degree of vertex; extending path.

1 Introduction and notation

In this paper, we consider only finite undirected graphs without loops and multiple edges. For notation and terminology not defined here we refer to [2]. Let G be a graph, and $V(G)$ and $E(G)$ denote the vertex set and the edge set of G respectively. For any $a \in V(G)$, $S, T \subset V(G)$ and any subgraph H of G, we put

$$N_S(a) = \{v \in S : va \in E(G)\},\ N_H(a) = N_{V(H)}(a),\ d_H(a) = |N_H(a)|,$$

$$N_S(T) = \bigcup_{v \in T} N_S(v),\ N_H(T) = N_{V(H)}(T),$$

and the order of H is the number of vertices in H, which is denoted by $|H|$ or $|V(H)|$. The degree of vertex a is defined to $|N_G(a)|$, and it is denoted by $d_G(a)$ or $d(a)$. Let

$$\delta(G) = min\{d(v) : v \in V(G)\},$$

$$\sigma_2(G) = min\{d(u) + d(v) : u, v \in V(G), uv \notin E(G)\},$$

and we denote by $G[S]$ the subgraph of G induced by S.

A path with end vertices u and v is called an (u, v)-path. An (u, v)-path P is said to be extendable if there is an (u, v)-path P' in G such that

[1]The work was supported by Doctoral Fund of Shandong Province (BS2010SW030)
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$V(P) \subset V(P')$ and $|V(P')| = |V(P)| + 1$. In this case we say that P' is one of the extending paths of P. A path is a hamiltonian path of G if it contains all the vertices of G. A graph G is said to be path extendable if every nonhamiltonian path in G is extendable.

In recent years, many results on hamiltonian properties of graphs have appeared. In this paper, we are interested in sufficient conditions for path extendability and cycle extendability of graphs. For some known results on path extendability and cycle extendability of graphs we refer to [5, 6, 9, 10, 11, 12] etc. Many of these are related to traditional conditions on degree, neighborhood union, connectivity, independence number, etc.

We mention two classical results in order of increasing generality.

Theorem 1 *(Dirac [4]). If G is a graph of order n, $(n \geq 3)$, such that $\delta(G) \geq \frac{n}{2}$, then G is hamiltonian.*

Theorem 2 *(Ore [8]). If G is a graph of order n, $(n \geq 3)$, such that $\sigma_2(G) \geq n$, then G is hamiltonian.*

In 1990, Hendry found some sufficient conditions for fully cycle extendability and obtained the following result.

Theorem 3 *(Hendry [6]). If G is a graph of order n, $(n \geq 3)$, such that $\delta(G) \geq \frac{n+1}{2}$, then G is fully cycle extendable.*

A graph G is said to be fully cycle extendable if

(1) for any vertex $v \in V(G)$, there exists a cycle C such that $|C| = 3$ and $v \in V(C)$; and

(2) for any nonhamiltonian cycle C in G, there exists a cycle C' in G such that $V(C) \subset V(C')$ and $|V(C')| = |V(C)| + 1$.

Theorem 4 *(Teng and Wang [10]). If G is a graph of order n such that $\delta(G) \geq \frac{n}{2} + 1$, then G is path extendable.*

The bound for the minimum degree $\delta(G)$ in Theorem 3 and Theorem 4 is sharp.

Other results about degree condition for hamiltonian properties we refer to [1, 3, 7] etc. In this paper, we show the following result.

Theorem 5. *Let G be a graph of order n, if $\sigma_2(G) \geq n + k (k \in N)$, then every path P of G with $\frac{n}{k+2} + 2 \leq |P| < n$ is extendable. The bound $\frac{n}{k+2} + 2$ is sharp.*

The sharpness can be seen from the following graph.

Let G_1 and G_2 be two complete graphs without common vertex, and we denote their vertex sets by $V(G_1) = \{a_1, a_2, a_3\}$ and $V(G_2) = \{b_1, b_2, \cdots, b_{2k+1}\}(k \geq 1)$ respectively. Let G_0 be a graph, where $V(G_0) = V(G_1) \cup V(G_2)$ and $E(G_0) = E(G_1) \cup E(G_2) \cup \{a_2 b_i : i = 1, 2, \cdots, k\} \cup \{a_1 b_{k+j}, a_3 b_{k+j} : j = 1, 2, \cdots, k+1\}$. Let $n = 2k+4$. The order of graph G_0 is n, and $\sigma_2(G_0) = n + k$, but there exists a path $P = a_1 a_2 a_3$ of G with $|P| = \frac{n}{k+2} + 1 = 3$ which is not extendable. Thus we can conclude that the bound $\frac{n}{k+2} + 2$ is sharp.

2 Proof of Theorem 5

Suppose that G satisfies the conditions of Theorem 5. If G is complete, then Theorem 5 obviously holds. Hence we suppose that G is not complete, then $\sigma_2(G) \leq 2(n-2)$. Let $P = v_1 v_2 \cdots v_p$ be an (u, v)-path$(u = v_1, v = v_p)$ of G with $\frac{n}{k+2} + 2 \leq |P| < n$. From the condition of the theorem it is easy to see that G is connected. We put

$$R = V(G) - V(P),\ T = N_P(R),$$

$$E(R,T) = \{xy \in E(G) : x \in R, y \in T\},$$

then it is obviously that $R \neq \phi \neq T$.

For $v_i, v_j \in V(P)(i < j)$, we put $v_i^{+l} = v_{i+l}$, $v_i^{-l} = v_{i-l}$. If there is no doubt about the path, we only write v_i^+, v_i^- instead of v_i^{+1}, v_i^{-1}. We let $v_i P v_j$ be the subpath $v_i v_{i+1} \cdots v_j$, and $v_j \overline{P} v_i = v_j v_{j-1} \cdots v_i$. In the following proof we always suppose that P is not extendable.

Note that $n + k \leq \sigma_2(G) \leq 2(n-2)$, so we have $k \leq n-4$, then

$$|P| \geq \frac{n}{k+2} + 2 \geq \frac{n}{(n-4)+2} + 2 = 3 + \frac{2}{n-2}.$$

For $|P|$ is integer, then $|P| \geq 4$.

Claim 1. Let $xy \in E(R,T)(x \in R, y \in T)$, then
(a) $xy^-, xy^+ \notin E(G)$; and
(b) If $z \in N_P(x)(z \neq y)$, then $y^- z^-, y^+ z^+ \notin E(G)$.

Proof. Otherwise, we can get an extending path of P, a contradiction. The proof of Claim 1 is complete.

Claim 2. $\forall w \in V(P)$, $N_R(w) \neq \phi$.

Proof. Otherwise, there must exist $xy \in E(R,T)(x \in R, y \in T)$ such that $N_R(y^-) = \phi$ or $N_R(y^+) = \phi$. Without loss of generality, we assume that $N_R(y^+) = \phi$, then x and y^+ are nonadjacent vertices in G, and $d(y^+) = d_P(y^+)$. We put

$$N^-_{u^+Pv}(y^+) = \{z^- : z \in N_{u^+Pv}(y^+)\},$$

then $N_P(x) \cap N^-_{u^+Pv}(y^+) = \phi$(by Claim 1(b)), hence there are at least $|N^-_{u^+Pv}(y^+)|$ vertices in P not adjacent to x. Note that $|N^-_{u^+Pv}(y^+)| = |N_{u^+Pv}(y^+)| \geq d_P(y^+) - 1$, we can get

$$d_P(x) \leq |P| - |N^-_{u^+Pv}(y^+)| \leq |P| - (d_P(y^+) - 1).$$

Thus $d_P(x) + d_P(y^+) \leq |P| + 1$, and

$$\begin{aligned} n + k &\leq \sigma_2(G) \leq d(x) + d(y^+) \\ &= d_R(x) + [d_P(x) + d_P(y^+)] \\ &\leq (|R| - 1) + (|P| + 1) = n, \end{aligned}$$

a contradiction. The proof of Claim 2 is complete.

Claim 3. There must exist $v_i, v_j \in V(P)(i < j)$ such that $\{v_i, v_j\} \neq \{u, v\}$, $N_R(v_i) \cap N_R(v_j) \neq \phi$ and
(a) If $v_i \neq u$, then $d_P(v_i^-) + d_P(v_j^-) \leq |P| + 1$, $N_R(v_i^-) \cap N_R(v_j^-) \neq \phi$; and
(b) If $v_j \neq v$, then $d_P(v_i^+) + d_P(v_j^+) \leq |P| + 1$, $N_R(v_i^+) \cap N_R(v_j^+) \neq \phi$.

Proof. Otherwise, for any $\{v_s, v_t\} \subset V(P)$, and $\{v_s, v_t\} \neq \{u, v\}$, we have $N_R(v_s) \cap N_R(v_t) = \phi$, then

$$\forall x \in R, \ d_{uPv^-}(x) < 2, \ d_{u^+Pv}(x) < 2. \tag{1}$$

Thus we have

$$\forall v_s, v_t \in V(uPv^-) = V(v_1Pv_{p-1})(s \neq t), \quad N_R(v_s) \cap N_R(v_t) = \phi,$$

and

$$|R| \geq \sum_{l=1}^{p-1} |N_R(v_l)|.$$

Let $y \in V(uPv^-)$ such that $|N_R(y)| = min\{|N_R(w)| : w \in V(uPv^-)\}$, then

$$|R| \geq \sum_{l=1}^{p-1} |N_R(v_l)| \geq (|P| - 1)|N_R(y)|,$$

hence

$$\begin{aligned} |N_R(y)| & \leq \frac{|R|}{|P|-1} = \frac{n-|P|}{|P|-1} < \frac{n}{|P|-1} - 1 \\ & \leq \frac{n}{(\frac{n}{k+2}+2)-1} - 1 < \frac{n}{\frac{n}{k+2}} - 1 = k+1. \end{aligned} \quad (2)$$

Since $|P| \geq 4$, there must be $y^+ \neq v$ or $y^- \neq u$. Without loss of generality, we assume that $y^+ \neq v$. By Claim 2, we have $N_R(y^+) \neq \phi$. Let $x \in N_R(y^+)$, then $xy \notin E(G)$(by Claim 1(a)). Note that $y^+ \in N_P(x)$ and $y^+ \notin \{u, v\}$, and from (1) we have

$$|N_P(x)| = 1. \quad (3)$$

From (2) and (3), we obtain

$$\begin{aligned} n+k & \leq \sigma_2(G) \leq d(x) + d(y) \\ & = (|N_R(x)| + |N_P(x)|) + (|N_R(y)| + |N_P(y)|) \\ & < (|R|-1) + 1 + k + 1 + (|P|-1) \\ & = |R| + |P| + k = n + k, \end{aligned}$$

a contradiction, hence there exists $v_i, v_j \in V(P)(i < j)$ such that $\{v_i, v_j\} \neq \{u, v\}$ and $N_R(v_i) \cap N_R(v_j) \neq \phi$.

(a) First, $v_j^- \neq v_i$(by Claim 1(a)). For $w \in N_P(v_i^-)$, by Claim 1(b), we have $w \neq v_j^-$, so

$$w \in V(uPv_i^{-2}) \cup V(v_iPv_j^{-2}) \cup V(v_jPv).$$

If $w \in V(uPv_i^{-2})$, then $w^+ \notin N_P(v_j^-)$(otherwise, P can be extended);
If $w \in V(v_iPv_j^{-2})$, then $w^- \notin N_P(v_j^-)$(otherwise, P can be extended);
If $w \in V(v_jPv^-)$, then $w^+ \notin N_P(v_j^-)$(otherwise, P can be extended).

So the number of the vertices in P which are not adjacent to v_j^- is at least

$$|N^+_{uPv_i^{-2}}(v_i^-)| + |N^-_{v_iPv_j^{-2}}(v_i^-)| + |N^+_{v_jPv^-}(v_i^-)|$$

$$= |N_{uPv^-}(v_i^-)| \geq |N_{uPv}(v_i^-)| - 1 = d_P(v_i^-) - 1,$$

hence $d_P(v_j^-) \leq |P| - (d_P(v_i^-) - 1)$, that is,

$$d_P(v_i^-) + d_P(v_j^-) \leq |P| + 1.$$

Suppose that $N_R(v_i^-) \cap N_R(v_j^-) = \phi$. Note that $N_R(v_i) \cap N_R(v_j) \neq \phi$, and that $x \in N_R(v_i) \cap N_R(v_j)$ implies $x \notin N_R(v_i^-) \cup N_R(v_j^-)$(otherwise, P can be extended), we get

$$\begin{aligned} & |N_R(v_i^-)| + |N_R(v_j^-)| \\ = \ & |N_R(v_i^-) \cup N_R(v_j^-)| \\ \leq \ & |R| - |N_R(v_i) \cap N_R(v_j)| \leq |R| - 1, \end{aligned}$$

and

$$\begin{aligned} n+k &\le \sigma_2(G) \le d(v_i^-) + d(v_j^-) \\ &= [d_R(v_i^-) + d_P(v_i^-)] + [d_R(v_j^-) + d_P(v_j^-)] \\ &= [d_R(v_i^-) + d_R(v_j^-)] + [d_P(v_i^-) + d_P(v_j^-)] \\ &\le |R| - 1 + |P| + 1 = n, \end{aligned}$$

a contradiction.

(b) It can be proved in the similar way with (a).

The proof of Claim 3 is complete.

Claim 4. Let $v_i, v_j \in V(P)(i < j)$ such that $\{v_i, v_j\} \neq \{u, v\}$, $N_R(v_i) \cap N_R(v_j) \neq \phi$.

(a) If $v_i \neq u$, then $d_P(v_i) + d_P(v_j) = |P| + 1 = d_P(v_i^-) + d_P(v_j^-)$,

$$d_R(v_i) + d_R(v_i^-) = |R| = d_R(v_j) + d_R(v_j^-),$$

$$d_R(v_i) = d_R(v_j^-),\ d_R(v_j) = d_R(v_i^-);$$

(b) If $v_j \neq v$, then $d_P(v_i) + d_P(v_j) = |P| + 1 = d_P(v_i^+) + d_P(v_j^+)$,

$$d_R(v_i) + d_R(v_i^+) = |R| = d_R(v_j) + d_R(v_j^+),$$

$$d_R(v_i) = d_R(v_j^+),\ d_R(v_j) = d_R(v_i^+).$$

Proof. (a) For v_i, v_j, by Claim 3(a), we have

$$d_P(v_i^-) + d_P(v_j^-) \le |P| + 1, \tag{4}$$

and

$$N_R(v_i^-) \cap N_R(v_j^-) \neq \phi. \tag{5}$$

Note (5), for v_i^-, v_j^-, by Claim 3(b), we have

$$d_P(v_i) + d_P(v_j) \le |P| + 1. \tag{6}$$

By Claim 1(a), we have $N_R(v_i) \cap N_R(v_i^-) = \phi$, $N_R(v_j) \cap N_R(v_j^-) = \phi$, so

$$d_R(v_i) + d_R(v_i^-) \le |R|,\ d_R(v_j) + d_R(v_j^-) \le |R|. \tag{7}$$

Under the condition $N_R(v_i) \cap N_R(v_j) \neq \phi$, and from (5) and Claim 1(b), we get that $v_i^- v_j^-, v_i v_j \notin E(G)$, so

$$\sigma_2(G) \le d(v_i^-) + d(v_j^-),\ \sigma_2(G) \le d(v_i) + d(v_j).$$

Consider $d_R(v_i^-)$ and $d_R(v_j)$:
If $d_R(v_i^-) < d_R(v_j)$, then from (4) and (7), we have

$$\begin{aligned} n+k &\leq \sigma_2(G) \leq d(v_i^-) + d(v_j^-) \\ &= [d_R(v_i^-) + d_P(v_i^-)] + [d_R(v_j^-) + d_P(v_j^-)] \\ &< [d_R(v_j) + d_R(v_j^-)] + [d_P(v_i^-) + d_P(v_j^-)] \\ &\leq |R| + |P| + 1 = n+1, \end{aligned}$$

a contradiction.

If $d_R(v_i^-) > d_R(v_j)$, then from (6) and (7), we also have

$$\begin{aligned} n+k &\leq \sigma_2(G) \leq d(v_i) + d(v_j) \\ &= [d_R(v_i) + d_P(v_i)] + [d_R(v_j) + d_P(v_j)] \\ &< [d_R(v_i) + d_R(v_i^-)] + [d_P(v_i) + d_P(v_j)] \\ &\leq |R| + |P| + 1 = n+1, \end{aligned}$$

a contradiction.

Hence

$$d_R(v_i^-) = d_R(v_j). \tag{8}$$

Consider $d_R(v_i)$ and $d_R(v_j^-)$ similarly, we can prove

$$d_R(v_i) = d_R(v_j^-). \tag{9}$$

On the other hand, if $d_R(v_i) + d_R(v_i^-) \leq |R| - 1$, from (9) and (4), we have

$$\begin{aligned} n+k &\leq \sigma_2(G) \leq d(v_i^-) + d(v_j^-) \\ &= [d_R(v_i^-) + d_P(v_i^-)] + [d_R(v_j^-) + d_P(v_j^-)] \\ &= [d_R(v_i^-) + d_R(v_i)] + [d_P(v_i^-) + d_P(v_j^-)] \\ &\leq |R| - 1 + |P| + 1 = n. \end{aligned}$$

If $d_R(v_j) + d_R(v_j^-) \leq |R| - 1$, from (9) and (6), we have

$$\begin{aligned} n+k &\leq \sigma_2(G) \leq d(v_i) + d(v_j) \\ &= [d_R(v_i) + d_P(v_i)] + [d_R(v_j) + d_P(v_j)] \\ &= [d_R(v_j) + d_R(v_j^-)] + [d_P(v_i) + d_P(v_j)] \\ &\leq |R| - 1 + |P| + 1 = n. \end{aligned}$$

Note the above two contradictions, and from (7) we have

$$d_R(v_i) + d_R(v_i^-) = |R|, \; d_R(v_j) + d_R(v_j^-) = |R|. \tag{10}$$

From (4),(9) and (10), we have

$$\begin{aligned} n+1 &\leq n+k \leq \sigma_2(G) \leq d(v_i^-) + d(v_j^-) \\ &= [d_R(v_i^-) + d_P(v_i^-)] + [d_R(v_j^-) + d_P(v_j^-)] \\ &= [d_R(v_i^-) + d_R(v_i)] + [d_P(v_i^-) + d_P(v_j^-)] \\ &= |R| + d_P(v_i^-) + d_P(v_j^-) \\ &\leq |R| + |P| + 1 = n+1, \end{aligned}$$

hence

$$d_P(v_i^-) + d_P(v_j^-) = |P| + 1. \tag{11}$$

From (6),(9) and (10), we have

$$\begin{aligned} n+1 &\le n+k \le \sigma_2(G) \le d(v_i) + d(v_j) \\ &= [d_R(v_i) + d_P(v_i)] + [d_R(v_j) + d_P(v_j)] \\ &= [d_R(v_j^-) + d_R(v_j)] + [d_P(v_i) + d_P(v_j)] \\ &= |R| + d_P(v_i) + d_P(v_j) \\ &\le |R| + |P| + 1 = n+1, \end{aligned}$$

hence

$$d_P(v_i) + d_P(v_j) = |P| + 1. \tag{12}$$

(b) It can be proved in the similar way with (a).

The proof of Claim 4 is complete.

Claim 5. $|P| \ge 5$.

Proof. We have shown that $p \ge 4$. If $p = 4$, by Claim 3 and Claim 1, we have

$$N_R(v_1) \cap N_R(v_3) \ne \phi \ne N_R(v_2) \cap N_R(v_4), \;\; v_1v_3, v_2v_4 \notin E(G).$$

Hence $d_P(v_1) \le 2$, and $d_P(v_3) = 2$.

By Claim 4, we get

$$2 + 2 \ge d_P(v_1) + d_P(v_3) = |P| + 1 = 5,$$

a contradiction. The proof of Claim 5 is complete.

Let $\{v_i, v_j\} \subset V(P) (i < j)$(by Claim 3 and Claim 1(a) its existence can be ensured) such that:

(1) $\{v_i, v_j\} \ne \{u, v\}$; and

(2) $N_R(v_i) \cap N_R(v_j) \ne \phi$; and

(3) i and $|v_i P v_j|$ are as small as possible.

Claim 6. $i = 1, j = i + 2 = 3$.

Proof. If $i \ne 1$, that is, $v_i \ne u$, by Claim 3(a), we have $N_R(v_i^-) \cap N_R(v_j^-) \ne \phi$, which contradicts the choice of i.

By Claim 1(a) and Claim 4(b), we have

$$j \ge i + 2 = 3, \;\; N_R(v_1) \cap N_R(v_2) = \phi, \;\; d_R(v_1) + d_R(v_2) = |R|.$$

Hence $R = N_R(v_1) \cup N_R(v_2)$.

If $j > i + 2 = 3$, then $v_3 \neq v_j$. Since $|v_i P v_j|$ is as small as possible, we obtain that $N_R(v_3) \cap N_R(v_1) = \phi$. By Claim 1(a), we have $N_R(v_3) \cap N_R(v_2) = \phi$, so $N_R(v_3) = \phi$, which contradicts Claim 2.

The proof of Claim 6 is complete.

Claim 7. $\forall v_l \in V(P), d_R(v_l) = \frac{|R|}{2}$.

Proof. By Claim 6, we have $N_R(v_1) \cap N_R(v_3) \neq \phi$. From Claim 3(b), we have

$$N_R(v_m) \cap N_R(v_{m+2}) \neq \phi, \ m = 1, 2, \cdots, p-2. \tag{13}$$

For v_1, v_3, using Claim 4(b), we have

$$d_R(v_1) = d_R(v_4), \ d_R(v_2) = d_R(v_3).$$

Note that $|P| \geq 5$ and (13), for v_2, v_4, using Claim4(b) again, we have

$$d_R(v_2) = d_R(v_5), \ d_R(v_3) = d_R(v_4).$$

Then

$$d_R(v_1) = d_R(v_4) = d_R(v_3) = d_R(v_2) = d_R(v_5).$$

Since $d_R(v_1) + d_R(v_2) = |R|$, we have

$$\frac{|R|}{2} = d_R(v_1) = d_R(v_4) = d_R(v_3) = d_R(v_2) = d_R(v_5).$$

If $|P| \neq 5$, for v_3, v_5, using Claim 4(b) again, we have $d_R(v_6) = d_R(v_3) = \frac{|R|}{2}$. In a similar way, we show that

$$d_R(v_l) = \frac{|R|}{2}, \ l = 1, 2, \cdots, |P|.$$

The proof of Claim 7 is complete.

Claim 8. $N_R(v_1) = N_R(v_3) = N_R(v_5) = \cdots = N_R(v_{2m+1}) = \cdots,$
$N_R(v_2) = N_R(v_4) = N_R(v_6) = \cdots = N_R(v_{2m}) = \cdots.$

Proof. By Claim 1(a) and Claim 7, we have

$$N_R(v_1) \cap N_R(v_2) = \phi, \ |N_R(v_1)| = |N_R(v_2)| = \frac{|R|}{2}, \ R = N_R(v_1) \cup N_R(v_2).$$

Since $N_R(v_3) \cap N_R(v_2) = \phi$, we get $N_R(v_3) \subset N_R(v_1)$. But $|N_R(v_3)| = |N_R(v_1)|$, so $N_R(v_3) = N_R(v_1)$, hence $R = N_R(v_2) \cup N_R(v_3)$.

Considering v_4 similarly, since $N_R(v_4) \cap N_R(v_3) = \phi$, we have $N_R(v_4) \subset N_R(v_2)$. But $|N_R(v_4)| = |N_R(v_2)|$, so $N_R(v_4) = N_R(v_2)$, hence $R = N_R(v_3) \cup N_R(v_4)$.

From above all, we get that

$$\begin{aligned} N_R(v_1) &= N_R(v_3) = N_R(v_5) &= \cdots = N_R(v_{2m+1}) = \cdots, \\ N_R(v_2) &= N_R(v_4) = N_R(v_6) &= \cdots = N_R(v_{2m}) = \cdots. \end{aligned}$$

The proof of Claim 8 is complete.

Let $S_1 = \{v_s \in V(P) : s \textit{ is odd number}\}$ and $S_2 = \{v_t \in V(P) : t \textit{ is even number}\}$.

If $|P| = 2m$, then $|S_1| = |S_2| = m$. By Claim 8 and Claim 1(b), we get that S_1 is an independent set, so

$$N_P(v_1) \subset S_2, \; d_P(v_1) \leq |S_2| = m.$$

Similarly, we get $d_P(v_3) \leq m$, hence

$$d_P(v_1) + d_P(v_3) \leq 2m = |P|.$$

But by Claim 4(b), we have $d_P(v_1) + d_P(v_3) = |P| + 1$, a contradiction.

If $|P| = 2m + 1$, then $|S_1| = m + 1, |S_2| = m$. By Claim 8 and Claim 1(b), we get that $S_1 - \{v_1\}$ and $S_1 - \{v_{2t+1}\}$ are independent sets, so

$$N_P(v_1) \subset S_2 \cup \{v_{2t+1}\}, \; N_P(v_3) \subset S_2,$$

Hence

$$d_P(v_1) \leq |S_2| + 1 = m + 1. \; d_P(v_3) \leq |S_2| = m,$$

and

$$d_P(v_1) + d_P(v_3) \leq 2m + 1 = |P|.$$

This contradicts Claim 4(b).

The proof of Theorem 5 is complete.

Corollary. Let G be a graph of order n, if $\sigma_2(G) \geq \frac{3}{2}n - 1$, then G is path extendable.

Proof. If $n \leq 4$, it is easy to see that G is complete, in this case the corollary holds. Hence in the following proof we assume $n \geq 5$.

Since $\sigma_2(G) \geq \frac{3}{2}n - 1 \geq n + k$, where $k = \lfloor \frac{n}{2} - 1 \rfloor \geq 1$, by Theorem 5, we get that every path P of G with $4 \leq |P| < n$ is extendable. Thus we only need prove that every path P of G with $|P| = 2$ or $|P| = 3$ is extendable.

When $|P| = 2$, we assume $P = v_1v_2$, and P is not extendable. Let $R = V(G) - V(P)$, then $|R| = n-2$, and $N_R(v_1) \cap N_R(v_2) = \phi$(otherwise, P can be extended). Without loss of generality, we assume that $|N_R(v_1)| \leq |N_R(v_2)|$, then $|N_R(v_1)| \leq \frac{|R|}{2} = \frac{n-2}{2}$. Let $x \in N_R(v_2)$, then x and v_1 are nonadjacent vertices in G, hence

$$\begin{aligned}\tfrac{3}{2}n-1 &\leq \sigma_2(G) \leq d(v_1)+d(x)\\ &= [d_R(v_1)+d_P(v_1)]+[d_R(x)+d_P(x)]\\ &\leq \tfrac{|R|}{2}+1+(|R|-1)+1\\ &= \tfrac{3}{2}|R|+1 = \tfrac{3}{2}(n-2)+1 = \tfrac{3}{2}n-2,\end{aligned}$$

a contradiction.

When $|P| = 3$, we assume $P = v_1v_2v_3$, and P is not extendable. Let $R = V(G) - V(P)$, then $|R| = n-3$ and $N_R(v_1) \cap N_R(v_2) = \phi$(otherwise, P can be extended).

If $|N_R(v_1)| \leq |N_R(v_2)|$, then $|N_R(v_1)| \leq \frac{|R|}{2} = \frac{n-3}{2}$. Let $x \in N_R(v_2)$, then x and v_1 are nonadjacent vertices in G, hence

$$\begin{aligned}\tfrac{3}{2}n-1 &\leq \sigma_2(G) \leq d(v_1)+d(x)\\ &= [d_R(v_1)+d_P(v_1)]+[d_R(x)+d_P(x)]\\ &\leq \tfrac{|R|}{2}+2+(|R|-1)+1\\ &= \tfrac{3}{2}|R|+2 = \tfrac{3}{2}(n-3)+2 = \tfrac{3}{2}n-\tfrac{5}{2},\end{aligned}$$

a contradiction.

If $|N_R(v_1)| > |N_R(v_2)|$, then $|N_R(v_2)| < \frac{|R|}{2} = \frac{n-3}{2}$. Let $y \in N_R(v_1)$, then y and v_2 are nonadjacent vertices in G, hence

$$\begin{aligned}\tfrac{3}{2}n-1 &\leq \sigma_2(G) \leq d(v_2)+d(y)\\ &= [d_R(v_2)+d_P(v_2)]+[d_R(y)+d_P(y)]\\ &< \tfrac{|R|}{2}+2+(|R|-1)+2\\ &= \tfrac{3}{2}|R|+3 = \tfrac{3}{2}(n-3)+3 = \tfrac{3}{2}n-\tfrac{3}{2}.\end{aligned}$$

a contradiction.

The proof of Corollary is complete.

The bound $\frac{3}{2}n-1$ is sharp, and the sharpness can be seen from the following graph.

Let H_1 and H_2 be two complete graphs without common vertex, and we denote their vertex sets respectively by

$$V(H_1) = \{c_1, c_2\}, \quad V(H_2) = \{d_1, d_2, \cdots, d_{2m}\} (m \geq 1).$$

Let H_0 be a graph, with

$$V(H_0) = V(H_1) \cup V(H_2),$$

and

$$E(H_0) = E(H_1) \cup E(H_2) \cup \{c_1d_i, c_2d_{m+i} : i = 1, 2, \cdots, m\}.$$

Let $n = 2m + 2$. The order of graph H_0 is n, and $\sigma_2(H_0) = \frac{3}{2}n - 2$, but there exists a path $P = c_1c_2$ which is not extendable. From this we can conclude that the bound $\frac{3}{2}n - 1$ is sharp.

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