Full Orientability of the Square of a Cycle

Fengwei Xu, Weifan Wang *
Department of Mathematics
Zhejiang Normal University, Jinhua 321004, China
Ko-Wei Lih
Institute of Mathematics
Academia Sinica, Nankang, Taipei 11529, Taiwan

Abstract

Let D be an acyclic orientation of a simple graph G. An arc of D is called *dependent* if its reversal creates a directed cycle. Let $d(D)$ denote the number of dependent arcs in D. Define $d_{\min}(G)$ ($d_{\max}(G)$) to be the minimum (maximum) number of $d(D)$ over all acyclic orientations D of G. We call G *fully orientable* if G has an acyclic orientation with exactly k dependent arcs for every k satisfying $d_{\min}(G) \leqslant k \leqslant d_{\max}(G)$. In this paper, we prove that the square of a cycle C_n is fully orientable except $n = 6$.

Key words: Cycle; Square; Digraph; Acyclic orientation; Full orientability

1 Introduction

Only simple graphs are considered in this paper unless otherwise stated. For a graph G, we denote its vertex set and edge set by $V(G)$ and $E(G)$, respectively. An *orientation* D of G assigns a direction to each edge of G. D is called *acyclic* if there does not exist any directed cycle. Suppose that D

*Corresponding author. Email: wwf@zjnu.cn; Research supported partially by NSFC (No. 10771197) and ZJNSF (No. Z6090150)

is an acyclic orientation of G. An arc of D is called *dependent* if its reversal creates a directed cycle. Let $d(D)$ denote the number of dependent arcs of D. We use $d_{\min}(G)$ and $d_{\max}(G)$ to denote the minimum and maximum number of $d(D)$ over all acyclic orientations D of G, respectively. It is known [2] that $d_{\max}(G) = |E(G)| - |V(G)| + c$ for a graph G having c components.

An interpolation question asks whether G has an acyclic orientation with exactly k dependent arcs for each k satisfying $d_{\min}(G) \leqslant k \leqslant d_{\max}(G)$. The graph G is called *fully orientable* if its interpolation question has an affirmative answer. West [7] showed that complete bipartite graphs are fully orientable.

A *k-coloring* of a graph G is a mapping f from $V(G)$ to the set $\{1, 2, \ldots, k\}$ such that $f(x) \neq f(y)$ for each edge $xy \in E(G)$. The *chromatic number* $\chi(G)$ is the smallest integer k such that G has a k-coloring. The *girth* $g(G)$ is the minimum length of a cycle in a graph G if there is any, and ∞ if G possesses no cycles.

Fisher et al. [2] showed that G is fully orientable if $\chi(G) < g(G)$, and $d_{\min}(G) = 0$ in this case. Since it is well-known [3] that every planar graph G with $g(G) \geqslant 4$ is 3-colorable, planar graphs of girth at least 4 are fully orientable.

The full orientability for several graph classes has been investigated recently. Lih, Lin, and Tong [6] showed that outerplanar graphs are fully orientable. To generalize this result, Lai, Chang, and Lih [4] proved that 2-degenerate graphs are fully orientable. Here a graph G is called *2-degenerate* if every subgraph H of G contains a vertex of degree at most 2 in H. Lai and Lih [5] gave further examples of fully orientable graphs, such as subdivisions of Halin graphs and graphs of maximum degree at most three. Let $K_{r(n)}$ denote the complete r-partite graph each of whose partite sets has n vertices. Chang, Lin, and Tong [1] proved that $K_{r(n)}$ is not fully orientable if $r \geqslant 3$ and $n \geqslant 2$. These are the only known graphs that are not fully orientable.

Suppose that G is a connected graph. For $m \geqslant 2$, the mth *power* of G, denoted G^m, is the graph defined by $V(G^m) = V(G)$ and two distinct vertices u and v are adjacent in G^m if and only if their distance in G is at most m. In particular, G^2 is called the *square* of G.

It is well-known that a directed Hamiltonian path exists for any acyclic orientation of the complete graph K_n on n vertices. This implies that

$d_{\min}(K_n) = d_{\max}(K_n) = \frac{1}{2}(n-1)(n-2)$, hence K_n is fully orientable ([7]). Throughout this paper, we use $C_n = v_0v_1\cdots v_{n-1}v_0$ to represent a cycle of length $n \geqslant 3$. It is easy to see that $C_n^2 \cong K_n$ if $3 \leqslant n \leqslant 5$, and hence is fully orientable. If $n = 6$, then $C_n^2 \cong K_{3(2)}$. By the result of [1], C_6^2 is not fully orientable and $d(D) \in \{4, 6, 7\}$ for any orientation D of C_6^2. In this paper, we shall prove that C_n^2 is fully orientable except $n = 6$.

2 Results

For a given graph G, let $\pi_T(G)$ be the minimum number of edges that can be deleted from G so that the new graph is triangle-free, i.e., having no K_3 as a subgraph. The following lemma appeared in [4].

Lemma 1 *For any graph G, $d_{\min}(G) \geqslant \pi_T(G)$.*

Lemma 2 *For $n \geqslant 7$, $\pi_T(C_n^2) = \lceil \frac{n}{2} \rceil$.*

Proof. When $n \geqslant 7$, C_n^2 contains exactly n distinct triangles. Since every edge of C_n^2 lies in at most two triangles, we have $\pi_T(C_n^2) \geqslant \lceil \frac{n}{2} \rceil$.

On the other hand, let $S = \{v_1v_2, v_3v_4, \ldots, v_{n-1}v_0\}$ if n is even, and $S = \{v_0v_1, v_1v_2, v_3v_4, \ldots, v_{n-2}v_{n-1}\}$ if n is odd. Obviously, $|S| = \lceil \frac{n}{2} \rceil$ and $G - S$ is triangle-free. Thus, $\pi_T(C_n^2) \leqslant |S| = \lceil \frac{n}{2} \rceil$. ∎

In a digraph D with vertex set $V(D)$ and arc set $E(D)$, we use $u \to v$ to denote the arc with tail u and head v. The *indegree* $d_D^-(v)$ of a vertex v in D is the number of arcs with head v; the *outdegree* $d_D^+(v)$ of v in D is the number of arcs with tail v. Let $R(D)$ denote the set of dependent arcs in D.

Theorem 3 *If $n \geqslant 7$, then $d_{\min}(C_n^2) = \pi_T(C_n^2) + 1$.*

Proof. In the first part, we are going to prove that $d_{\min}(C_n^2) \geqslant \pi_T(C_n^2)+1$. Assume to the contrary that $d_{\min}(C_n^2) < \pi_T(C_n^2) + 1$. It follows from Lemmas 1 and 2 that $d_{\min}(C_n^2) = \pi_T(C_n^2) = \lceil \frac{n}{2} \rceil$. Let D be an acyclic orientation of C_n^2 with $d(D) = d_{\min}(C_n^2)$. Let F be the set of all underlying edges of the arcs in $R(D)$. Thus, $|F| = \pi_T(C_n^2) = \lceil \frac{n}{2} \rceil$ and $C_n^2 - F$ is triangle-free. We use C to denote the closed walk $v_0, v_1, \ldots, v_{n-1}, v_0$ in D.

The proof is divided into two cases, depending on the parity of n.

Case 1. Assume $n = 2k$ for some $k \geqslant 4$.

Since $C_n^2 - F$ is triangle-free and $|F| = k$, it is easy to see from the construction of C_n^2 that $F = \{v_1v_2, v_3v_4, \ldots, v_{n-1}v_0\}$ or $F = \{v_0v_1, v_2v_3, \ldots, v_{n-2}v_{n-1}\}$. Without loss of generality, we may assume the former.

Claim. No $v \in V(D)$ satisfies $d_C^+(v) = d_C^-(v) = 1$.

Assume to the contrary that we had $v_{i-1} \to v_i \to v_{i+1}$ (indices modulo n) for some i in D. Then we would have $v_{i-1} \to v_{i+1}$ in D since D is acyclic, and hence $v_{i-1} \to v_{i+1}$ is dependent, contradicting the assumption that $v_{i-1}v_{i+1} \notin F$.

It follows from the Claim that every vertex $v \in V(D)$ satisfies $d_C^+(v) = 0$ and $d_C^-(v) = 2$ or $d_C^+(v) = 2$ and $d_C^-(v) = 0$. Without loss of generality, we may suppose that $d_C^+(v_i) = 2$ and $d_C^-(v_i) = 0$ for each odd i, and $d_C^+(v_i) = 0$ and $d_C^-(v_i) = 2$ for each even i, i.e., C is oriented as $v_1 \to v_2 \leftarrow v_3 \to v_4 \leftarrow \cdots \to v_{n-2} \leftarrow v_{n-1} \to v_0 \leftarrow v_1$. Therefore, $R(D) = \{v_1 \to v_2, v_3 \to v_4, \ldots, v_{n-1} \to v_0\}$. Since $v_i \leftarrow v_{i+1}$ is not dependent for each even i, the edge $v_{i-1}v_{i+1}$ must be directed as $v_{i-1} \to v_{i+1}$. Consequently, a directed cycle $v_1 \to v_3 \to \cdots \to v_{n-1} \to v_1$ is constructed, contradicting the acyclicity of D.

Case 2. Assume $n = 2k + 1$ for some $k \geqslant 3$.

In this case, $|R(D)| = |F| = k + 1$. Note that $C_n^2 - F$ is triangle-free. Hence, C_n^2 has exactly $2k+1$ distinct triangles, and every edge v_iv_j belongs to exactly one triangle (or two triangles) depending on the distance between v_i and v_j is 2 (or 1) in C, then F contains at least k edges in C and hence at most one edge outside C. Since n is odd, there must exist some i such that $v_{i-1} \to v_i \to v_{i+1}$. So, $v_{i-1} \to v_{i+1}$ is a dependent arc. Hence, F contains exactly one edge outside C. We may assume that $F = \{v_1v_2, v_3v_4, \ldots, v_{n-2}v_{n-1}, v_{n-1}v_1\}$. We may also assume that $v_{n-1} \to v_1$. (The case for $v_1 \to v_{n-1}$ can be handled in a similar way.)

We examine the direction of v_1v_2. First assume that $v_1 \to v_2$. Since $v_1 \to v_2$ is the only arc of F in the triangle $v_1v_2v_3v_1$, we have $v_1 \to v_3 \to v_2$. Similarly, in the triangle $v_2v_3v_4v_2$, we have $v_2 \to v_4$ and $v_3 \to v_4$. In this way, it leads to $v_4 \leftarrow v_5 \to v_6 \leftarrow \cdots \leftarrow v_{n-2} \to v_{n-1} \leftarrow v_0$. If $v_1 \to v_0$, then a directed 3-cycle $v_1 \to v_0 \to v_{n-1} \to v_1$ is produced. If $v_0 \to v_1$, then $v_0 \to v_1$ would be a dependent arc of D. Contradictions are obtained in both cases.

Next, assume that $v_2 \to v_1$. Since $v_2 \to v_1$ is the only arc of F in the triangle $v_1v_2v_3v_1$, we have $v_2 \to v_3 \to v_1$. Similarly, in the triangle $v_2v_3v_4v_2$, we have $v_4 \to v_2$ and $v_4 \to v_3$. In this way, it leads to $v_3 \leftarrow v_4 \to v_5 \to v_6 \leftarrow \cdots \leftarrow v_{n-1} \to v_0$. Since $v_{n-1} \to v_0$ is not dependent, we have $v_0 \to v_1$. Since $v_2 \to v_3$ is not dependent, we have $v_3 \to v_1$. Similarly, we have $v_5 \to v_3$; $v_7 \to v_5$; $\cdots$; $v_0 \to v_{n-2}$; $v_2 \to v_0$; $v_4 \to v_2$; $v_6 \to v_4$; $\cdots$; $v_{n-1} \to v_{n-3}$. However, the existence of the directed path $v_0 \to v_{n-2} \to v_{n-4} \cdots \to v_5 \to v_3 \to v_1$ makes $v_0 \to v_1$ a dependent arc, contrary to our assumption.

In the second part, we are going to prove that $d_{\min}(C_n^2) \leqslant \pi_T(C_n^2) + 1$. In fact, an acyclic orientation D_0 of G will be constructed so that $d(D_0) = \pi_T(C_n^2) + 1$. The construction is divided into two cases, depending on the parity of n.

Case 1. Assume $n = 2k$ for some $k \geqslant 4$.

Let D_0 be defined as follows.

$v_1 \to v_{n-1}$; $v_1 \to v_0$; $v_2 \to v_0 \to v_{n-1}$; $v_{n-2} \to v_0$;

$v_{2i-1} \to v_{2i+1}$ for each $i = 1, 2, \ldots, k-1$;

$v_{2i} \to v_{2i+2}$ for each $i = 1, 2, \ldots, k-2$;

$v_{2i-1} \leftarrow v_{2i} \to v_{2i+1}$ for each $i = 1, 2, \ldots, k-1$.

By a close examination, we can see that D_0 is an acyclic orientation of C_n^2 such that $R(D_0) = \{v_1 \to v_{n-1}, v_2 \to v_0, v_2 \to v_3, v_4 \to v_5, \ldots, v_{n-2} \to v_{n-1}\}$. Therefore, $d(D_0) = |R(D_0)| = k + 1 = \pi_T(C_n^2) + 1$.

Case 2. Assume $n = 2k + 1$ for some $k \geqslant 3$.

Let D_0 be defined as follows.

$v_2 \to v_1 \to v_{n-1}$; $v_3 \to v_1 \to v_0$; $v_2 \to v_0 \to v_{n-1}$; $v_{n-2} \to v_{n-1}$; $v_{n-2} \to v_0$;

$v_{2i+1} \to v_{2i} \to v_{2i+2}$ for each $i = 1, 2, \ldots, k-1$;

$v_{2i} \leftarrow v_{2i-1} \to v_{2i+1}$ for each $i = 2, \ldots, k-1$.

By a close examination, we can see that D_0 is an acyclic orientation of C_n^2 such that $R(D_0) = \{v_3 \to v_1, v_1 \to v_{n-1}, v_2 \to v_0, v_3 \to v_4, v_5 \to v_6, v_7 \to v_8, \ldots, v_{n-2} \to v_{n-1}\}$. Therefore, $d(D_0) = |R(D_0)| = k + 2 = \pi_T(C_n^2) + 1$. This completes our proof. ∎

Theorem 4 *For $n \geqslant 7$, C_n^2 is fully orientable.*

Proof. For every graph G, there exists an acyclic orientation D so that $d(D) = d_{\max}(G)$ in [2]. So the present theorem is established if, for each integer s, $\pi_T(C_n^2) + 1 = m < s \leqslant n$, an acyclic orientation D_{s-m} of C_n^2 is constructed to satisfy $d(D_{s-m}) = s$. In fact, such a sequence of acyclic orientations D_{s-m} can be recursively constructed from the D_0 defined in the proof of Theorem 3. We divide our construction into two cases, depending on the parity of n.

Case 1. Assume $n = 2k$ for some $k \geqslant 4$.

By Lemma 2, $\pi_T(C_n^2) = k$. First consider the range $k+2 \leqslant s \leqslant 2k-2$. Assume that D_{s-k-2} has already been constructed.

Let D_{s-k-1} be the acyclic orientation of C_n^2 obtained from D_{s-k-2} by reversing the arc $v_{2(s-k-1)-1} \to v_{2(s-k-1)+1}$. It is easy to check that $R(D_{s-k-1}) = R(D_{s-k-2}) \cup \{v_{2(s-k-1)} \to v_{2(s-k-1)-1}\}$. Hence $d(D_{s-k-1})$ $= d(D_{s-k-2}) + 1 = s$.

If $s = 2k-1$, let D_{k-2} be the acyclic orientation of C_n^2 obtained from D_{k-3} by reversing the arcs $v_1 \to v_{2k-1}, v_1 \to v_0, v_{2k-3} \to v_{2k-1}$. It is easy to check that $R(D_{k-2}) = R(D_{k-3}) \setminus \{v_1 \to v_{2k-1}\} \cup \{v_0 \to v_1, v_{2k-2} \to v_{2k-3}\}$ and $d(D_{k-2}) = d(D_{k-3}) + 2 - 1 = 2k-1$.

If $s = 2k$, let D_{k-1} be the acyclic orientation of C_n^2 obtained from D_{k-2} by reversing the arc $v_{2k-1} \to v_1$. It is easy to check that $R(D_{k-1}) = R(D_{k-2}) \setminus \{v_0 \to v_1\} \cup \{v_0 \to v_{2k-1}, v_{2k-5} \to v_{2k-3}\}$ and $d(D_{k-1}) = d(D_{k-2}) + 2 - 1 = 2k$.

Case 2. Assume $n = 2k+1$ for some $k \geqslant 3$.

By Lemma 2, $\pi_T(C_n^2) = k+1$. First consider the range $k+2 \leqslant s \leqslant 2k-2$. Assume that D_{s-k-3} has already been constructed.

Let D_{s-k-2} be the acyclic orientation of C_n^2 obtained from D_{s-k-3} by reversing the arc $v_{2(s-k-2)} \to v_{2(s-k-2)+2}$. It is easy to check that $R(D_{s-k-2}) = R(D_{s-k-3}) \cup \{v_{2(s-k-2)+1} \to v_{2(s-k-2)}\}$. Hence $d(D_{s-k-2})$ $= d(D_{s-k-3}) + 1 = s$.

If $s = 2k$, let D_{k-2} be the acyclic orientation of C_n^2 obtained from D_{k-3} by reversing the arcs $v_0 \to v_{2k}$. It is easy to check that $R(D_{k-2}) = R(D_{k-3}) \setminus \{v_1 \to v_{2k}\} \cup \{v_{2k-1} \to v_0, v_1 \to v_0\}$ and $d(D_{k-2}) = d(D_{k-3}) - 1 + 2 = 2k$.

If $s = 2k+1$, let D_{k-1} be the acyclic orientation of C_n^2 obtained from D_{k-3} by reversing the arc $v_{2k-2} \to v_{2k}$. It is easy to check that $R(D_{k-1}) =$

$R(D_{k-3}) \cup \{v_{2k-1} \to v_{2k-2}, v_{2k-4} \to v_{2k-2}\}$ and $d(D_{k-1}) = d(D_{k-3}) + 2 = 2k+1$. ∎

To conclude this paper, we would like to pose the following problem.

Problem. For any given integer $k \geqslant 2$, does there exist a smallest constant $\alpha(k)$ such that C_n^k is fully orientable whenever $n \geqslant \alpha(k)$?

Since $C_{2k+2}^k \cong K_{(k+1)(2)}$ and $K_{(k+1)(2)}$ is not fully orientable when $k \geq 2$ by the result of [1], we have $\alpha(k) \geqslant 2k+3$. The only solved case of this problem is $\alpha(2) = 7$ by Theorem 4.

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