On Finding Lagrangians of 3-uniform Hypergraphs

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Abstract

It is known that determining the Lagrangian of a general r-uniform hypergraph is useful in practice and is non-trivial when $r \geq 3$. In this paper, we explore the Lagrangians of 3-uniform hypergraphs with edge sets having restricted structures. In particular, we establish a number of optimization problems for finding the largest Lagrangian of 3-uniform hypergraphs with the number of edges $m = \binom{k}{3} - a$ where a=3 or 4. We also verify that the largest Lagrangian has the colex ordering structure for 3-uniform hypergraphs when the number of edges is small.

Key Words: Cliques, Colex ordering, Left-compressing, Lagrangians of r-uniform Hypergraphs.

1 Introduction

For a set V and a positive integer r we denote by $V^{(r)}$ the family of all r-subsets of V. An r-uniform graph or r-graph G consists of a set $V(G)$ of vertices and a set $E(G) \subseteq V(G)^{(r)}$ of edges. An edge $e = \{a_1, a_2, \ldots, a_r\}$ will be simply denoted by $a_1 a_2 \ldots a_r$. An r-uniform graph H is a *subgraph*

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of an r-uniform graph G, denoted by $H \subset G$ if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. Let $K_t^{(r)}$ denote the complete r-graph on t vertices, that is the r-graph on t vertices containing all possible edges. A complete r-graph on t vertices is also called a clique with order t. Let $[n]^{(r)}$ represent the complete r-uniform graph on the vertex set $\{1, 2, 3, \ldots, n\}$. When $r = 2$, an r-uniform graph is a simple graph. When $r \geq 3$, an r-graph is often called a hypergraph. Let N be the set of all positive integers.

We now give the definition of the Lagrangian of an r-uniform graph below. More studies of Lagrangians can be found in [4], [5], [7], [10], and [16].

Definition 1.1 *For an r-uniform graph G with vertex set $\{1, 2, \ldots, n\}$, edge set $E(G)$ and a vector $\vec{x} = (x_1, \ldots, x_n) \in R^n$, define*

$$\lambda(G, \vec{x}) = \sum_{\{i_1, \ldots, i_r\} \in E(G)} x_{i_1} x_{i_2} \ldots x_{i_r}.$$

The value x_i is called the *weight* of the vertex i.

Definition 1.2 *Let $S = \{\vec{x} = (x_1, x_2, \ldots, x_n) : \sum_{i=1}^{n} x_i = 1, x_i \geq 0$ for $i = 1, 2, \ldots, n\}$. The Lagrangian of G, denote by $\lambda(G)$, is defined as*

$$\lambda(G) = \max\{\lambda(G, \vec{x}) : \vec{x} \in S\}.$$

We call $\vec{x} = (x_1, x_2, \ldots, x_n) \in R^n$ a legal weighting for G if $\sum_{i=1}^{n} x_i = 1, x_i \geq 0$ for $i = 1, 2, \ldots, n$. A vector $\vec{y} \in S$ is called an *optimal vector* for G if $\lambda(G, \vec{y}) = \lambda(G)$.

The following fact is easily implied by the definition of the Lagrangian.

Fact 1.1 *Let G_1, G_2 be r-uniform graphs and $G_1 \subset G_2$. Then*

$$\lambda(G_1) \leq \lambda(G_2).$$

In 1941, Turán [17] provided an answer to the following question: What is the maximum number of edges in a graph with n vertices not containing a complete subgraph of order k, for a given k? This is the well-known Turán theorem. Later, in another classical paper, T. S. Motzkin and E. G. Straus [7] provided a novel proof of Turan's theorem using a continuous characterization of the clique number of a graph in terms of Lagrangians of a graph. They determined the following simple expression for the Lagrangians of a 2-graph.

Theorem 1.2 *(Motzkin and Straus [7]) If G is a 2-graph in which a largest clique has order t then* $\lambda(G) = \lambda(K_t^{(2)}) = \frac{1}{2}(1 - \frac{1}{t})$.

The Motzkin-Straus result and its extension were successfully employed in optimization to provide heuristics for the maximum clique problem (see [1],[2], [3], [6],[9], [13]). Furthermore, the Motzkin-Straus theorem has been generalized to vertex-weighted graphs [see [6]] and edge-weighted graphs [8].

An attempt to generalize the Motzkin-Straus theorem to hypergraphs is due to Sós and Straus [15]. Recently, in [12] Rota Buló and Pelillo generalized the Motzkin and Straus' result to r-graphs in some way using a continuous characterization of maximal cliques. Due to the difficulty of Turán's problem [17] for $r \geq 3$, determining the Lagrangian of a general r-graph is non-trivial when $r \geq 3$ (see [16]). Indeed the obvious generalization of Motzkin and Straus' result is false because there are many examples of r-graphs that do not achieve their Lagrangian on any proper subhypergraph. Frankl and Füredi [4] asked the following question. Given $r \geq 3$ and $m \in N$ how large can the Lagrangians of an r-graph with m edges be? In order to state their conjecture on this problem we require the following definition. For distinct $A, B \in N^{(r)}$ we say that A is less than B in the *colex ordering* if $max(A \triangle B) \in B$. For example we have $246 < 156$ in $N^{(3)}$ since $max(\{2,4,6\} \triangle \{1,5,6\}) \in \{1,5,6\}$. The following conjecture of Frankl and Füredi (if it is true) proposes a solution to the above question.

Conjecture 1.3 *(Frankl and Füredi [4]) The r-graph with m edges formed by taking the first m sets in the colex ordering of* $N^{(r)}$ *has the largest Lagrangian of all r-graphs with m edges. In particular, the r-graph with* $\binom{k}{r}$ *edges and the largest Lagrangian is* $[k]^{(r)}$.

This conjecture is true when $r = 2$ by Theorem 1.2. For the case $r = 3$, Talbot in [16] proved the following.

Theorem 1.4 *(Talbot) Let m and k be integers satisfying*

$$\binom{k-1}{3} \leq m \leq \binom{k-1}{3} + \binom{k-2}{2} - (k-1).$$

Then Conjecture 1.3 is true for $r = 3$ *and this value of m. Conjecture 1.3 is also true for* $r = 3$ *and* $m = \binom{k}{3} - 1$ *or* $m = \binom{k}{3} - 2$.

The truth of Frankl and Füredi's conjecture is not known in general for $r \geq 4$. Even in the case $r = 3$, Theorem 1.4 does not cover the case when $\binom{k-1}{3} + \binom{k-2}{2} - (k-2) \leq m \leq \binom{k}{3} - 3$ in this conjecture.

This paper is organized as follows. We first summarize some useful results in the next section. Then we establish a number of optimization problems for finding the largest Lagrangian of 3-graphs when the number of edges $m = \binom{k}{3} - a$ where a=3 or 4. The solutions of these optimization problems are tested when $m = \binom{k}{3} - 3$ for $k = 6, \cdots, 100$ and $m = \binom{k}{3} - 4$ for $k = 7, \cdots, 50$. Also, we find the Lagrangians of 3-graphs with near colex ordering structures, and using them we verify Conjecture 1.3 for 3-graphs when the number of edges $m \le 50$.

2 Useful Results

For an r-graph $G = (V, E)$ we denote the $(r-1)$-neighborhood of a vertex $i \in V$ by $E_i = \{A \in V^{(r-1)} : A \cup \{i\} \in E\}$. Similarly, we will denote the $(r-2)$-neighborhood of a pair of vertices $i, j \in V$ by $E_{ij} = \{B \in V^{(r-2)} : B \cup \{i, j\} \in E\}$. We denote the complement of E_i by $E_i^c = \{A \in V^{(r-1)} : A \cup \{i\} \in V^{(r)} \backslash E\}$. Also, we will denote the complement of E_{ij} by $E_{ij}^c = \{B \in V^{(r-2)} : B \cup \{i, j\} \in V^{(r)} \backslash E\}$. Denote

$$E_{i \backslash j} = (E_i \cap E_j^c) \setminus \{jk, k \in E_{ij}\}.$$

We will impose one additional condition on any optimal weighting $\vec{x} = (x_1, x_2, \ldots, x_n)$ for an r-graph G:

$$|\{i : x_i > 0\}| \text{ is minimal, i.e. if } \vec{y} \text{ is a legal weighting for } G \text{ satisfying } |\{i : y_i > 0\}| < |\{i : x_i > 0\}|, \text{ then } \lambda(G, \vec{y}) < \lambda(G). \quad (1)$$

When the theory of Lagrange multipliers is applied to find the optimum of $\lambda(G, \vec{x})$, subject to $\sum_{i=1}^{n} x_i = 1$, notice that $\lambda(E_i, \vec{x})$ corresponds to the partial derivative of $\lambda(G, \vec{x})$ with respect to x_i. The following Lemma gives some necessary condition of an optimal vector of $\lambda(G)$ by applying the theory of Lagrange multipliers [14].

Lemma 2.1 *(Frankl and Rödl [5]) Let $G = (V, E)$ be an r-graph and $\vec{x} = (x_1, x_2, \ldots, x_n)$ be an optimal legal weighting for G with $k \le n$ non-zero weights satisfying condition (1). Then for every $\{i, j\} \in [k]^{(2)}$, (a) $\lambda(E_i, \vec{x}) = \lambda(E_j, \vec{x}) = r\lambda(G)$, (b) there is an edge in E containing both i and j.*

We say that an r-graph $G = (V, E)$ is *left compressed* if $(i_1, \ldots, i_r) \in E$ implies $(j_1, \ldots, j_r) \in E$ provided $j_p \le i_p$ for every $p, 1 \le p \le r$. For example, the following 3-graph G is left compressed: the vertex set is $\{1, 2, 3, 4, 5\}$ and the edge set consists of 7 edges $\{123, 124, 134, 234, 125, 135, 145\}$.

Remark 2.2 *(a) In Lemma 2.1, part(a) implies that*

$$x_j\lambda(E_{ij},\vec{x}) + \lambda(E_{i\setminus j},\vec{x}) = x_i\lambda(E_{ij},\vec{x}) + \lambda(E_{j\setminus i},\vec{x}).$$

In particular, if G is left compressed, then

$$(x_i - x_j)\lambda(E_{ij},\vec{x}) = \lambda(E_{i\setminus j},\vec{x}) \tag{2}$$

for any i, j satisfying $1 \leq i < j \leq k$ since $E_{j\setminus i} = \emptyset$.

(b)If G is left-compressed, then an optimal legal weighting for G $\vec{x} = (x_1, x_2, \ldots, x_n)$ must satisfy

$$x_1 \geq x_2 \geq \ldots \geq x_n \geq 0 \tag{3}$$

by (2).

Denote

$$\lambda_m^r = \max\{\lambda(G) : G \text{ is an } r-\text{graph with } m \text{ edges}\}.$$

We denote the r-graph with m edges formed by taking the first m elements in the colex ordering of $N^{(r)}$ by $C_{r,m}$. The next three lemmas are proved by Talbot in [16].

Lemma 2.3 *(Talbot [16]) There exists a left compressed r-graph G with m edges such that $\lambda(G) = \lambda_m^r$.*

Lemma 2.4 *(Talbot [16]) For any integers m, k, and r satisfying $\binom{k-1}{r} \leq m \leq \binom{k-1}{r} + \binom{k-2}{r-1}$, we have $\lambda(C_{r,m}) = \lambda([k-1]^{(r)})$.*

Lemma 2.5 *(Talbot [16]) Let G be a 3-graph with m edges satisfying $\lambda(G) = \lambda_m^3$. Suppose that $\vec{x} = (x_1, x_2, \ldots, x_n)$ is an optimal legal weighting for G satisfying $x_1 \geq x_2 \geq \cdots \geq x_k > x_{k+1} = \cdots = x_n = 0$. Then the number of edges in G satisfies*

$$|E| \geq \binom{k-1}{3} + \binom{k-2}{2} - (k-2).$$

Theorem 1.4 is implied by Lemmas 2.4 and 2.5.

3 Special Cases

In this section, we formulate several optimization problems in two or four independent variables. These problems completely characterize the Lagrangian of 3-graphs with $m = \binom{k}{3} - 3$ and $m = \binom{k}{3} - 4$ edges. Also, solutions of these problems can be used to verify Conjecture 1.3 for 3-graphs with $m = \binom{k}{3} - 3$ and $m = \binom{k}{3} - 4$ edges. Note that $[k]^{(3)} = [k-1]^{(3)} \cup \{kij, \text{ where } ij \in [k-2]^{(2)}\} \cup \{1(k-1)k, 2(k-1)k, \ldots, (k-2)(k-1)k\}$. We first gave two computational optimization problems for the case when $m = \binom{k}{3} - 3$.

Lemma 3.1 *Let G be a 3-graph with vertex set $\{1, 2, \ldots, k\}$ and edge set*

$$\begin{aligned} E(G) &= [k-1]^{(3)} \cup \{kij, \text{ where } ij \in [k-2]^{(2)}\} \\ &\quad \cup \{1(k-1)k, 2(k-1)k, \ldots, (k-5)(k-1)k\} \end{aligned}$$

Then $\lambda(G) = Z(A) > \lambda([k-1]^{(3)})$ where $Z(A)$ is given by solving the maximum value in Problem A.

Problem A. *Let $k \geq 6$. Find the maximum value $Z(A)$ of $f(a,b,c)$ where*

$$\begin{aligned} f(a,b,c) &= \binom{k-5}{3}a^3 + 3\binom{k-5}{2}a^2b + 2\binom{k-5}{2}a^2c + 6(k-5)abc \\ &\quad + 3(k-5)ab^2 + 6b^2c + b^3 + (k-5)ac^2 \end{aligned}$$

under the constraint

$$a \geq b \geq c \geq 0, \quad (k-5)a + 3b + 2c = 1. \tag{4}$$

Proof. Observe that G is left-compressed. Let $\vec{x} = (x_1, x_2, x_3, x_4, \ldots, x_k)$ be an optimal legal weighting for G satisfying conditions (1) on Page 4 and (3) on Page 5. We first notice that $x_k > 0$ since, if $x_k = 0$, then $\lambda(G, \vec{x}) = \lambda([k-1]^{(3)}) = \frac{(k-2)(k-3)}{6(k-1)^2}$, also the subgraph of G with edge set $[k-1]^{(3)} \cup \{kij, \text{ where } ij \in [k-2]^{(2)}\} \cup \{1(k-1)k\}$ has the Lagrangian bigger than $\lambda([k-1]^{(3)})$ by taking $x_1 = \cdots = x_{k-2} = \frac{1}{k-1}$ and $x_{k-1} = x_k = \frac{1}{2(k-1)}$ (In this case, $\lambda(G, \vec{x}) = \binom{k-2}{3}(\frac{1}{k-1})^3 + \frac{2}{2(k-1)}\binom{k-2}{2}(\frac{1}{k-1})^2 + \frac{k-5}{4(k-1)^3} = \lambda([k-1]^{(3)}) + \frac{k-5}{4(k-1)^3}$). Now we may assume each $x_i > 0$ for $1 \leq i \leq k$. Observe that $E_{i \setminus j} = \emptyset$ for $1 \leq i < j \leq k-5$, $k-4 \leq i < j \leq k-2$, and $k-1 \leq i < j \leq k$. By (2),

$$x_1 = \cdots = x_{k-5} \stackrel{\text{def}}{=} a, \quad x_{k-4} = x_{k-3} = x_{k-2} \stackrel{\text{def}}{=} b, \quad x_{k-1} = x_k \stackrel{\text{def}}{=} c.$$

Note that $(k-5)a+3b+2c=1$. In viewing of the edge set $E(G)$, $\lambda(G)$ is the maximum value of

$$\begin{aligned} f(a,b,c) &= \binom{k-5}{3}a^3+3\binom{k-5}{2}a^2b+2\binom{k-5}{2}a^2c+6(k-5)abc \\ &\quad +3(k-5)ab^2+6b^2c+b^3+(k-5)ac^2 \end{aligned}$$

under the constraint (4). This completes the proof. ■

Lemma 3.2 *Let G be a 3-graph with vertex set $\{1,2,\ldots,k\}$ and edge set*

$$\begin{aligned} E(G) &= [k-1]^{(3)} \cup \{kij, \text{ where } ij \in [k-2]^{(2)} \setminus \{(k-3)(k-2)\}\} \\ &\quad \cup\{1(k-1)k, 2(k-1)k, \ldots, (k-4)(k-1)k\}. \end{aligned}$$

Then $\lambda(G) = \max\{\frac{(k-2)(k-3)}{6(k-1)^2}, Z(B)\}$ where $Z(B)$ is given by solving the maximum value in Problem B.

Problem B. *Let $k \geq 6$. Find the maximum value $Z(B)$ of $g(a,b,c)$ where $g(a,b,c) = \binom{k-4}{3}a^3+3\binom{k-4}{2}a^2b+3(k-4)ab^2+b^3+\binom{k-4}{2}a^2c+3(k-4)abc$ under the constraint*

$$a \geq b \geq c \geq 0, \quad (k-4)a+3b+c=1. \tag{5}$$

Proof. Observe that G is left-compressed. Let $\vec{x}=(x_1,x_2,x_3,x_4,\ldots,x_k)$ be an optimal legal weighting for G satisfying conditions (1) on Page 4 and (3) on Page 5. If $x_k=0$, then $\lambda(G,\vec{x})=\lambda([k-1]^{(3)})=\frac{(k-2)(k-3)}{6(k-1)^2}$. If $x_k>0$, then each $x_i>0$. Observe that $E_{i\setminus j}=\emptyset$ for $1 \leq i < j \leq k-4$, and $k-3 \leq i < j \leq k-1$. By (2),

$$x_1=\cdots=x_{k-4} \stackrel{\text{def}}{=} a, \quad x_{k-3}=x_{k-2}=x_{k-1} \stackrel{\text{def}}{=} b.$$

Let $x_k \stackrel{\text{def}}{=} c$. Note that $(k-4)a+3b+c=1$. In viewing of the edge set E, $\lambda(G)$ is the maximum value of

$$\begin{aligned} g(a,b,c) &= \binom{k-4}{3}a^3+3\binom{k-4}{2}a^2b+3(k-4)ab^2+b^3 \\ &\quad +c[\binom{k-4}{2}a^2+2(k-4)ab]+(k-4)abc \\ &= \binom{k-4}{3}a^3+3\binom{k-4}{2}a^2b+3(k-4)ab^2+b^3+\binom{k-4}{2}a^2c \\ &\quad +3(k-4)abc \end{aligned}$$

under the constraint (5). This completes the proof. ■

Remark 3.3 *Notice that in Problems A and B, when $k = 6$, we set these terms $\binom{k-4}{3}$, $\binom{k-5}{3}$, and $\binom{k-5}{2}$ value 0. Also, when $k = 7$, we set this term $\binom{k-5}{3}$ value 0.*

Now we state several more similar computational lemmas regarding the case $m = \binom{k}{3} - 4$.

Lemma 3.4 *Let G be a 3-graph with vertex set $\{1, 2, \ldots, k\}$ and edge set*

$$\begin{aligned} E(G) &= [k-1]^{(3)} \cup \{kij, \text{ where } ij \in [k-2]^{(2)}\} \\ &\quad \cup \{1(k-1)k, 2(k-1)k, \ldots, (k-6)(k-1)k\} \end{aligned}$$

Then $\lambda(G) = Z(C) > \lambda([k-1]^{(3)})$ where $Z(C)$ is given by solving the maximum value in Problem C.

Problem C. *Let $k \geq 7$. Find the maximum value $Z(C)$ of $h(a, b, c)$ where*

$$\begin{aligned} h(a,b,c) &= \binom{k-6}{3}a^3 + 4\binom{k-6}{2}a^2b + 2\binom{k-6}{2}a^2c + 8(k-6)abc \\ &\quad + 6(k-6)ab^2 + 12b^2c + 4b^3 + (k-6)ac^2 \end{aligned}$$

under the constraint

$$a \geq b \geq c \geq 0, \quad (k-6)a + 4b + 2c = 1. \tag{6}$$

Proof. Observe that G is left-compressed. Let $\vec{x} = (x_1, x_2, x_3, x_4, \ldots, x_k)$ be an optimal legal weighting for G satisfying conditions (1) on Page 4 and (3) on Page 5. We first notice that $x_k > 0$ since, if $x_k = 0$, then $\lambda(G, \vec{x}) = \lambda([k-1]^{(3)}) = \frac{(k-2)(k-3)}{6(k-1)^2}$, also the subgraph of G with edge set $[k-1]^{(3)} \cup \{kij, \text{ where } ij \in [k-2]^{(2)}\} \cup \{1(k-1)k\}$ has the Lagrangian bigger than $\lambda([k-1]^{(3)})$ by taking $x_1 = \cdots = x_{k-2} = \frac{1}{k-1}$ and $x_{k-1} = x_k = \frac{1}{2(k-1)}$. So we may assume that each $x_i > 0$. Observe that $E_{i \setminus j} = \emptyset$ for $1 \leq i < j \leq k-6$, $k-5 \leq i < j \leq k-2$, and $k-1 \leq i < j \leq k$. By (2),

$$x_1 = \cdots = x_{k-6} \stackrel{\text{def}}{=} a, \quad x_{k-5} = x_{k-4} = x_{k-3} = x_{k-2} \stackrel{\text{def}}{=} b, \quad x_{k-1} = x_k = c.$$

Note that $(k-6)a + 4b + 2c = 1$. In viewing of the edge set E, $\lambda(G)$ is the maximum value of

$$\begin{aligned} h(a,b,c) &= \binom{k-6}{3}a^3 + 4\binom{k-6}{2}a^2b + 2\binom{k-6}{2}a^2c + 8(k-6)abc \\ &\quad + 6(k-6)ab^2 + 12b^2c + 4b^3 + (k-6)ac^2 \end{aligned}$$

under the constraint (6). This completes the proof. ∎

Lemma 3.5 *Let G be a 3-graph with vertex set $\{1,2,\ldots,k\}$ and edge set*

$$\begin{aligned} E(G) &= ([k-1]^{(3)} \setminus \{(k-3)(k-2)(k-1)\}) \cup \{kij,\ \ ij \in [k-2]^{(2)} \\ &\quad \setminus\{(k-3)(k-2)\}\} \cup \{1(k-1)k, 2(k-1)k, \ldots, (k-4)(k-1)k\} \end{aligned}$$

Then $\lambda(G) \leq \max\{\frac{(k-2)(k-3)}{6(k-1)^2}, Z(D)\}$, where $Z(D)$ is given by the maximum value in Problem D.

Problem D *Solve for the maximum value $Z(D)$ of $u(a,b)$ where*

$$u(a,b) = \binom{k-4}{3}a^3 + 4\binom{k-4}{2}a^2b + 6(k-4)ab^2$$

under the constraint

$$a \geq b \geq 0, (k-4)a + 4b = 1. \tag{7}$$

Proof. Observe that G is left-compressed. Let $\vec{x} = (x_1, x_2, x_3, x_4, ..., x_k)$ be an optimal legal weighting for G satisfying conditions (1) on Page 4 and (3) on Page 5. If $x_k = 0$, then $\lambda(G, \vec{x}) \leq \lambda([k-1]^{(3)}) = \frac{(k-2)(k-3)}{6(k-1)^2}$. If $x_k > 0$, then each $x_i > 0$. Observe that $E_{i \setminus j} = \emptyset$ for $1 \leq i < j \leq k-4$, and $k-3 \leq i < j \leq k$. By (2),

$$x_1 = \cdots = x_{k-4} \stackrel{\text{def}}{=} a, \quad x_{k-3} = x_{k-2} = x_{k-1} = x_k \stackrel{\text{def}}{=} b.$$

Note that $(k-4)a + 4b = 1$. In viewing of the edge set $E(G)$, $\lambda(G)$ is the maximum value of

$$u(a,b) \quad = \quad \binom{k-4}{3}a^3 + 4\binom{k-4}{2}a^2b + 6(k-4)ab^2$$

under the constraint (7). This completes the proof. ■

Remark 3.6 *(Computation Result for Problem D). Let $k \geq 7$. The maximum value $Z(D)$ in Problem D can be solved in terms of k: Let $A(k) = 3\binom{k-4}{3} - 3\binom{k-4}{2}(k-4) + \frac{9}{8}(k-4)^3$, $B(k) = 2\binom{k-4}{2} - \frac{3}{2}(k-4)^2$, $C(k) = \frac{3}{8}(k-4)$. Then $Z(D) = u(a,b)$ where $a = X(k) = \frac{-B(k)-\sqrt{B^2(k)-4A(k)C(k)}}{2A(k)}$ and $b = \frac{1-(k-4)a}{4}$.*

Lemma 3.7 *Let G be a 3-graph with vertex set $\{1,2,\ldots,k\}$ and edge set*

$$\begin{aligned} E(G) &= [k-1]^{(3)} \cup \{kij, \text{ where } ij \in [k-2]^{(2)} \setminus \{(k-2)(k-3)\}\} \\ &\quad \cup\{1(k-1)k, 2(k-1)k, \ldots, (k-5)(k-1)k\} \end{aligned}$$

Then $\lambda(G) = \max\{\frac{(k-2)(k-3)}{6(k-1)^2}, Z(E)\}$, *where* $Z(E)$ *is given by the maximum value in Problem E.*

Problem E. *Let* $k \geq 7$. *Find the maximum value* $Z(E)$ *of* $v(a,b,c,d,e)$ *where*

$$\begin{aligned} v(a,b,c,d,e) &= \binom{k-5}{3}a^3 + \binom{k-5}{2}a^2(b+2c+d) \\ &+(k-5)a(c^2+2bc+2cd+bd) + 2bcd + bc^2 + c^2d \\ &+e[\binom{k-5}{2}a^2 + (k-5)a(b+2c+d) + 2bc] \end{aligned}$$

under the constraint

$$a \geq b \geq c \geq d \geq e \geq 0, (k-5)a + b + 2c + d + e = 1, \tag{8}$$

Proof. Observe that G is left-compressed. Let $\vec{x} = (x_1, x_2, x_3, x_4, ..., x_k)$ be an optimal legal weighting for G satisfying conditions (1) on Page 4 and (3) on Page 5. If $x_k = 0$, then $\lambda(G, \vec{x}) = \lambda([k-1]^{(3)}) = \frac{(k-2)(k-3)}{6(k-1)^2}$. If $x_k > 0$, then each $x_i > 0$. Observe that $E_{i\setminus j} = \emptyset$ for $1 \leq i < j \leq k-5$, and $k-3 \leq i < j \leq k-2$. By (2),

$$x_1 = \cdots = x_{k-5} \stackrel{\text{def}}{=} a, \quad x_{k-3} = x_{k-2} \stackrel{\text{def}}{=} c.$$

Let $x_{k-4} \stackrel{\text{def}}{=} b$, $x_{k-1} \stackrel{\text{def}}{=} d$, and $x_k \stackrel{\text{def}}{=} e$. Note that $(k-5)a+b+2c+d+e = 1$. In viewing of the edge set E, $\lambda(G)$ is the maximum value of

$$\begin{aligned} v(a,b,c,d,e) &= \binom{k-5}{3}a^3 + \binom{k-5}{2}a^2(b+2c+d) \\ &+(k-5)a(c^2+2bc+2cd+bd) + 2bcd + bc^2 + c^2d \\ &+e[\binom{k-5}{2}a^2 + (k-5)a(b+2c+d) + 2bc] \end{aligned}$$

under the constraint (8). This completes the proof. ∎

Remark 3.8 *Note that in Problem D, when* $k \leq 6$, *we set* $\binom{k-4}{3}$ *value 0. In Problems C and E, when* $k = 7$, *we set these terms* $\binom{k-6}{3}$, $\binom{k-6}{2}$ $\binom{k-5}{3}$, *and* $\binom{k-5}{2}$ *value 0. Also, when* $k = 8$, *we set this term* $\binom{k-6}{3}$ *value 0.*

Theorem 3.9 *(1) Let* $k \geq 6$. *If the maximum value* $Z(A) \geq Z(B)$, *then Conjecture 1.3 is true for 3-graphs with* $m = \binom{k}{3} - 3$ *edges.*

(2) Let $k \geq 7$. *If the maximum values* $Z(C) \geq max\{Z(D), Z(E)\}$, *then Conjecture 1.3 is true for 3-graphs with* $m = \binom{k}{3} - 4$ *edges.*

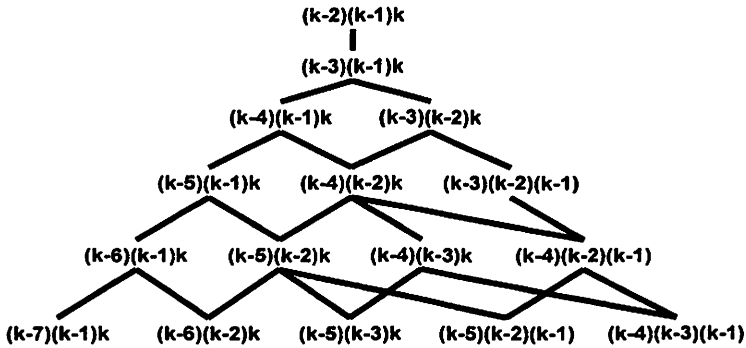

Figure 1:

Proof. Let $m = \binom{k}{3} - a$ where $a = 3$ or 4, and $G = (V, E)$ be a 3-graph satisfying $\lambda(G) = \lambda^3_m$. By Lemma 2.3, we may assume that G is left compressed. Let $\vec{x} = (x_1, x_2, x_3, x_4, \ldots)$ be an optimal legal weighting for G satisfying conditions (1) on Page 4 and (3) on Page 5. By Lemma 2.5, $x_i = 0$ for $i \geq k+1$ since otherwise, $|E| \geq \binom{k}{3} + \binom{k-1}{2} - (k-1) > m$. So we may assume that $V(G) = \{1, 2, \ldots, k\}$. By Lemmas 3.1 and 3.4, we may assume that $x_k > 0$ and this graph G with $m = \binom{k}{3} - a$ edges can be formed from $[k]^{(3)}$ by removing a edges appropriately. By left compressing partial order property, if an edge in Figure 1 is not in E, then all its predecessors can not be in E either. Therefore those a edges that will be removed from $[k]^{(3)}$ must include the edges $(k-2)(k-1)k$ and $(k-3)(k-1)k$ which are the predecessors of $a-2$ edges in the Hasse graph of the left compressing partial order (see Figure 1).

To prove Theorem 3.9(1), the edge set E has only two possible cases, i.e., either removing the edge set $\{(k-2)(k-1)k, (k-3)(k-1)k, (k-4)(k-1)k\}$ or removing the edge set $\{(k-2)(k-1)k, (k-3)(k-1)k, (k-3)(k-2)k\}$ from $[k]^{(3)}$ (see Figure 1). We discuss each case below.

Case (1a). Remove $\{(k-2)(k-1)k, (k-3)(k-1)k, (k-4)(k-1)k\}$ from $[k]^3$. In this case, $E = E(C_{3,m}) = [k-1]^{(3)} \cup \{kij,$ where $ij \in [k-2]^{(2)}\} \cup \{1(k-1)k \ldots, (k-5)(k-1)k\}$. Using Lemma 3.1, $\lambda(G) = Z(A)$.

Case (1b). Remove $\{(k-2)(k-1)k, (k-3)(k-1)k, (k-3)(k-2)k\}$ from $[k]^3$. In this case, $E = [k-1]^{(3)} \cup \{kij,$ where $ij \in [k-2]^{(2)} \setminus \{(k-3)(k-2)\}\} \cup \{1(k-1)k \ldots, (k-4)(k-1)k\}$. Using Lemma 3.2, $\lambda(G) = \max\{\frac{(k-2)(k-3)}{6(k-1)^2}, Z(B)\}$.

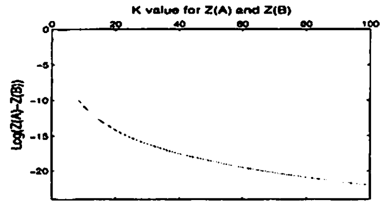

Figure 2:

It is clear that if $Z(A) \geq Z(B)$, then $\lambda(G) = \lambda(C_{3,m})$ and Conjecture 1.3 will be true for 3-graphs with $m = \binom{k}{3} - 3$ edges. This completes the proof of Theorem 3.9(1).

To prove Theorem 3.9(2), the edge set E has only three possible cases, i.e., either removing $\{(k-2)(k-1)k, (k-3)(k-1)k, (k-4)(k-1)k, (k-5)(k-1)k\}$ or removing $\{(k-2)(k-1)k, (k-3)(k-1)k, (k-3)(k-2)k, (k-3)(k-2)(k-1)\}$ or removing $\{(k-2)(k-1)k, (k-3)(k-1)k, (k-4)(k-1)k, (k-3)(k-2)k\}$ from $[k]^3$ (see Figure 1). We discuss each case below.

Case (2a). Remove $\{(k-2)(k-1)k, (k-3)(k-1)k, (k-4)(k-1)k, (k-5)(k-1)k\}$ from $[k]^3$. In this case, $E = E(C_{3,m}) = [k-1]^{(3)} \cup \{kij,$ where $ij \in [k-2]^{(2)}\} \cup \{1(k-1)k \ldots, (k-6)(k-1)k\}$. Using Lemma 3.4, $\lambda(G) = Z(C)$.

Case (2b). Remove $\{(k-2)(k-1)k, (k-3)(k-1)k, (k-3)(k-2)k, (k-3)(k-2)(k-1)\}$ from $[k]^3$. In this case, $E = [k-1]^{(3)} \setminus \{(k-3)(k-2)(k-1)\} \cup \{kij,$ where $ij \in [k-2]^{(2)} \setminus \{(k-3)(k-2)\}\} \cup \{1(k-1)k, \ldots, (k-4)(k-1)k\}$. Using Lemma 3.5, $\lambda(G) \leq \max\{\frac{(k-2)(k-3)}{6(k-1)^2}, Z(D)\}$.

Case (2c). Remove $\{(k-2)(k-1)k, (k-3)(k-1)k, (k-4)(k-1)k, (k-3)(k-2)k\}$ from $[k]^3$. In this case, $E = [k-1]^{(3)} \cup \{kij,$ where $ij \in [k-2]^{(2)} \setminus \{(k-2)(k-3)\}\} \cup \{1(k-1)k \ldots, (k-5)(k-1)k\}$. Using Lemma 3.7, $\lambda(G) = \max\{\frac{(k-2)(k-3)}{6(k-1)^2}, Z(E)\}$.

It is clear that if $Z(C) \geq max\{Z(D), Z(E)\}$, then $\lambda(G) = \lambda(C_{3,m})$ and Conjecture 1.3 will be true for 3-graphs with $m = \binom{k}{3} - 4$ edges. This completes the proof of Theorem 3.9(2). ■

Remark 3.10 *Using the software MatLab, we test the truth of the condition that $Z(A) \geq Z(B)$ in Theorem 3.9 for k from 6 to 100 and the truth of the condition that $Z(C) \geq \max\{Z(D), Z(E)\}$ in Theorem 3.9 for k from 7 to 50. We plot our findings as two figures (Figure 2 and Figure 3) as follows.*

In Figure 3, we let $Max(H)$, $Max(U)$, and $Max(V)$ represent $Z(C)$,

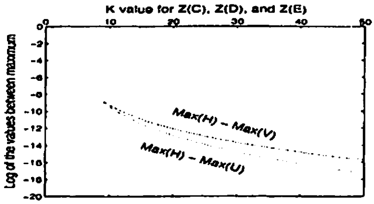

Figure 3:

$Z(D)$, and $Z(E)$, respectively. From Figure 2, we see that $Z(A) > Z(B)$. From Figure 3, we can also see that $Z(C) > Z(D) > Z(E)$. Using the same software, we can also verify that $\lambda(G) = Z(A)$, $\lambda(G) = Z(B)$, $\lambda(G) = Z(C)$, $\lambda(G) = Z(D)$, $\lambda(G) = Z(E)$ for $k \geq 7$ in Lemmas 3.1, 3.2, 3.4, 3.5, and 3.7 respectively.

4 Verify Conjecture 1.3 When $r = 3$ and $m \leq 50$

In this section, we first state two results which estimate the Lagrangian of some 3-graphs G with $m = \binom{k}{3} + \binom{k-1}{2}$ edges and its edges differ the first m edges in the colex ordering only by a few edges. These two results will be used in the verification of Proposition 4.3. Note that the first m edges (in 3-graphs) in the colex ordering is

$$[k]^{(3)} \cup \{(k+1)ij, \text{where } ij \in [k-1]^{(2)}\} = C_{3,m},$$

and in this case, $\lambda(C_{3,m}) = \lambda([k]^{(3)}])$ by Lemma 2.4.

Lemma 4.1 *Let G be a 3-graph with vertex set $\{1, 2, \ldots, k+1\}$ and edge set*

$$\begin{aligned} E(G) &= [k]^{(3)} \cup \{(k+1)ij, \text{where } ij \in [k-1]^{(2)} \\ &\quad \backslash \{(k-2)(k-1), \ldots, (k-b-1)(k-1)\}\} \\ &\quad \cup \{1k(k+1), \ldots, bk(k+1)\}, \end{aligned}$$

where $b \leq \lfloor \frac{k}{2} \rfloor - 1$. Then $\lambda(G) = \lambda([k]^{(3)}) = \frac{(k-1)(k-2)}{6k^2}$.

Proof. Observe that G is left-compressed. Let $\vec{x} = (x_1, x_2, x_3, x_4, \ldots, x_{k+1})$ be an optimal legal weighting for G satisfying conditions (1) on Page 4 and

(3) on Page 5. If $x_{k+1} = 0$, then $\lambda(G) = \lambda([k]^{(3)}) = \frac{(k-1)(k-2)}{6k^2}$ by direct calculation and we are done. So we can assume that $x_{k+1} > 0$. Observe that $E_{i\setminus j} = \emptyset$ for $1 \le i < j \le b$, and $k-b-1 \le i < j \le k-2$. By (2),

$$x_1 = \cdots = x_b. \tag{9}$$

$$x_{k-2} = x_{k-3} = \cdots = x_{k-b-1}. \tag{10}$$

Taking $i = k$ and $j = k+1$ in (2), we have

$$x_{k-1}(x_{k-2} + x_{k-3} + \cdots + x_{k-b-1}) = (x_k - x_{k+1})(x_1 + \cdots + x_b).$$

Applying Equations (9) and (10) to the above equation, we get

$$x_{k-1}x_{k-2} = x_1(x_k - x_{k+1}). \tag{11}$$

Taking $i = 1$ and $j = k$ in (2), we have

$$(x_1 - x_k)(x_2 + x_3 + \cdots + x_{k-1} + x_{k+1}) = x_{k+1}(x_{b+1} + x_{b+2} + \cdots + x_{k-1}).$$

Notice that

$$x_2 + x_3 + \cdots + x_{k-1} + x_{k+1} \ge x_{b+1} + x_{b+2} + \cdots + x_{k-1}.$$

So, $x_1 - x_k \le x_{k+1}$, i. e.,

$$x_1 \le x_k + x_{k+1}. \tag{12}$$

Combining equations (11) and (12), we get

$$x_{k-2}x_{k-1} \le (x_k + x_{k+1})(x_k - x_{k+1}) = x_k^2 - x_{k+1}^2.$$

On the other hand, $x_{k-2}x_{k-1} \ge x_k^2$ since $x_{k-2} \ge x_{k-1} \ge x_k$. Therefore,

$$x_{k+1} = 0.$$

This completes the proof. ■

The proof of the following lemma is very similar to the proof of the above lemma. We use $\{k-1, k-2, \ldots, k-l\}^{(2)}$ to represent all possible pairs from vertex set $\{k-1, k-2, \ldots, k-l\}$.

Lemma 4.2 *Let G be a 3-graph with vertex set $\{1, 2, \ldots, k+1\}$ and edge set*

$$\begin{aligned} E(G) \;=\; & [k]^{(3)} \\ & \cup\{(k+1)ij, \text{where } ij \in [k-1]^{(2)} \setminus \{k-1, k-2, \ldots, k-l\}^{(2)}\} \\ & \cup\{1k(k+1), \ldots, bk(k+1)\}, \end{aligned}$$

where $b = \binom{l}{2} < k - l$. Then $\lambda(G) = \lambda([k]^{(3)}) = \frac{(k-1)(k-2)}{6k^2}$.

Proof. Observe that G is left-compressed. Let $\vec{x} = (x_1, x_2, x_3, x_4, ..., x_{k+1})$ be an optimal legal weighting for G satisfying conditions (1) on Page 4 and (3) on Page 5. If $x_{k+1} = 0$, then $\lambda(G) = \lambda([k]^{(3)}) = \frac{(k-1)(k-2)}{6k^2}$ by direct calculation and we are done. So assume that $x_{k+1} > 0$, then each $x_i > 0$. Observe that $E_{i\setminus j} = \emptyset$ for $1 \le i < j \le b$, and $k - l \le i < j \le k-1$. By (2),

$$x_1 = \cdots = x_b; \tag{13}$$

$$x_{k-1} = x_{k-2} = \cdots = x_{k-l}. \tag{14}$$

Taking $i = k$ and $j = k+1$ in (2), and combining (14), we have

$$(x_k - x_{k+1})(x_1 + \cdots + x_b) = \binom{l}{2} x_{k-1}^2 = b x_{k-1}^2.$$

Applying Equation (13) to the above equation, we get

$$x_1(x_k - x_{k+1}) = x_{k-1}^2. \tag{15}$$

Taking $i = 1$ and $j = k$ in (2), we get

$$(x_1 - x_k)(x_2 + x_3 + \cdots + x_{k-1} + x_{k+1}) = x_{k+1}(x_{b+1} + x_{b+2} + \cdots + x_{k-1}).$$

Notice that $x_2 + x_3 + \cdots + x_{k-1} + x_{k+1} \ge x_{b+1} + x_{b+2} + \cdots + x_{k-1}$. So, $x_1 - x_k \le x_{k+1}$, i. e.,

$$x_1 \le x_k + x_{k+1}. \tag{16}$$

Combining equations (15) and (16), we get

$$x_{k-1}^2 \le (x_k + x_{k+1})(x_k - x_{k+1}) = x_k^2 - x_{k+1}^2.$$

On the other hand, $x_{k-1}^2 \ge x_k^2$ since $x_{k-1} \ge x_k$. Therefore,

$$x_{k+1} = 0.$$

This completes the proof. ■

Proposition 4.3 *Conjecture 1.3 is true for 3-graphs with $m \le 50$ edges.*

Before our verification, we would like to point out that in the proof of Proposition 4.3, some verification (Cases 1, 2, 4.1, 4.2, 7.1, 7.2, and 7.4 below depend on mathematical arguments since in those cases some 3-graphs whose edge set is not the set of the first m edges in colex ordering also achieve the maximum value $\lambda(C_{3,m})$. Note that we can use computer software to test cases 3, 4.3, 5, 6, 7.3, and 7.5 below, since, in these cases, those maximum values are strictly less than the corresponding maximum values $\lambda(C_{3,m})$, therefore, the accuracy can be guaranteed.

Proof. Verification goes by case analysis. First we point out, Theorem 1.4 cover the following cases: 1. when the number of edges $m = 1, 2, 3, 4$: $1 = \binom{3}{3}$, $2 = \binom{4}{3} - 2$, $3 = \binom{4}{3} - 1$, and $4 = \binom{4}{3}$; 2. when the number of edges $m = 8, 9, 10, 11$: $8 = \binom{5}{3} - 2$, $9 = \binom{5}{3} - 1$, $10 = \binom{5}{3}$, and $11 = \binom{6-1}{3} + \binom{6-2}{2} - (6-1)$; 3. when the number of edges $m = 18, 19, 20, 21, 22, 23, 24$: $18 = \binom{6}{3} - 2$, $19 = \binom{6}{3} - 1$, $20 = \binom{7-1}{3} \leq m = 20, 21, 22, 23, 24 \leq \binom{7-1}{3} + \binom{7-2}{2} - (7-1)$; 4. when the number of edges is in the range of $33 \leq m \leq 43$: $33 = \binom{7}{3} - 2$, $34 = \binom{7}{3} - 1$, $35 = \binom{8-1}{3} \leq m \leq 43 = \binom{8-1}{3} + \binom{8-2}{2} - (8-1)$. We now discuss the rest of m not covered by Theorem 1.4 in several cases.

Case 1. $m = 5, 6, 7$. The proof in this case goes as follows. Note that $\binom{4}{3} < m = 5, 6, 7 \leq \binom{4}{3} + \binom{3}{2}$. By Lemma 2.4, $\lambda(C_{3,m}) = \lambda([4]^{(3)}) = \binom{4}{3}/4^3 = \frac{1}{16}$ for $m = 5, 6, 7$. Since λ_m^3 increases as m increases, it is sufficient to show the case for $m = 7$.

Let $G = (V, E)$ be a 3-graph satisfying $\lambda(G) = \lambda_7^3$ for $|E| = 7$. By Lemma 2.3, we may assume that G is left compressed. Let $\vec{x} = (x_1, x_2, x_3, x_4, \ldots)$ be an optimal legal weighting for G satisfying conditions (1) on Page 4 and (3) on Page 5. By Lemma 2.5, $x_i = 0$ for $i \geq 6$ since otherwise, $|E| \geq \binom{5}{3} + \binom{4}{2} - 4 > 7$. So we may assume that $V(G) = \{1, 2, 3, 4, 5\}$. If $x_5 = 0$, then $\lambda(G) \leq \lambda([4]^{(3)}) = \frac{1}{16}$ and we are done. So assume that $x_5 \neq 0$. By Lemma 2.1, the edge $145 \in E$. Since G is left compressed, all triples containing 1 (there are 6 of them) are in E and another edge is 234. So $E = \{123, 124, 134, 135, 125, 145, 234\}$.

Observe that $E_{i \setminus j} = \emptyset$ for $2 \leq i < j \leq 4$. By (2), $x_2 = x_3 = x_4 \overset{\text{def}}{=} a$. Let $x_5 \overset{\text{def}}{=} b$. Notice that $\lambda(E_1, \vec{x}) = \lambda(E_5, \vec{x})$ implies that $3a^2 + 3ab = 3ax_1$, so $x_1 = a + b$. Therefore, $a + b + 3a + b = 1$ implies that $b = \frac{1-4a}{2}$.

Then by Lemma 2.1,

$$\lambda(G) = \frac{1}{3}\lambda(E_1, \vec{x}) = a^2 + ab = -a^2 + \frac{a}{2} = -(a - \frac{1}{4})^2 + \frac{1}{16} \leq \frac{1}{16}.$$

This completes the proof of this case.

Case 2. $m = 12, 13, 14, 15, 16$. The proof in this case goes as follows. Note that $\binom{6-1}{3} < m = 12, 13, 14, 15, 16 \leq \binom{6-1}{3} + \binom{6-2}{2}$. By Lemma 2.4, $\lambda(C_{3,m}) = \lambda([5]^{(3)}) = \binom{5}{3}/5^3 = \frac{2}{25}$ for $m = 12, 13, 14, 15, 16$. Since λ_m^3 increases as m increases, it is sufficient to show that $\lambda_m^3 = 2/25$ for $m = 16$.

Let $G = (V, E)$ be a 3-graph satisfying $\lambda(G) = \lambda_{16}^3$ for $|E| = 16$. By Lemma 2.3, we can assume that G is left compressed. Let $\vec{x} = (x_1, x_2, x_3, x_4,$

...) be an optimal legal weighting for G satisfying conditions (1) on Page 4 and (3) on Page 5. By Lemma 2.5, $x_i = 0$ for $i \geq 7$ since otherwise $|E| \geq \binom{6}{3} + \binom{5}{2} - 5 > 16$. If $x_6 = 0$, then $\lambda(G) \leq \lambda([5]^{(3)}) = \frac{2}{25}$ and we are done. So assume that $x_6 > 0$. By Lemma 2.1, the edge $156 \in E$. Note that E is formed by removing 4 edges from $[6]^{(3)}$. Since G is left compressed and $156 \in E$, then either the set $\{456, 356, 346, 345\}$ or the set $\{456, 356, 256, 346\}$ is removed from $[6]^{(3)}$ in viewing of Figure 1 (take $k = 6$ there).

Subcase 2.1: The set $\{456, 356, 346, 345\}$ is removed from $[6]^{(3)}$. In this case,

$$E(G) = [5]^{(3)} \setminus \{345\} \cup \{6ij, \text{ where } ij \in [4]^{(2)} \setminus \{34\}\} \cup \{156, 256\}$$

Applying Lemma 3.5 by taking $k = 6$, it is enough to show that

$$Z(D) = \max\{4a^2b + 12ab^2\},$$

under the constraint $2a + 4b = 1$ with $a \geq b \geq 0$ is $\leq \frac{2}{25}$. Using Remark 3.6 by taking $k = 6$, we have $Z(D) = u(a, b)$ where $a = \frac{4-\sqrt{7}}{6}$ and $b = \frac{-1+\sqrt{7}}{12}$. By a direct calculation, $u(a, b) < \frac{2}{25}$. This completes the proof of this subcase.

Subcase 2.2: The set $\{456, 356, 256, 346\}$ is removed from $[6]^{(3)}$. In this case,

$$E = [5]^{(3)} \cup \{6ij, \text{where } ij \in [4]^{(2)} \setminus \{34\}\} \cup \{156\}.$$

Applying Lemma 4.1 by taking $b = 1$ and $k = 5$, then we get $\lambda(G) = \frac{2}{25}$. This completes the proof of this subcase.

Case 3. $m = 17$. Taking $k = 6$ in Theorem 3.9(1) and using Figure 2, we are done.

Case 4. $m = 25, 26, 27, 28, 29, 30$. The proof in this case goes as follows. Note that $\binom{7-1}{3} < m = 25, 26, 27, 28, 29, 30 \leq \binom{7-1}{3} + \binom{7-2}{2}$. By Lemma 2.4, $\lambda(C_{3,m}) = \lambda([6]^{(3)}) = \binom{6}{3}/6^3 = \frac{5}{54}$ for $m = 25, 26, 27, 28, 29, 30$. Since λ_m^3 increases as m increases, it is sufficient to show that $\lambda_m^3 = 5/54$ for $m = 30$.

Let $G = (V, E)$ be a 3-graph satisfying $\lambda(G) = \lambda_{30}^3$ for $|E| = 30$. By Lemma 2.3, we can assume that G is left compressed. Let $\vec{x} = (x_1, x_2, x_3, x_4, ...)$ be an optimal legal weighting for G satisfying conditions (1) on Page 4 and (3) on Page 5. By Lemma 2.5, $x_i = 0$ for $i \geq 8$ since otherwise $|E| \geq \binom{7}{3} + \binom{6}{2} - 6 > 30$. If $x_7 = 0$, then $\lambda(G) \leq \lambda([6]^{(3)}) = \frac{5}{54}$ and we are done. So assume that $x_7 > 0$. By Lemma 2.1, the edge $167 \in E$.

Note that E is formed by removing 5 edges from $[7]^{(3)}$. Since G is left compressed and $167 \in E$, then the set $\{567, 467, 367, 267, 457\}$, or the set $\{567, 467, 367, 457, 357\}$, or the set $\{567, 467, 367, 457, 456\}$ is removed from $[7]^{(3)}$ in viewing of Figure 1 (take $k = 7$ there). We will discuss each case below. Note that

$$[7]^{(3)} = [6]^{(3)} \cup \{7ij, \text{where } ij \in [5]^{(2)}\} \cup \{167, 267, 367, 467, 567\}.$$

Subcase 4.1: $\{567, 467, 367, 267, 457\}$ is not contained in E. In this case,

$$E = [6]^{(3)} \cup \{7ij, \text{ where } ij \in [5]^2 \setminus \{45\}\} \cup \{167\}.$$

Applying Lemma 4.1 by taking $b = 1$ and $k = 6$, then we get $\lambda(G) = \frac{5}{54}$. This completes the proof of this subcase.

Subcase 4.2: $\{567, 467, 367, 457, 357\}$ is not contained in E. In this case,

$$E = [6]^{(3)} \cup \{7ij, \text{ where } ij \in [5]^2 \setminus \{45, 35\}\} \cup \{167, 267\}.$$

Applying Lemma 4.1 by taking $b = 2$ and $k = 6$, then we get $\lambda(G) = \frac{5}{54}$. This completes the proof of this subcase.

Subcase 4.3: $\{567, 467, 367, 457, 456\}$ is not contained in E. In this case,

$$E = [6]^{(3)} \setminus \{456\} \cup \{7ij, \text{ where } ij \in [5]^2 \setminus \{45\}\} \cup \{167, 267\}.$$

Observe that $E_1 \cap E_2^c = \emptyset$. By (2), $x_1 = x_2 \stackrel{\text{def}}{=} a$. Let $x_3 \stackrel{\text{def}}{=} b$. Observe that $E_4 \cap E_5^c = \emptyset$ and $E_6 \cap E_7^c = \emptyset$. By (2),

$$x_4 = x_5 \stackrel{\text{def}}{=} c, \quad x_6 = x_7 \stackrel{\text{def}}{=} d.$$

Note that $2a + b + 2c + 2d = 1$. In viewing of edges in E,

$$\lambda(G) = a^2(b+2c+2d)+2a(2bc+2bd+c^2+4cd+d^2)+(bc^2+4bcd) \stackrel{\text{def}}{=} F(a, b, c, d).$$

Using the software Matlab, we see that the maximum value of $F(a, b, c, d)$ under the constraint

$$2a + b + 2c + 2d = 1, a \geq b \geq c \geq d \geq 0$$

is $\leq 0.09140413 < \frac{5}{54}$.

Case 5. $m = 31$. Taking $k = 7$ in Theorem 3.9(2) and using Figure 3, we are done.

Case 6. $m = 32$. Taking $k = 7$ in Theorem 3.9(1) and using Figure 2, we are done.

Case 7. $44 \le m \le 50$. The proof in this case goes as follows. Note that $\binom{8-1}{3} < 44 \le 50 = \binom{8-1}{3} + \binom{8-2}{2}$. By Lemma 2.4, $\lambda(C_{3,m}) = \lambda([7]^{(3)}) = \binom{7}{3}/7^3 = \frac{5}{49}$ for $44 \le m \le 50$. Since λ_m^3 increases as m increases, it is sufficient to show that $\lambda_m^3 = 5/49$ for $m = 50$.

Let $G = (V, E)$ be a 3-graph satisfying $\lambda(G) = \lambda_{50}^3$ for $|E| = 50$. By Lemma 2.3, we can assume G is left compressed. Let $\vec{x} = (x_1, x_2, x_3, x_4, \ldots)$ be an optimal legal weighting for G satisfying conditions (1) on Page 4 and (3) on Page 5. By Lemma 2.5, $x_i = 0$ for $i \ge 9$ since otherwise $|E| \ge \binom{8}{3} + \binom{7}{2} - 7 > 50$. If $x_8 = 0$, then $\lambda(G) \le \lambda([7]^{(3)}) = \frac{5}{49}$ and we are done. So assume that $x_8 > 0$. By Lemma 2.1, the edge $178 \in E$. Note that E is formed by removing 6 edges from $[7]^{(3)}$. Since G is left compressed and $178 \in E$, then the set $\{678, 578, 478, 378, 278, 568\}$, or the set $\{678, 578, 478, 378, 568, 468\}$, or the set $\{678, 578, 478, 378, 568, 567\}$, or the set $\{678, 578, 478, 568, 468, 458\}$, or the set $\{678, 578, 478, 568, 567, 468\}$ is removed from $[8]^{(3)}$ in viewing of Figure 1 (take $k = 8$ there).

We will discuss the possible cases below. Note that

$$[8]^{(3)} = [7]^{(3)} \cup \{8ij, \text{where } ij \in [6]^{(2)}\} \cup \{178, 278, 378, 478, 578, 678\}$$

Subcase 7.1. The set $\{678, 578, 478, 378, 278, 568\}$ is removed from $[8]^{(3)}$. In this case,

$$E = [7]^{(3)} \cup \{8ij, ij \in [6]^{(2)} \setminus \{56\}\} \cup \{178\}.$$

Applying Lemma 4.1 by taking $b = 1$ and $k = 7$, then we get $\lambda(G) = \frac{5}{49}$. This completes the proof of this subcase.

Subcase 7.2. The set $\{678, 578, 478, 378, 568, 468\}$ is removed from $[8]^{(3)}$. In this case,

$$E = [7]^{(3)} \cup \{8ij, ij \in [6]^{(2)} \setminus \{56, 46\}\} \cup \{178, 278\}.$$

Applying Lemma 4.1 by taking $b = 2$ and $k = 7$, then we get $\lambda(G) = \frac{5}{49}$. This completes the proof of this subcase.

Subcase 7.3. The set $\{678, 578, 478, 378, 568, 567\}$ is removed from $[8]^{(3)}$. In this case,

$$E = [7]^{(3)} \setminus \{567\} \cup \{8ij, ij \in [6]^{(2)} \setminus \{56\}\} \cup \{178, 278\}.$$

Observe that $E_1 \cap E_2^c = \emptyset$. By (2), $x_1 = x_2 \stackrel{\text{def}}{=} a$. Similarly, $E_3 \cap E_4^c = \emptyset$, $E_5 \cap E_6^c = \emptyset$, and $E_7 \cap E_8^c = \emptyset$ imply that $x_3 = x_4 \stackrel{\text{def}}{=} b$, $x_5 = x_6 \stackrel{\text{def}}{=} c$, $x_7 = x_8 \stackrel{\text{def}}{=} d$. Note that $2a + 2b + 2c + 2d = 1$. In viewing of edges in E,

$$\lambda(G) \;=\; a^2(2b + 2c + d) + 2a(b^2 + 4bc + c^2 + 2bd + 2cd)$$

$$\begin{aligned}
&+ && (2bc^2+2b^2c+b^2d+4bcd)+d(a^2+4ab+4ac+b^2+4bc)+2ad^2 \\
&= && a^2(2b+2c)+2a(b^2+4bc+c^2)+(2bc^2+2b^2c) \\
& && +2d(a^2+4ab+4ac+b^2+4bc)+2ad^2 \\
&\stackrel{\text{def}}{=} && X(a,b,c,d).
\end{aligned}$$

Using the software Matlab, testing result shows that the maximum value of $X(a,b,c,d)$ under the constraint

$$2a+2b+2c+2d=1, a\geq b\geq c\geq d\geq 0.$$

is $\leq 0.1011297 < \frac{5}{49}$.

Subcase 7.4. The set $\{678, 578, 478, 568, 468, 458\}$ is removed from $[8]^{(3)}$. In this case,

$$E=[7]^{(3)}\cup\{8ij, ij\in[6]^{(2)}\setminus\{56,46,45\}\}\cup\{178,278,378\}.$$

Applying Lemma 4.2 by taking $l=3$ (then $b=3$) and $k=7$, then we get $\lambda(G)=\frac{5}{49}$. This completes the proof of this subcase.

Subcase 7.5. The set $\{678, 578, 478, 568, 567, 468\}$ is removed from $[8]^{(3)}$. In this case,

$$E=[7]^{(3)}\setminus\{567\}\cup\{8ij, ij\in[6]^{(2)}\setminus\{56,46\}\}\cup\{178,278,378\}.$$

Observe that $E_{i\setminus j}=\emptyset$ for $1\leq i<j\leq 3$. By (2), $x_1=x_2=x_3\stackrel{\text{def}}{=}a$. Let $x_4\stackrel{\text{def}}{=}b$, $x_5\stackrel{\text{def}}{=}c$, $x_6\stackrel{\text{def}}{=}d$, $x_7\stackrel{\text{def}}{=}e$, $x_8\stackrel{\text{def}}{=}f$. Note that $3a+b+c+d+e+f=1$. In viewing of edges in E,

$$\begin{aligned}
\lambda(G) &= a^3+3a^2(b+c+d+e)+3a(bc+bd+be+cd+ce+de) \\
&\quad +bcd+bce+bde+f[3a^2+3a(b+c+d)+bc]+3aef \\
&\stackrel{\text{def}}{=} Y(a,b,c,d,e,f).
\end{aligned}$$

Using the Software Matlab, testing result shows that the maximum value of $Y(a,b,c,d,e,f)$ under the constraint $3a+b+c+d+e+f=1, a\geq b\geq c\geq d\geq e\geq f\geq 0$ is $\leq 0.1010085 < \frac{5}{49}$. This completes the verification. ■

Remark 4.4 *It is possible to verify the conjecture for bigger value m by case analysis. Since as m gets bigger, case analysis will be more tedious, we do not give any details for any further values m although it is possible.*

Remark 4.5 *As a generalization of Lemmas 4.1 and 4.2, we proved the following result in [11] recently.*

Theorem. *Let m and l be positive integers satisfying $\binom{l-1}{3} \le m \le \binom{l-1}{3} + \binom{l-2}{2}$. Let G be a 3-graph with m edges and G contain a clique of size $l-1$. Then $\lambda(G) = \lambda([l-1]^{(3)})$.*

We believe that if G is a 3-graph with m edges and G contains no clique of size $l-1$, where $\binom{l-1}{3} \le m \le \binom{l-1}{3} + \binom{l-2}{2}$, then $\lambda(G)$ should be strictly less than $\lambda([l-1]^{(3)})$. If one can verify this, then combining with the above Theorem, one can verify Conjecture 1.3 for all m, where $\binom{l-1}{3} \le m \le \binom{l-1}{3} + \binom{l-2}{2}$. This would extend Theorem 1.4.

In [11], we also generalized Lemmas 4.1 and 4.2 to r-uniform hypergraphs. This gives us hope that the above Theorem might be generalized to r-uniform hyergraphs.

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