Some considerations on the n-th commutativity degrees of finite groups

A. Erfanian, B. Tolue
Department of Mathematics and
Center of Excellence in Analysis on Algebraic Structures,
Ferdowsi University of Mashhad, Mashhad, Iran.
erfanian@math.um.ac.ir and b.tolue@gmail.com
N. H. Sarmin
Department of Mathematics, Faculty of Science,
Universiti Teknologi Malaysia, Skudai, Malaysia.
nhs@utm.my

Abstract

Let G be a finite group and n a positive integer. The n-th commutativity degree $P_n(G)$ of G is the probability that the n-th power of a random element of G commutes with another random element of G. In 1968, P. Erdös and P.Turan investigated the case $n = 1$, involving only methods of combinatorics. Later several authors improved their studies and there is a growing literature on the topic in the last 10 years. We introduce the relative n-th commutativity degree $P_n(H, G)$ of a subgroup H of G. This is the probability that an n-th power of a random element in H commutes with an element in G. The influence of $P_n(G)$ and $P_n(H, G)$ on the structure of G is the purpose of the present work.

[0] **2000 Mathematics Subject Classification**(*Amer. Math. Soc.*) **Primary**: 20P05; **Secondary** : 20D60
Keywords : Commutativity degree, n-th nilpotency degree, isoclinism, extra special p-groups.

1 Introduction

All the groups which we consider are finite. For every group G, the n-th commutativity degree $P_n(G)$ of G is the probability that the n-th power of a random element of G commutes with another random element of G. More precisely,

$$P_n(G) = \frac{|\{(x,y) \in G \times G : [x^n, y] = 1\}|}{|G|^2}. \qquad (1)$$

(1) has been recently introduced in [1] by N. M. M. Ali and N. Sarmin. They computed (1) for some values of n and some 2-generators 2-groups of nilpotency class 2. The importance of $P_n(G)$ is due to the fact that $d(G) = P_1(G)$ is the *commutativity degree* of G, introduced by P. Erdös and P. Turan in [3]. Such a work became a classic reference for the studies of several authors, as testified for instance by [4, 5, 7, 11]. There are many generalizations of $d(G)$. The *n-th nilpotency degree* $d^n(G)$ of G was studied [4] and [11]. The *mutually commuting n-tuples degree* $d_n(G)$ of G was studied in [5]. Among these two notions, we will see that $P_n(G)$ can be placed and this justifies our interest to deal with it. The main results of the present paper are as the following.

Theorem A. Let G be a non-abelian group and p be the smallest prime dividing the order of G. Then the following statements are equivalent:

(i) $\frac{G}{Z(G)} \simeq \mathbb{Z}_p \times \mathbb{Z}_p$;

(ii) G is isoclinic with an extra special p-group of order p^3;

(iii) $P_n(G) = \dfrac{p^2+p-1}{p^3}$, for all n whenever it is not divisible by p.

Theorem B. If G and H are two isoclinic groups, then $P_n(G) = P_n(H)$ for every $n \geq 1$.

Section 2 is devoted to prove some basic properties of $P_n(G)$. Successively we will give the details of the proofs of Theorems A and B in Section 3. Terminology and notations are standard and can be found in [10].

2 Some Basic Results

The following two definitions have been already mentioned above.

Definition 2.1. *Let G be a group. For every $n \geq 1$*

$$d^n(G) = \frac{|\{(x_1, \dots, x_{n+1}) \in G^{n+1} : [x_1, \dots, x_{n+1}] = 1\}|}{|G|^{n+1}} \tag{2}$$

is called n-th nilpotency degree of G and

$$d_n(G) = \frac{|\{(x_1, \dots, x_{n+1}) \in G^{n+1} : x_i x_j = x_j x_i\}|}{|G|^{n+1}} \tag{3}$$

is called mutually commuting n-tuples degree of G.

Obviously for $n = 1$ in (2) and (3) we find the commutativity degree $d(G)$ in [7, 11]. There are some significant results on $d^n(G)$ and $d_n(G)$ in [4, 5, 11]. In these works it was studied the general concept of *relative n-th nilpotency degree* $d^n(H,G)$ of a subgroup H of G. By using the idea given in [4], we may introduce the following notion.

Definition 2.2. *Let H be a subgroup of a group G.*

$$P_n(H,G) = \frac{|\{(h,g) \in H \times G : [h^n, g] = 1\}|}{|H||G|} \tag{4}$$

is called relative n-th commutativity degree of G.

Clearly, if $H = G$ then $P_n(G) = P_n(H,G)$. We note that $P_n(H,G)$ and $P_n(G)$ are sometimes equal to one. For instance, if G is abelian or has exponent dividing n then $P_n(G) = P_n(H,G) = 1$. We may also easily see that if G is a nilpotent group of class 2 in which its derive subgroup has exponent dividing n, then we will again have $P_n(G) = P_n(H,G) = 1$. Of course, we may have $P_n(H,G) = 1$ and $P_n(G) < 1$ for some groups G and subgroups H of G and positive integer n (take $H \subseteq Z(G)$, for instance).

Some lemmas are necessary for the proof of the main theorems. Let us start with an initial fact, which compares $P_n(H,G)$ and $P_n(H)$.

Lemma 2.3. *Let H be a subgroup of a group G. Then $P_n(H,G) \leq P_n(H)$, for every $n \geq 1$. The equality holds if $G = HZ(G)$.*

Proof. We have

$$P_n(H,G) = \frac{1}{|H||G|}\sum_{h\in H}|C_G(h^n)| = \frac{1}{|H|}\sum_{h\in H}\frac{|C_G(h^n)|}{|G|} \leq \frac{1}{|H|}\sum_{h\in H}\frac{|C_H(h^n)|}{|H|}$$

$$= \frac{1}{|H|^2}\sum_{h\in H}|C_H(h^n)| = P_n(H)$$

by [4, Lemma 3.2]. If $G = HZ(G)$, then $[G : C_G(x)] = [H : C_H(x)]$ for every $x \in G$. So, $P_n(H,G) = P_n(H)$ for every $n \geq 1$.

Lemma 2.4. *Let H be a proper subgroup of a group G. Then $\frac{1}{[G:H]}P_n(H,G) < P_n(G)$ for every $n \geq 1$.*

Proof. We can see that

$$P_n(G) = \frac{1}{|G|^2}\sum_{g\in G}|C_G(g^n)| = \frac{1}{|G|^2}[\sum_{g\in H}|C_G(g^n)| + \sum_{g\in G-H}|C_G(g^n)|]$$

$$= \frac{1}{|G|^2}[|H||G|P_n(H,G) + \sum_{g\in G-H}|C_G(g^n)|] > \frac{|H|}{|G|}P_n(H,G).$$

This leads to the desired result.

Lemma 2.5. *Let H and K be subgroups of a group G such that K is contained in H. Then $P_n(H,G) \geq \frac{1}{[H:K]}P_n(K,G) \geq \frac{1}{[G:K]}P_n(K,H)$ for every $n \geq 1$.*

Proof. The proof is similar to that of Lemma 2.4.

One of the main difference between $P_n(G)$, $d^n(G)$ and $d_n(G)$ is that $d^n(G)$ is always increasing and $d_n(G)$ is always decreasing for every $n \geq 1$, but $P_n(G)$ does not have the same growth. It is sometimes increasing and sometimes decreasing, up to the structure of G and the choice of n. For instance, if G is a dihedral group of order 8, then either $P_n(G) = 5/8$ if n odd, or $P_n(G) = 1$ if n even. More generally, if G is a nilpotent group of class 2, then either $P_n(G) = 1$ if exp (G') divides n, or $P_n(G) = P_r(G)$ otherwise, where $r = n - exp(G')t$ for some $t \geq 1$. In other words, if $n \equiv r\ mod(exp(G'))$, then $P_n(G) = P_r(G)$, since $P_0(G) = 1$. Of course, it is always valid that $P_n(G) \geq P_1(G) = d(G)$ for every $n \geq 1$. The same situation is true for $P_n(H,G)$.

The next result compares factor groups with respect to (1).

Lemma 2.6. *Let N be a normal subgroup of a group G. Then $\frac{1}{|N|}P_n(\frac{G}{N}) \leq P_n(G) \leq P_n(\frac{G}{N})$, for every $n \geq 1$.*

Proof. By [4, Lemma 3.8 and 10, Corollary 2.24], it is clear that $|\frac{C_G(x)N}{N}| \leq |C_{\frac{G}{N}}(xN)| \leq |C_G(x)|$ for every $x \in G$. Thus we have

$$|N|^2|\frac{G}{N}|^2P_n(\frac{G}{N}) = |N|^2 \sum_{xN\in\frac{G}{N}} |C_{\frac{G}{N}}(x^nN)| \geq \sum_{x\in G} |C_{\frac{G}{N}}(x^nN)||C_N(x^n)| \geq |G|^2P_n(G)$$

$$= \sum_{x\in G} |C_G(x^n)| \geq \sum_{x\in G} |C_{\frac{G}{N}}(x^nN)| = |N| \sum_{xN\in\frac{G}{N}} |C_{\frac{G}{N}}(x^nN)| = |N||\frac{G}{N}|^2P_n(\frac{G}{N}).$$

It is easy to check that if N is the identity subgroup then both inequalities will be equalities. Moreover, if $N \cap G' = 1$, then the second inequality is actually an equality.

We can extend Lemma 2.6 to the case of $P_n(H,G)$ as follows.

Theorem 2.7. *Let N and H be subgroups of a group G such that N is normal in G and $N \subseteq H$. Then $\frac{1}{|N|}P_n(\frac{H}{N}, \frac{G}{N}) \leq P_n(H,G) \leq P_n(\frac{H}{N}, \frac{G}{N})$. Moreover, the equality on the right hand side holds when $N \cap G' = 1$.*

Proof. The proof of the first part is very similar to that of Lemma 2.6 so we omit it here. For the second part, we observe that, if $N \cap G' = 1$, then $|\frac{C_G(h^n)N}{N}| = |C_{\frac{G}{N}}(h^nN)|$ and this implies $P_n(H,G) = P_n(\frac{H}{N}, \frac{G}{N})$.

It is not actually easy to determine the exact value of $P_n(G)$ for every group G and for every positive integer n. But we may compute it for some known groups. The following example gives a specific formula for $P_n(D_{2m})$, where $m \geq 2$ and $n \geq 1$.

Example 2.8. *Let D_{2m} be the dihedral group of order $2m$ and n be a positive integer with $(m,n) = t$. Then*
(i) if m is odd, then $P_n(D_{2m})$ is $\frac{3m^2+tm}{4m^2}$ or $\frac{m^2+(t+2)m}{4m^2}$, whenever n is even or odd, respectively.
(ii) if m and n are even, then $P_n(D_{2m})$ is $\frac{3m^2+tm}{4m^2}$ or $\frac{3m^2+2tm}{4m^2}$, whenever $\frac{m}{t}$ is odd or even, respectively.
(iii) if m is even and n is odd, then $P_n(D_{2m})$ is $\frac{m^2+(2t+4)m}{4m^2}$.

Proof. (*i*) Suppose $D_{2m} = \{e, a, a^2 \cdots, a^{m-1}, b, ab, a^2b, \cdots, a^{m-1}b\}$ and m is an odd number. We have to count the number of pairs $(x,y) \in$

D_{2m} with $[x^n, y] = e$. Obviously for identity element x we have $2m$ pairs. Put $x = a^j$, $1 \leq j \leq m-1$. It is clear that x^n commutes with all $y = a^i$, where $0 \leq i \leq m-1$ and so we have $m(m-1)$ pairs here. Consider $y = a^i b$, $0 \leq i \leq m-1$, then $[(a^j)^n, a^i b] = a^{-2jn} = e$ if $m|2jn$. Since $(m,n) = t$ and m is odd, we should have $\frac{m}{t}|j$. Thus j can be multiple of $\frac{m}{t}$ whenever it is between one and $m-1$, i.e. $j = \frac{m}{t}, \frac{2m}{t}, \cdots, \frac{km}{t}$ such that $\frac{km}{t} < m$. Hence for such j we have $(t-1)m$ elements. Now, assume $x = a^j b$ and $0 \leq j \leq m-1$. We know that x has order 2. If n is an even number so there are $2m^2$ pairs in this case. If n is an odd number, then $x^n = x$ and it will only commute with identity element and itself. Thus we have $2m$ pairs here. This completes (i). The proof of (ii) and (iii) are very similar to what we have just done for (i).

We used GAP in [6] to verify the values in Example 2.8 for D_{10} when n is small enough. Some details are the following:

$$P_8(D_{10}) = 8/10 \ , \ P_{10}(D_{10}) = 1 \ , \ P_{11}(D_{10}) = 4/10 \ , \ P_{15}(D_{10}) = 6/10 \ .$$

The next result follows easily from the definitions and can be extended to a finite number of groups. We omit its proof.

Proposition 2.9. *If G_1 and G_2 are two groups, then $P_n(G_1 \times G_2) = P_n(G_1)P_n(G_2)$.*

We end this section recalling a notion in [8], useful in the proof of Theorem B.

Definition 2.10. *Let G and H be two groups; a pair (φ, ψ) is called an isoclinism of groups G and H if φ is an isomorphism from $G/Z(G)$ to $H/Z(H)$, ψ is also an isomorphism from G' to H' and $\psi([g_1, g_2]) = [h_1, h_2]$ whenever $h_i \in \varphi(g_i Z(G))$, for all $g_i \in G$, $h_i \in H$, $i \in \{1,2\}$ such that the following diagram commutes*

$$\begin{array}{ccc} \frac{G}{Z(G)} \times \frac{G}{Z(G)} & \longrightarrow & \frac{H}{Z(H)} \times \frac{H}{Z(H)} \\ \downarrow & & \downarrow \\ G' & \longrightarrow & H' \end{array}$$

If there is an isoclinism from G to H, we say that G and H are *isoclinic* and denote it by $G \sim H$. It can be easily checked that $\sim$ is an equivalence relation in the universe of all finite groups. Moreover, two isomorphic groups are obviously isoclinic, while the converse is not true. For instance, Q_8 and D_8 are isoclinic but not isomorphic (see [11] for details).

Finally we recall that G is an *extra special p-group* of order p^{2m+1} if $G' = Z(G) \simeq \mathbb{Z}_p$ and $G/Z(G)$ is an elementary abelian p-group of rank $2m$.

3 Proof of Main Theorems

The next bound plays an important role in the proof of Theorem A.

Lemma 3.1. *Let G be a group, p be a prime and $G/Z(G)$ an elementary abelian p-group of rank s. Then $P_n(G) = 1$ if p divides n. Otherwise,*

$$\frac{p^s + p^{s-1} - 1}{p^{2s-1}} \leq P_n(G) \leq \frac{p^s + p - 1}{p^{s+1}}.$$

Proof. Since $\frac{G}{Z(G)}$ is an elementary abelian p-group, $(xZ(G))^p = Z(G)$ and therefore $x^p \in Z(G)$. Moreover, G is nilpotent of class 2. First, suppose that $p|n$, then there is a positive integer t such that $n = pt$. So, for every arbitrary pair (x, y) in G^2 we can see that

$$[x^n, y] = [x^{pt}, y] = [x^p, y]^{(x^p)^{t-1}}[(x^p)^{t-1}, y] = \ldots = [x^p, y] = 1.$$

Hence $P_n(G) = 1$. Now, assume that p does not divide n. Then it is clear that if $x \notin Z(G)$ then $x^n \notin Z(G)$. Thus we can find a lower and upper bound for $|C_G(x)|$ when $x \notin Z(G)$. Obviously, $Z(G) \neq C_G(x)$ and so

$$p^s = [G : Z(G)] = [G : C_G(x)][C_G(x) : Z(G)] \geq [G : C_G(x)]p.$$

Therefore $|C_G(x)| \geq \frac{|G|}{p^{s-1}}$. Similarly, we have $|C_G(x)| \leq \frac{|G|}{p}$. Thus

$$P_n(G) = \frac{1}{|G|^2} \sum_{x \in G} |C_G(x^n)| = \frac{1}{|G|^2}[\sum_{x \in Z(G)} |C_G(x^n)| + \sum_{x \notin Z(G)} |C_G(x^n)|]$$

$$= \frac{1}{|G|^2}[|G||Z(G)| + \sum_{x \notin Z(G)} |C_G(x^n)|],$$

and consequently

$$\frac{|Z(G)|}{|G|} + \frac{|G| - |Z(G)|}{|G|^2}\frac{|G|}{p^{s-1}} \leq P_n(G) \leq \frac{|Z(G)|}{|G|} + \frac{|G| - |Z(G)|}{|G|^2}\frac{|G|}{p}.$$

The result follows.

If $s = 2$ in Lemma 3.1 and p does not divide n, then the lower and upper bound coincides. This means that $P_n(G) = \frac{p^2+p-1}{p^3}$. Furthermore, if $p = 2$ i.e. $\frac{G}{Z(G)} \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$, then either $P_n(G) = 1$ if n is even or $P_n(G) = 5/8$ if n is odd. Already D_8 and Q_8 satisfy this circumstance. Moreover, consider the group

$$G = \langle a, b | a^9 = b^3, bab^{-1} = a^4 \rangle. \tag{5}$$

We can check that G is a metacyclic group and $\frac{G}{Z(G)} \simeq \mathbb{Z}_3 \times \mathbb{Z}_3$. Thus by the above remark we have $P_n(G) = 11/27$ for all n not divisible by 3 and $P_n(G) = 1$ otherwise. For example $P_5(G) = P_{10}(G) = 11/27$, and $P_6(G) = P_9(G) = 1$.

Mimicking the techniques of Lemma 3.1 and use Lemma 3.2 in [4] to prove the following generalization.

Theorem 3.2. *Let H be a subgroup of a group G and p be a prime such that $\frac{H}{Z(G)\cap H}$ is isomorphic to $\underbrace{\mathbb{Z}_p \times \ldots \times \mathbb{Z}_p}_{s\text{-times}}$.*

(i) If p divides n, then $P_n(H,G) = 1$.

(ii) If p does not divide n, then $\frac{1}{p^s} + \frac{|H|}{|G|}\frac{p^s-1}{p^{2s-1}} \leq P_n(H,G)$.

(iii) If p does not divide n and $Z(G) = Z(H)$ then $P_n(H,G) \leq \frac{p^s+p-1}{p^{s+1}}$.

We remind the fact that $|C_H(x)| \leq \frac{|H|}{p}$ for $x \notin Z(H)$ and $\frac{|C_G(x)|}{|G|} \leq \frac{|C_H(x)|}{|H|}$ for $x \in G$.

Now, we prove Theorem A.

Proof of Theorem A. (i) implies (iii) by Lemma 3.1 for $s = 2$.

We claim that (iii) implies (i). By hypothesis the value of probability is valid for $n = 1$ so

$$\frac{p^2+p-1}{p^3} = d(G) = \frac{1}{|G|^2}\sum_{x\in G}|C_G(x)| = \frac{1}{|G|^2}\Big(\sum_{x\in Z(G)}|C_G(x)| + \sum_{x\in G-Z(G)}|C_G(x)|\Big)$$

$$\leq \frac{1}{|G|^2}\Big(|G||Z(G)| + \frac{|G|}{p}(|G| - |Z(G)|)\Big).$$

This would imply that $|G/Z(G)| \leq p^2$ and since p is the smallest prime that divides the order of G, we have $|G/Z(G)| = 1$ or p or q $(p < q)$ or p^2. If $|G|/|Z(G)| = 1$ or p or q then G is abelian and this is a contradiction. Hence $|G|/|Z(G)| = p^2$ and noncyclic, as claimed.

We claim that (ii) implies (i). G is isoclinic with an extra special p-group of order p^3 and so $G/Z(G) \simeq \mathbb{Z}_p \times \mathbb{Z}_p$.

We claim that (i) implies (ii). Assume that $G/Z(G) \simeq \mathbb{Z}_p \times \mathbb{Z}_p$. One can easily find that $|G'| = p$ (see [10]). Now, if H is an extra special p-group of order p^3 then we have $H/Z(H) \simeq \mathbb{Z}_p \times \mathbb{Z}_p$ and $Z(H) = H' \simeq \mathbb{Z}_p$. Thus, $G/Z(G) \simeq H/Z(H)$ and $G' \simeq H'$. This completes the proof.

As mentioned in Lemma 3.1, if $G/Z(G)$ is an elementary abelian p-group of rank s, then we will have a lower and upper bound for $P_n(G)$. The following theorem gives the exact formula for $P_n(G)$ when G is an extra special p-group.

Theorem 3.3. *Let G be an extra special p-group of rank $2k$. If p does not divide n, then $P_n(G) = \frac{p^{2k}+p-1}{p^{2k+1}}$. Otherwise $P_n(G) = 1$.*

Proof. If p divides n then $P_n(G) = 1$ by Lemma 3.1. Assume n is not divisible by p. We claim that $|G| = p|C_G(x)|$ for every $x \notin Z(G)$. Fix an element $x \in G$ and consider the map $\varphi : y \in G \mapsto [x, y] \in G'$. φ is a homomorphism of groups whose kernel is $C_G(x)$. Since x is not in the center, $G/C_G(x)$ has order p. Now, if $x \notin Z(G)$, then $x^n \notin Z(G)$ because $G/Z(G)$ is an elementary abelian p-group of rank $2k$ and p does not divide n. Hence,

$$P_n(G) = \frac{1}{|G|^2} \sum_{x \in G} |C_G(x^n)| = \frac{1}{|G|^2}[\sum_{x \in Z(G)} |C_G(x^n)| + \sum_{x \notin Z(G)} |C_G(x^n)|] =$$

$$\frac{1}{|G|^2}[|Z(G)||G| + (|G| - |Z(G)|)p^{2k}] = \frac{|Z(G)|}{|G|} + (\frac{1}{|G|} - \frac{|Z(G)|}{|G|^2})p^{2k} = \frac{p^{2k}+p-1}{p^{2k+1}}.$$

We illustrate Theorem 3.3 with an example.

Example 3.4. *Consider the group $G = \langle a, b, c | a^3 = b^3 = c^3 = 1, bac = ab, ca = ac, cb = bc \rangle$ One can easily see that $|G| = 27$, $|G'| = |Z(G)| = 3$ and $G/Z(G)$ is an elementary abelian group of rank 2. Thus G is an extra special 3-group of order 27. Now, using GAP, we computed $P_n(G)$ for some values of n. For instance, if $n = 1, 2, 4, 5, 7, 8, 10$ then $P_n(G) = 11/27$ and* *for $n = 3, 6, 9$ we have $P_n(G) = 1$ thus verified some results of Theorem 3.3.*

Now we recall two known results of P. Lescot in [11] and J.C. Bioch in [2]. The first states that two isoclinic groups have the same commutativity degree, so that Theorem B generalizes it. The second will play an important role in the proof of Theorem B and is appended below.

Theorem 3.5. *Let G and H be groups. Then G is isoclinic to H if and only if there is a group X with normal subgroups $M \simeq Z(G)$ and $N \simeq Z(H)$ such that $G \simeq \frac{X}{N} \sim X \sim \frac{X}{M} \simeq H$.*

Proof. Assume G and H are isoclinic. It is enough to put $X = \{(g, h) \in G \times H | \varphi(gZ(G)) = hZ(H)\}$ and the proof follows (see [2] for more details). The converse is obvious.
Now we are able to prove Theorem B.

Proof of Theorem B. Assume that G and H are isoclinic groups. By Theorem 3.5, there is a group X with normal subgroups N and M such that $G \simeq \frac{X}{N} \sim X$ and similarly $H \simeq \frac{X}{M} \sim X$. On the other hand, we have $X' \simeq (\frac{X}{N})' = \frac{X'N}{N} \simeq \frac{X'}{N \cap X'}$ so $N \cap X' = 1$. Hence, by the remark given after the proof of Lemma 2.6 we have $P_n(G) = P_n(\frac{X}{N}) = P_n(X)$. Similarly, $P_n(H) = P_n(\frac{X}{M}) = P_n(X)$ and therefore $P_n(G) = P_n(H)$ as required.

Finally, we illustrate Theorem B using the following example.

Example 3.6. *Let $G_k = \langle a, b | a^3 = b^{2k}, b^{-1}ab = a^{-1} \rangle$, where k is any positive integer which is not divisible by 3. Then we can show that $G'_k \cap Z(G_k) = 1$ and G_k is isoclinic with the symmetric group S_3. Thus, by Theorem B we have $P_n(G_k) = P_n(S_3)$ for all $n \geq 1$. We computed the probability for the group for some values k and n and they verify Theorem B. Some details are following:*

$P_1(S_3) = P_1(G_2) = 1/2$, $P_2(S_3) = P_2(G_5) = 5/6$, $P_3(S_3) = P_3(G_7) = 2/3$,

$P_4(S_3) = P_4(G_{11}) = 5/6$, $P_5(S_3) = P_5(G_{13}) = 1/2$, $P_6(S_3) = P_6(G_5) = 1$,

$P_7(S_3) = P_7(G_8) = 1/2$, $P_8(S_3) = P_8(G_{10}) = 5/6$, $P_9(S_3) = P_9(G_{13}) = 2/3$.

Acknowledgement The authors are very grateful to the referee for his/her useful comments. Some parts of this research was done during the sabbatical period of the first author at the Department of Mathematics of the Universiti Teknologi Malaysia. Special thanks go to these institutions for their hospitality.

References

[1] N. M. M. Ali and N. H. Sarmin, *On some problems in group theory of probabilistic nature*, Technical Report, Department of Mathematics, Universiti Teknologi Malaysia, 2006, Johor, Malaysia.

[2] J. C. Bioch, On n-isoclinic groups, *Indag. Math.* **38** (1976), 400–407.

[3] P. Erdös and P. Turan, On some problems of statistical group theory, *Acta Math. Acad. Sci. Hung.* **19** (1968), 413–435.

[4] A. Erfanian, P. Lescot and R. Rezaei, On the relative commutativity degree of a subgroup of a finite group, *Comm. Algebra* **35** (2007), 4183–4197.

[5] A. Erfanian and F. Russo, Probability of the mutually commuting n-tuples in some classes of compact groups, *Bull. Iran. Math. Soc.* **34** (2008), 27–38.

[6] GAP, *GAP-groups, algorithm and programming,* 2002, Software available online at http://www.gap-system.org.

[7] W. H. Gustafson, What is the probability that two groups elements commute?, *Amer. Math. Monthly* **80** (1973) 1031-1304.

[8] P.Hall, The classification of prime-power groups, *J. reine ang. Math.* **182** (1940), 130-141.

[9] N. S. Hekster, On the structure of n-isoclinism classes of groups, *J. Pure Appl. Algebra* **40** (1986), 63–65

[10] M. Isaacs, *Character Theory of Finite Groups*, Academic Press, New York, 1976.

[11] P. Lescot, Isoclinism classes and commutativity degrees of finite groups, *J. Algebra* **177** (1995), 847–869.