Properties of total restrained domination vertex critical graphs*

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Abstract

A graph G with no isolated vertex is total restrained domination vertex critical if for any vertex v of G that is not adjacent to a vertex of degree one, the total restrained domination number of $G - v$ is less than the total restrained domination number of G. We call these graphs γ_{tr}-vertex critical. If such a graph G has total restrained domination number k, then we call it $k - \gamma_{tr}$-vertex critical. In this paper, we study some properties in γ_{tr}-vertex critical graphs of minimum degree at least two.

*The research was in part supported by a grant from IPM (No. 88050037).

Keywords: Domination; Total restrained domination; Critical; Matching.
2010 Mathematical Subject Classification: 05C69.

1 Introduction

For notation and graph theory terminology we in general follow [10]. Specifically, let $G = (V, E)$ be a graph with vertex set V of order n and edge set E. We denote the degree of a vertex v in G by $d_G(v)$. For a set $S \subseteq V$, the subgraph induced by S is denoted by $G[S]$. The (open) neighborhood of vertex $v \in V$ is denoted by $N(v) = \{u \in V \mid uv \in E\}$ while $N[v] = N(v) \cup \{v\}$. For a set $S \subseteq V$, $N(S) = \bigcup_{v \in S} N(v)$ and $N[S] = N(S) \cup S$. The set S is a *dominating set* if $N[S] = V$, and a *total dominating set* if $N(S) = V$. For sets $A, B \subseteq V$, we say that A *dominates* B if $B \subseteq N[A]$ while A *totally dominates* B if $B \subseteq N(A)$. The minimum cardinality of a total dominating set is the *total domination number*, denoted $\gamma_t(G)$.

An *end-vertex* in a graph G is a vertex of degree one, and a *support vertex* is one that is adjacent to an end-vertex. A (vertex) *cut-set* in a connected graph G is a subset S of vertices such that $G - S$ is disconnected. The *connectivity* of G, written $\kappa(G)$, is the minimum size of a vertex set S such that $G - S$ is disconnected or has just one vertex. A graph G is *k-connected* if its connectivity is at least k. For a subset S of vertices, we denote by $c(G - S)$, the number of components of $G - S$. We also use $o(G - S)$ for the number of odd components of $G - S$, (see [15]).

A set of pair-wise independent edges in a graph G is called a *matching* or 1-*factor*. A matching is *perfect* if it is incident with every vertex of G. A graph G is called *factor-critical* if $G - v$ has a perfect matching for every vertex v.

Note that the removal of a vertex in a graph may decrease the domination number. A graph G is called *domination vertex critical* if $\gamma(G-v) < \gamma(G)$, for every vertex v in G. For references on domination vertex critical graphs see for example [1, 2, 3, 6, 8, 12].

Chen et al. [4], introduced the study of *total restrained domination*, which was further studied by some other peoples, for example Cyman et al. [5] and Hattingh et al. [9]. A set $S \subseteq V(G)$ is a *total restrained dominating set*, or just TRDS, if every vertex of G is adjacent to a vertex in S and every vertex in $V(G)\backslash S$ is also adjacent to a vertex in $V(G)\backslash S$. The *total restrained domination number* of G, denoted by $\gamma_{tr}(G)$, is the minimum cardinality of a TRDS of G.

Gera et al. [7], studied vertex and edge critical total restrained domination in graphs. For a graph G let $S(G)$ denotes the set of all support vertices of G. A graph G is *total restrained domination vertex critical*, or just γ_{tr}-vertex critical, if for any vertex v of $V(G) \setminus S(G)$, $\gamma_{tr}(G-v) < \gamma_{tr}(G)$. Similarly, G is *total restrained domination edge critical*, or just γ_{tr}-edge critical, if for any $e \notin E(G)$, $\gamma_{tr}(G+e) < \gamma_{tr}(G)$. They characterized all γ_{tr}-vertex critical trees, as well as those γ_{tr}-vertex critical graphs G for which $\gamma_{tr}(G) - \gamma_{tr}(G-v) = n-2$ for some $v \in V(G)$.

In this paper, we continue the study of γ_{tr}-vertex critical graphs with minimum degree at least two. We call a graph G, $k-\gamma_{tr}$-vertex critical if G is γ_{tr}-vertex critical and $\gamma_{tr}(G) = k$. We first present some basic lemmas. We then focus on $3-\gamma_{tr}$-vertex critical graphs and study connectivity and matching properties for these graphs.

All graphs in this paper are connected, and have minimum degree at lease two. Thus we henceforth do not state these properties in each result in this paper. We call a vertex v, a *total restrained domination critical* vertex, or just γ_{tr}-critical vertex, if $\gamma_{tr}(G-v) < \gamma_{tr}(G)$. Thus a graph G is γ_{tr}-vertex critical if each vertex v of G is a γ_{tr}-critical vertex. For a vertex v in a γ_{tr}-vertex critical graph G, we denote by S_v a minimum TRDS for $G - v$.

We make use of the following known results.

Theorem 1.1 (Chen, Ma and Sun [4]). *(1) For a path P_n on $n \geq 2$ vertices, $\gamma_{tr}(P_n) = n - 2\lfloor \frac{n-2}{4} \rfloor$,*
(2) For a cycle C_n on $n \geq 3$ vertices, $\gamma_{tr}(C_n) = n - 2\lfloor \frac{n}{4} \rfloor$.

Theorem 1.2 (Lovasz and Plummer, [11]). *A graph G is factor-critical if and only if $o(G-S) \leq |S|-1$ for all $S \subseteq V(G)$.*

Theorem 1.3 (Sumner, [13]). *Let G be a m-connected graph of even order and with no induced $K_{1,m+1}$. Then G has a perfect matching.*

Theorem 1.4 (Tutte, [14]). *A graph G has a perfect matching if and only if $o(G - S) \leq |S|$ for all $S \subseteq V(G)$.*

2 Some basic results

In this section we state some basic results on γ_{tr}-vertex critical graphs with minimum degree at least two. We begin with the following lemma.

Lemma 2.1. *Let G be a γ_{tr}-vertex critical graph and $v \in V(G)$. If $S_v \cap N_G(v) \neq \emptyset$, then $N_G(v) \subseteq S_v$.*

Proof. Let G be a γ_{tr}-vertex critical graph and let $S_v \cap N_G(v) \neq \emptyset$ for some vertex v. Let $u \in S_v \cap N_G(v)$. Assume to the contrary that $N_G(v) \nsubseteq S_v$. Let $w \in N_G(v) \backslash S_v$. We observe that v is totally dominated by u and v is adjacent to w. This implies that S_v is a TRDS for G, a contradiction. Hence $N_G(v) \subseteq S_v$. □

Corollary 2.2. *Let G be a γ_{tr}-vertex critical graph and $v \in V(G)$. Then,*
(1)If $S_v \cap N_G(v) \neq \emptyset$, then $\gamma_{tr}(G-v) = \gamma_{tr}(G) - 1$.
(2) If $e = uv \in E(G)$, and $N_G[u] \subseteq N_G[v]$, then $\gamma_{tr}(G-v) = \gamma_{tr}(G) - 1$.
(3) If $deg(v) = 2$, then $\gamma_{tr}(G-v) = \gamma_{tr}(G) - 1$.

Note that removing any vertex from a cycle C_n results in a path P_{n-1}. Also by Theorem 1.1, $\gamma_{tr}(P_n) = n - 2\left\lfloor\frac{n-2}{4}\right\rfloor$ and $\gamma_{tr}(C_n) = n - 2\left\lfloor\frac{n}{4}\right\rfloor$. Now it is straightforward to obtain the following.

Lemma 2.3. *A cycle C_n is γ_{tr}-vertex critical if and only if $n \equiv 3 (mod\ 4)$.*

The next result provides a forbidden condition for a graph G to be γ_{tr}-vertex critical.

Proposition 2.4. *If a graph G has non adjacent vertices u and v with $N_G(u) \subseteq N_G(v)$, then G is not γ_{tr}-vertex critical.*

Proof. Let u and v be two non adjacent vertices in a graph G with $N_G(u) \subseteq N_G(v)$. Assume to the contrary, that G is γ_{tr}-vertex critical. For u to be dominated by S_v, we have $S_v \cap N_G(v) \neq \emptyset$. By Lemma 2.1 we deduce that $N_G(v) \cup \{u\} \subseteq S_v$. Let $w \in N_G(u)$. Since $deg(u) \geq 2$, we find that $(S_v \setminus N_G[u]) \cup \{v, w\}$ is a TRDS for G of size less than $\gamma_{tr}(G)$, a contradiction. □

Corollary 2.5. *If a graph G has a vertex of degree 2 that belongs to a 4-cycle, then G is not γ_{tr}-vertex critical.*

3 $3 - \gamma_{tr}$-vertex critical graphs

In this section we present the main results of this paper. We focus on $3 - \gamma_{tr}$-vertex critical graphs with minimum degree at least two. First we study the diameter in these graphs.

Theorem 3.1. *A 3-γ_{tr}-vertex critical graph G has a diameter of at most* 3*. This bound is sharp.*

Proof. Let G be a 3-γ_{tr}-vertex critical graph with $diam(G) = d$. Let $x, y \in V(G)$ such that $d(x, y) = d$, and let P be a shortest path between x and y. For $i = 0, 1, 2, ..., d$, we let $V_i = \{v \in V(G) : d(x, v) = i\}$. Let $v \in V_1$ be the vertex on P. For x to be dominated by S_v, we have $S_v \cap V_1 \neq \emptyset$. Since G is not a complete graph, and G is a 3-γ_{tr}-vertex critical graph, we deduce that $S_v \subseteq V_1 \cup V_2$. It follows that $d \leq 3$.
To see the sharpness let H_1 be a copy of P_4 and let H_2 be a copy of $\overline{H_1}$. Let F be the graph obtained form $H_1 \cup H_2$ by adding all edges between H_1 and H_2 except for a perfect matching between corresponding vertices of H_1 and H_2, and then adding two new vertex x and y such that x is joined to every vertex in H_1 and y is joined to every vertex in H_2. It is easy to see that F is a $3 - \gamma_{tr}$-vertex critical graph of diameter 3. □

Lemma 3.2. *If G is a $3-\gamma_{tr}$-vertex critical graph and $G \neq K_3$, then $\delta(G) \geq 3$.*

Proof. Let G be a $3 - \gamma_{tr}$-vertex critical graph and $G \neq K_3$. Assume to the contrary, that $\delta(G) = 2$. Let x be a vertex with $deg(x) = 2$, and let $N(x) = \{y, z\}$. Without loss of generality assume that $deg_G(y) \geq deg_G(z)$. In order for S_y to dominate x, it follows that $z \in S_y$. Since S_y is a TRDS for $G - y$, we deduce that $S_y = \{x, z\}$. By Lemma 2.1, $N_G(y) \subseteq \{x, z\}$, and so $G = C_3$, a contradiction. Thus, $\delta(G) \geq 3$. □

Since for a vertex v in a γ_{tr}-vertex critical graph G, S_v dominates $G - v$, we obtain the following.

Observation 3.3. *Let G be a $3 - \gamma_{tr}$-vertex critical graph, and let S be a vertex cut-set with at least two vertices. Then for any vertex $v \in S$, $S_v \cap S \neq \emptyset$.*

In the next theorem we show that a $3-\gamma_{tr}$-vertex critical graph is 3-connected.

Theorem 3.4. *If G is a $3-\gamma_{tr}$-vertex critical graph, then $\kappa(G) \geq 3$.*

Proof. Let G be a $3-\gamma_{tr}$-vertex critical graph. We first show that G has no cut-vertex. Assume to the contrary, that G has a cut vertex x. Since $G[S_x]$ is connected, and $G-x$ is disconnected, then S_x does not dominate $G-x$. This is a contradiction. Thus G is 2-connected.

Now assume that $\kappa(G) = 2$. Let $S = \{x, y\}$ be a minimum vertex cut-set. By Observation 3.3, $x \in S_y$. If x is adjacent to y, then by Lemma 2.1 and Observation 3.3, $N_G(y) \subseteq S_y$, and so by Lemma 3.2, $|S_y| \geq 3$, a contradiction. Thus x is not adjacent to y. Also by Lemma 3.2 any component of $G-S$ has at least two vertices. We show that $N_G(x) \cap N_G(y) = \emptyset$. Suppose to the contrary, that $N_G(x) \cap N_G(y) \neq \emptyset$. Let $z \in N_G(x) \cap N_G(y)$. Since S_z dominates $G-z$, by Lemma 2.1 and Observation 3.3, $\{x, y\} \subseteq S_z$ and so $S_z = \{x, y\}$. This implies that x is adjacent to y, a contradiction. Thus $N_G(x) \cap N_G(y) = \emptyset$. Let G_1 and G_2 be two components of $G-S$. Since $y \in S_x$ and S_x dominates $G-x$, we may assume, without loss of generality, that y is adjacent to all vertices of G_1. But $x \in S_y$ and S_y dominates $G-y$. This implies that x is adjacent to some vertex in G_1. We conclude that $N_G(x) \cap N_G(y) \neq \emptyset$, a contradiction. Thus $\kappa(G) \geq 3$. □

We now study some properties of vertex cut-sets.

Lemma 3.5. *Let G be a $3-\gamma_{tr}$-vertex critical graph, and let S be a vertex cut-set with $|S| = 3$. If $G-S$ has a component of order 1, then $c(G-S) = 2$.*

Proof. Let G be a $3-\gamma_{tr}$-vertex critical graph, and let $S = \{x, y, z\}$ be a vertex cut-set. Let $G_1, G_2, ..., G_k$ be the components of $G - S$, where $V(G_1) = \{a\}$. From Lemma 3.2 we find that $N(a) = S$, and from Lemma 2.1 we find that $S_a \cap S = \emptyset$. Since $G[S_a]$ is connected, S_a is contained in a component $G_i \neq G_1$. But S_a dominates only G_i. It follows that $i = k = 2$. □

Lemma 3.6. *Let G be a $3 - \gamma_{tr}$-vertex critical graph, and let S be a cut-set with $|S| = 3$. If $G - S$ has a component of order 2, then $c(G - S) \leq 3$.*

Proof. Let G be a $3 - \gamma_{tr}$-vertex critical graph, and let $S = \{x, y, z\}$ be a vertex cut-set. Let $G_1, G_2, ..., G_k$ be the components of $G - S$, where $V(G_1) = \{z_1, z_2\}$. Assume to the contrary, that $k \geq 4$. From Lemma 3.5 we obtain that $|V(G_i)| \geq 2$ for $i = 2, 3, ..., k$. Moreover, z_1 is adjacent to z_2, since otherwise $G - S$ contains $k + 1$ components.

If there is a vertex $v \in V(G) \setminus S$ such that $S \subseteq N_G(v)$, then since $|S_v| = 2$, by Lemma 2.1 we obtain that $S_v \subseteq V(G_j)$ for some $j \in \{1, 2, ..., k\}$. But then S_v does not dominate $G - v$, a contradiction. We deduce that for any $v \in V(G) \setminus S$, $S \not\subseteq N_G(v)$. Since S_{z_1} dominate z_2, we obtain that $S_{z_1} \cap S \neq \emptyset$. Similarly, $S_{z_2} \cap S \neq \emptyset$. Thus using Lemma 3.2, we find that $|S_{z_1} \cap S| = |S_{z_2} \cap S| = 2$. Without loss of generality assume that $N_G(z_1) = \{z_2, x, y\}$ and $N_G(z_2) = \{z_1, y, z\}$. By Lemma 2.1, we find that $x \in S_{z_2}$. Further, $S_{z_2} \cap S = \{x\}$. We conclude that x is adjacent to all of the vertices of at least two components among $G_2, G_3 ...,$ and G_k. Similarly, $z \in S_{z_1}$ and z is adjacent to all of the vertices of at least two components among $G_2, G_3, ...,$ and G_k. Since $k \geq 4$, we obtain that there is a component G_r for some $r \in \{2, 3, ..., k\}$ such that x and z are adjacent to all vertices of G_r. Let $u_{r_1} \in V(G_r)$. It follows that $y \in S_{u_{r_1}}$. Since $|V(G_r)| \geq 2$, there is a vertex $u_{r_2} \in V(G_r) \setminus \{u_{r_1}\}$ such that $y \in N_G(u_{r_2})$. Thus $S \subseteq N_G(u_{r_2})$, a contradiction. □

Theorem 3.7. *Let G be a $3 - \gamma_{tr}$-vertex critical graph, and let S be a vertex cut-set such that any component of $G - S$ has at* *least three vertices. If $c(G-S) \geq 3$, then $|S| \geq 4$.*

Proof. Let G be a $3-\gamma_{tr}$-vertex critical graph and let S be a vertex cut-set such that $c(G-S) \geq 3$ and any component of $G-S$ has at least three vertices. Let $G_1, G_2, ..., G_k$ be the components of $G-S$. Assume to the contrary, that $|S| \leq 3$. By Theorem 3.4, we obtain that $|S| = 3$. Let $S = \{x, y, z\}$. If $G[S]$ is connected, then there is a vertex $w \in S$ such that w is adjacent to the two vertices of $S \backslash \{w\}$. Then by Lemmas 2.1, 3.2, and Observation 3.3, we obtain that $|S_w| \geq 3$, a contradiction. Thus $G[S]$ is disconnected. According to Observation 3.3 we may assume that $y \in S_x$. Since $G[S]$ is disconnected, we may assume that $S_x = \{y, y_1\}$ where y_1 belongs to a component of $G-S$. Since $c(G-S) \geq 3$, y is adjacent to all of the vertices of at least two components of $G-S$. Without loss of generality, assume that $z \in S_y$. As before, we find that z is adjacent to all of the vertices of at least two components of $G-S$. Since $k \geq 3$, we obtain that there is a component G_j for some $j \in \{1, 2, ..., k\}$ such that y and z are adjacent to all vertices of G_j. Thus $\{y, z\} \subseteq N_G(u)$ for any vertex $u \in V(G_j)$. Now we consider S_{u_1}, where $u_1 \in V(G_j)$. By Lemmas 2.1, and 3.2 we find that $x \in S_{u_1}$. Since $|V(G_j)| \geq 3$, we obtain that $N_G(x) \cap V(G_j) \neq \emptyset$, and thus $N_G(x) \cap N_G(y) \cap N_G(z) \neq \emptyset$. This produce a contradiction. □

In the rest of the paper we study matching properties in $3-\gamma_{tr}$-vertex critical graphs. By Theorems 3.4 and 1.3, we obtain the following:
(1) Any $3-\gamma_{tr}$-vertex critical claw-free graph of even order has a perfect matching,
(2) Any $3-\gamma_{tr}$-vertex critical $K_{1,4}$-free graph of even order has a perfect matching.

Theorem 3.8. *Any $3-\gamma_{tr}$-vertex critical claw-free graph of odd order is factor-critical.*

Proof. Let G be a $3 - \gamma_{tr}$-vertex critical claw-free graph of odd order. If G is not factor-critical, then by Theorem 1.2, there is a subset $S \subseteq V(G)$ such that $o(G - S) \geq |S|$. Since G is of odd order, we obtain $o(G - S) \geq |S| + 1$. By Theorem 3.4, $o(G - S) \geq 4$. Since $|S_v| = 2$ for any vertex v, we observe that G has a $K_{1,3}$ as an induced subgraph, a contradiction. □

Similarly the following is verified.

Theorem 3.9. *Any* $3 - \gamma_{tr}$*-vertex critical* $K_{1,4}$*-free graph of odd order is factor-critical.*

Now we study matching properties for $K_{1,5}$-free graphs.

Theorem 3.10. *Any* $3 - \gamma_{tr}$*-vertex critical* $K_{1,5}$*-free graph of even order has a perfect matching.*

Proof. Let G be a $3 - \gamma_{tr}$-vertex critical $K_{1,5}$-free graph of even order. Suppose to the contrary that G has no perfect matching. By Theorem 1.4, there is a subset $S \subseteq V(G)$ such that $o(G - S) \geq |S| + 1$. Since G is of even order, we conclude that $o(G - S) \geq |S| + 2$. Since by Theorem 3.4, $|S| \geq 3$, we find that $o(G - S) \geq 5$. But then Lemmas 3.5, 3.6 lead that any component of $G - S$ has at least three vertices. Now Theorem 3.7 implies that $|S| \geq 4$. We deduce that $o(G - S) \geq 6$. Let $G_1, G_2, ..., G_k$ be the odd components of $G - S$, where $k \geq 6$. We proceed with Fact 1.

Fact 1. $\Delta(G[S]) \leq 2$.

To see this let $x \in S$. By Observation 3.3, $S_x \cap S \neq \emptyset$. Let $y \in S_x \cap S$. If $S_x \not\subseteq S$, then y dominates the vertices of at least five components of $G - S$. This is a contradiction, since G is $K_{1,5}$-free. Thus, $S_x \subseteq S$. Let $S_x = \{y, z\}$. Assume that $c(G - S) \geq 8$. Since G is $K_{1,5}$-free, we obtain that y can not dominate the vertices of at least five components of $G - S$, and similarly z can not dominate the vertices of at least five components of $G - S$. Thus $c(G - S) = 8$, and we may assume without loss of generality that y dominates $V(G_1) \cup V(G_2) \cup V(G_3) \cup V(G_4)$, and z dominates $V(G_5) \cup V(G_6) \cup V(G_7) \cup V(G_8)$. Let $y_i \in N_G(y) \cap V(G_i)$ for $i = 1, 2, 3, 4$. Then $y_1, ..., y_4, z$ form a $K_{1,5}$, a contradiction. We deduce that, $c(G - S) \leq 7$. Now we observe that

$$2 + |S| \leq o(G - S) \leq c(G - S) \leq 7.$$

This implies that $|S| \leq 5$. But $N_G(x) \cap \{y, z\} = \emptyset$. So by Lemma 2.1 and Observation 3.3, x is adjacent to at most two vertices of S. So $deg_{G[S]}(x) \leq 2$, and so $\Delta(G[S]) \leq 2$. This completes the proof of Fact 1.

Let $v_1 \in V(G_1)$. Since $o(G - S) \geq 6$ and G is $K_{1,5}$-free, we find that $S_{v_1} \subseteq S$. If $|S| = 5$, then since S_{v_1} dominates S, we find that there is a vertex $w_1 \in S_{v_1}$ such that $deg_{G[S]}(w_1) \geq 3$, contradicting Fact 1. So suppose that $|S| = 4$. Let $S = \{x, y, z, u\}$. By Fact 1 we may assume that $N_{G[S]}(y) = \{z, u\}$. By Lemmas 2.1 and 3.2 we obtain that $S_y \cap S = \{x\}$. This produces a $K_{1,5}$ centered at x, a contradiction. □

Theorem 3.11. *Any $3 - \gamma_{tr}$-vertex critical $K_{1,5}$-free graph of odd order is factor-critical.*

Proof. Let G be a $3 - \gamma_{tr}$-vertex critical $K_{1,5}$-free graph of odd order. Suppose to the contrary that G is not factor-critical. By Theorem 1.2, there is a subset $S \subseteq V(G)$ such that $o(G - S) \geq |S|$. Since G is of odd order, we conclude that $o(G-S) \geq |S|+1$. By Theorem 3.4, $|S| \geq 3$, and so $o(G - S) \geq 4$. It follows from Lemmas 3.5 and 3.6 that any component of $G - S$ has at least three vertices. Now Theorem 3.7 implies that $|S| \geq 4$. We deduce that $o(G - S) \geq 5$. Let G_1, G_2, G_3, G_4, G_5 be five odd components of $G - S$. Let $x \in S$. By Observation 3.3, $S_x \cap S \neq \emptyset$. Let $y \in S_x \cap S$. If $S_x \nsubseteq S$, then y dominates the vertices of at least four components of $G-S$. We can then choose a vertex from each of these components, and together with the vertex in $S_x \setminus \{y\}$, form a $K_{1,5}$, which is a contradiction. So $S_x \subseteq S$. Similar to the proof of Theorem 3.10 we observe that $c(G-S) \leq 7$. Now we observe that

$$1 + |S| \leq o(G-S) \leq c(G-S) \leq 7.$$

This implies that $|S| \leq 6$. For $4 \leq |S| \leq 5$, with the same manner as in the proof of Theorem 3.10, we produce a contradiction. So we suppose that $|S| = 6$. Let $S = \{v_1, v_2, ..., v_6\}$.

If there is a vertex $w \in S$ such that $deg_{G[S]}(w) \geq 4$, then by Lemma 2.1 and Observation 3.3, either $|S_w| \geq 3$, or G has a $K_{1,5}$, both of which is a contradiction. So $\Delta(G[S]) \leq 3$. This implies that $G[S]$ has at most 9 edges.

For any vertex $u \in V(G)$, $S_u \subseteq S$ and $|S_u| = 2$. But $|V(G)| = |S| + |V(G) - S| \geq 5 + 5(3) = 20$. Since there are at most nine pairs of adjacent vertices in S, we deduce that there are two vertices u_1 and u_2 in G such that $S_{u_1} = S_{u_2}$. This means that S_{u_1} is a TRDS for G, a contradiction. □

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