Chordal Graphs are Fully Orientable

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Abstract

Suppose that D is an acyclic orientation of a graph G. An arc of D is called *dependent* if its reversal creates a directed cycle. Let $d_{\min}(G)$ ($d_{\max}(G)$) denote the minimum (maximum) of the number of dependent arcs over all acyclic orientations of G. We call G *fully orientable* if G has an acyclic orientation with exactly d dependent arcs for every d satisfying $d_{\min}(G) \leqslant d \leqslant d_{\max}(G)$. A graph G is called *chordal* if every cycle in G of length at least four has a chord. We show that all chordal graphs are fully orientable.

Keyword: acyclic orientation; full orientability; simplicial vertex; chordal graph.

1 Introduction

Let G be a finite graph without multiple edges or loops. We use $|G|$ and $\|G\|$ to denote the number of vertices and the number of edges of G, respectively. An orientation D of a graph G is obtained by assigning a fixed direction,

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either $x \to y$ or $y \to x$, on every edge xy of G. The original undirected graph is called the *underlying* graph of any such orientation.

An orientation D is called *acyclic* if there does not exist any directed cycle. A directed graph having no directed cycle is commonly known as a *directed acyclic graph*, or DAG for short. DAGs provide frequently used data structures in computer science for encoding dependencies. An equivalent way of describing a DAG is the existence of a particular type of ordering of the vertices called a *topological ordering.* A topological ordering of a directed graph G is an ordering of its vertices as $v_1, v_2, \ldots, v_{|G|}$ such that for every arc $v_i \to v_j$, we have $i < j$. The reader who is interested in knowing more about DAGs is referred to reference [1] which supplies a wealth of information on DAGs.

Suppose that D is an acyclic orientation of G. An arc of D, or its underlying edge, is called *dependent* (in D) if its reversal creates a directed cycle in the resulted orientation. Note that $u \to v$ is a dependent arc if and only if there exists a directed walk of length at least two from u to v. Let $d(D)$ denote the number of dependent arcs in D. Let $d_{\min}(G)$ and $d_{\max}(G)$ be, respectively, the minimum and the maximum values of $d(D)$ over all acyclic orientations D of G. It is known ([8]) that $d_{\max}(G) = \|G\| - |G| + c$ for a graph G having c connected components.

An interpolation question asks whether G has an acyclic orientation with exactly d dependent arcs for every d satisfying $d_{\min}(G) \leqslant d \leqslant d_{\max}(G)$. Following West ([20]), we call G *fully orientable* if its interpolation question has an affirmative answer. Note that forests are trivially fully orientable. It is also easy to see ([13]) that a graph is fully orientable if all of its connected components are. West [20] showed that complete bipartite graphs are fully orientable. Let $\chi(G)$ denote the *chromatic number* of G, i.e., the least number of colors to color the vertices of G so that adjacent vertices receive different colors. Let $g(G)$ denote the *girth* of G, i.e., the length of a shortest cycle of G if there is any, and ∞ if G is a forest. Fisher, Fraughnaugh, Langley, and West [8] showed that G is fully orientable if $\chi(G) < g(G)$. They also proved that $d_{\min}(G) = 0$ when $\chi(G) < g(G)$. In fact, $d_{\min}(G) =$

0 if and only if G is a *cover graph*, i.e., the underlying graph of the Hasse diagram of a partially ordered set. ([16], Fact 1.1).

A number of graph classes have been shown to consist of fully orientable graphs in recent years. Here, we give a brief summary of some results.

A graph is called *2-degenerate* if each of its subgraphs contains a vertex of degree at most two. Lai, Chang, and Lih [12] have established the full orientability of 2-degenerate graphs that generalizes a previous result for outerplanar graphs ([15]). A *Halin graph* is a plane graph obtained by drawing a tree without vertices of degree two in the plane, and then drawing a cycle through all leaves in the plane. A *subdivision of an edge* of a graph is obtained by replacing that edge by a path consisting of new internal vertices. A *subdivision* of a graph is obtained through a sequence of subdivisions of edges. Lai and Lih [13] showed that subdivisions of Halin graphs and graphs with maximum degree at most three are fully orientable. In [15], Lai, Lih, and Tong proved that a graph G is fully orientable if $d_{\min}(G) \leqslant 1$. This generalizes the results in [8] mentioned before.

The main purpose of this paper is to show that the class of fully orientable graphs includes the important class of chordal graphs.

Let C be a cycle of a graph G. An edge e of G is called a *chord* of C if the two endpoints of e are non-consecutive vertices on C. A graph G is called *chordal* if each cycle in G of length at least four possesses a chord. Chordal graphs are variously known as *triangulated graphs* [2], *rigid-circuit graphs* [6], and *monotone transitive graphs* [17] in the literature. Chordal graphs can be characterized in a number of different ways. (For instance, [3], [6], [9], [10], and [17]).

Chordal graphs have applications in areas such as the solution of sparse symmetric systems of linear equations [18], data-base management systems [19], knowledge based systems [7], and computer vision [5]. The importance of chordal graphs primarily lies in the phenomenon that many NP-complete problems can be solved by polynomial-time algorithms for chordal graphs.

We need the following characterization of chordal graphs to prove our main result. A complete subgraph of a graph G is called a *clique* of G.

A vertex v of a graph G is said to be *simplicial* if v together with all its adjacent vertices induce a clique in G. An ordering $v_1, v_2, \ldots, v_n$ of all the vertices of G forms a *perfect elimination ordering* of G if each v_i, $1 \leqslant i \leqslant n$, is simplicial in the subgraph induced by $v_i, v_{i+1}, \ldots, v_n$.

Theorem 1 [18] *A graph G is a chordal graph if and only if it has a perfect elimination ordering.*

The reader is referred to Golumbic's classic [11] for more information on chordal graphs.

2 Results

Up to the naming of vertices, any acyclic orientation D of K_n produces the topological ordering $v_1, \ldots, v_n$ such that the arc $v_i \to v_j$ belongs to D if and only if $i < j$. Moreover, $v_i \to v_j$ is a dependent arc in D if and only if $j - i > 1$. A vertex is called a *source* (or *sink*) if it has no ingoing (or outgoing) arc. The following observation is very useful in the sequel. Let D be an acyclic orientation of the complete graph K_n where $n \geqslant 3$. The number of dependent arcs in D incident to a vertex v is $n-2$ if v is the source or the sink of D and is $n-3$ otherwise.

In this section, we assume that the clique Q of a graph G has q vertices. Let G' be the graph obtained from G by adding a new vertex v adjacent to all vertices of Q. We see that $d_{\max}(G') = \|G'\| - |G'| + 1 = (\|G\| + q) - (|G| + 1) + 1 = (\|G\| - |G| + 1) + q - 1 = d_{\max}(G) + q - 1$. Furthermore, we have the following.

Lemma 2 **(1)** *If G has an acyclic orientation D with $d(D) = d$, then G' has an acyclic orientation D' with $d(D') = d + q - 1$.*

(2) *We have $d_{\min}(G')$ equal to $d_{\min}(G) + q - 2$ or $d_{\min}(G) + q - 1$.*

Proof. The statements hold trivially when $q = 1$. Assume $q \geqslant 2$.

(1) Let D' be the extension of D into G' by making v into a source. Clearly, D' is an acyclic orientation. Let $v_1, \ldots, v_q$ be the topological ordering of vertices of Q with respect to D. Suppose that $x \to y$ is a dependent arc in D'.

Case 1. If this arc is in D, then it is already dependent in D since v is a source in D'.

Case 2. If this arc is $v \to v_1$, then, for some $2 \leqslant i \leqslant q$, a directed path $v \to v_i \to z_1 \to \cdots \to z_i \to v_1$ of length at least three would be produced such that $z_1, \ldots, z_i$ are all vertices in G. It follows that $v_1 \to v_i \to z_1 \to \cdots \to z_i \to v_1$ is a directed cycle in D, contradicting to the acyclicity of D.

Case 3. If this arc is $v \to v_k$ for $2 \leqslant k \leqslant q$, then it is a dependent arc in D' since $v \to v_{k-1} \to v_k$ is a directed path of length two.

Therefore, $d(D') = d + q - 1$.

(2) By statement **(1)**, we have $d_{\min}(G') \leqslant d_{\min}(G) + q - 1$. Let D' be an acyclic orientation of G' with $d(D') = d_{\min}(G')$. Since the subgraph induced by Q and $\{v\}$ is a clique of order $q + 1$, the number of dependent arcs in D' incident to v is $q - 1$ or $q - 2$. Let D be the restriction of D' to $V(G)$. Then we have $d_{\min}(G') = d(D') \geqslant d(D) + q - 2 \geqslant d_{\min}(G) + q - 2$. ∎

Since every number d satisfying $d_{\min}(G) + q - 1 \leqslant d \leqslant d_{\max}(G) + q - 1$ is achievable as $d(D')$ for some acyclic orientation D' of G' by **(1)**, the following is a consequence of **(2)**.

Corollary 3 *If G is fully orientable, so is G'.*

The above theorem amounts to preserving full orientability by the addition of a simplicial vertex. Hence, by successively applying it to the reverse of a perfect elimination ordering of a connected chordal graph, every such graph is fully orientable. Our main result thus follows.

Theorem 4 *If G is a chordal graph, then G is fully orientable.*

Remark. Adding a simplicial vertex may not increase the maximum and the minimum numbers of dependent edges by the same amount. For instance, any acyclic orientation of a triangle gives rise to exactly one dependent arc. However, the graph K_4 minus an edge, which is obtained from a triangle by adding a simplicial vertex, has minimum value one and maximum value two.

Now we want to give a characterization to tell which case in **(2)** of Lemma 2 will happen. A dependent arc in Q is said to be *non-trivial* with respect to the acyclic orientation D if it is dependent in D but not in the induced orientation $D[Q]$. Equivalently, any directed cycle obtained by reversing that arc contains vertices not in Q.

Lemma 5 *Assume $q \geqslant 2$. There is an acyclic orientation D of G such that Q has a dependent arc that is non-trivial with respect to D if and only if D can be extended to an acyclic orientation D' of G' with $d(D') = d(D)+q-2$.*

Proof. ($\Rightarrow$) Assume that D is an acyclic orientation of G such that Q has a dependent arc that is non-trivial with respect to D. Let $v_1, \ldots, v_q$ be the topological ordering of the vertices of Q with respect to D. The arcs in the set $\{v_i \rightarrow v_j \mid j - i > 1\}$ are dependent arcs that are not non-trivial with respect to D.

By our assumption, we can find $1 \leqslant k < q$ such that $v_k \rightarrow v_{k+1}$ is a dependent arc in D. We obtain an extension D' of D into G' by defining $v_a \rightarrow v$ for all $a \leqslant k$ and $v \rightarrow v_b$ for all $b > k$. This D' must be acyclic for otherwise a directed path would be produced in D, contradicting the acyclicity of D. The set $\{vv_r \mid r \neq k, k+1\}$ gives rise to a set of dependent arcs in D' and both vv_k and vv_{k+1} are not dependent in D'. Moreover, an edge of G is dependent in D if and only if it is dependent in D'. Therefore, $d(D') = d(D) + q - 2$.

($\Leftarrow$) Assume that D can be extended to an acyclic orientation D' of G' with $d(D') = d(D) + q - 2$. If the vertex v is a source or a sink, then $d(D') = d(D) + q - 1$, contradicting our assumption. Without loss of generality, we may suppose that, for some $1 \leqslant k < q$, $v_a \rightarrow v$ for all $1 \leqslant a \leqslant k$ and $v \rightarrow v_{k+1}$. The acyclicity of D' implies that $v \rightarrow v_b$ for all $b > k$. Hence, the arc $v_k \rightarrow v_{k+1}$ is dependent in D' for $v_k \rightarrow v \rightarrow v_{k+1}$ is a directed path of length two. Since the $q-2$ arcs vv_r $(r \neq k, k+1)$ incident to v are already dependent in D', it forces $v_k \to v_{k+1}$ to be a dependent arc in D. Therefore, $v_k \to v_{k+1}$ is non-trivial with respect to D. ∎

Corollary 6 *Assume $q \geqslant 2$. There is an acyclic orientation D of G such that $d(D) = d_{\min}(G)$ and Q has a dependent arc that is non-trivial with respect to D if and only if $d_{\min}(G') = d_{\min}(G) + q - 2$.*

Remark. For the complete graph K_n on n vertices, it is well-known ([20]) that $d_{\min}(K_n) = d_{\max}(K_n) = (n-1)(n-2)/2$. Hence, the condition in Theorem 5 and Corollary 6 that Q has a dependent arc that is non-trivial with respect to D can be replaced by the condition that Q has more than $(q-1)(q-2)/2$ arcs that are dependent in D.

In contrast to the addition of a simplicial vertex, the deletion of a simplicial vertex may destroy full orientability. The following example attests to this possibility.

Let $K_{r(n)}$ denote the complete r-partite graph each of whose partite sets has n vertices. It is proved in [4] that $K_{r(n)}$ is not fully orientable when $r \geqslant 3$ and $n \geqslant 2$. Any acyclic orientation of $K_{3(2)}$ has 4, 6, or 7 dependent arcs. Figure 1 shows an acyclic orientation of $K_{3(2)}$ with 6 dependent arcs. Two dependent arcs appear in the innermost triangle 146. Let K' be the graph obtained from $K_{3(2)}$ by adding a vertex v adjacent to vertices 1, 4, and 6. By Lemma 2, there exist acyclic orientations of K' with 6, 8, or 9 dependent arcs. Actually, $d_{\max}(K') = 9$. Applying Lemma 5 to Figure 1, we obtain an acyclic orientation of K' with 7 dependent arcs. Any acyclic orientation of $K_{3(2)}$ with 4 dependent arcs cannot have two dependent arcs from the triangle 146 since there are three triangles each of which is edge-disjoint from the triangle 146 and we know that every triangle must have one dependent arc. It follows from Corollary 6 that $d_{\min}(K') = 6$. Hence, K' is fully orientable. The deletion of the simplicial vertex v from K' produces $K_{3(2)}$ that is not fully orientable.

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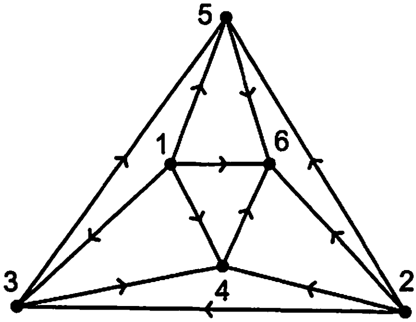

Figure 1: An acyclic orientation of $K_{3(2)}$ with 6 dependent arcs.

References

[1] J. Bang-Jensen and G. Gutin, *Digraphs: Theory, Algorithms and Applications*, 2nd ed., Springer-Verlag, London, 2009.

[2] C. Berge, Some classes of perfect graphs, in *Graph Theory and Theoretical Physics*, (F. Harary, ed.), pp. 155–166, Academic Press, New York, 1967.

[3] P. Buneman, A characterization of rigid circuit graphs, *Discrete Math.* **9** (1974) 205–212.

[4] G. J. Chang, C.-Y. Lin, and L.-D. Tong, Independent arcs of acyclic orientations of complete r-partite graphs, *Discrete Math.* **309** (2009) 4280–4286.

[5] F. R. K. Chung and D. Mumford, Chordal completions of planar graphs, *J. Combin. Theory Ser. B* **31** (1994) 96–106.

[6] G. Dirac, On rigid circuit graphs, *Abh. Math. Sem. Univ. Hamburg* **25** (1961) 71–76.

[7] R. E. England, J. R. S. Blair, and M. G. Thomason, Independent computations in a probabilistic knowledge-based system, Technical Report

CS-91-128, Department of Computer Science, The University of Tennessee, Knoxville, Tennessee, 1991.

[8] D. C. Fisher, K. Fraughnaugh, L. Langley, and D. B. West, The number of dependent arcs in an acyclic orientation, *J. Combin. Theory Ser. B* **71** (1997) 73–78.

[9] D. R. Fulkerson and O. A. Gross, Incidence matrices and interval graphs, *Pacific J. Math.* **15** (1965) 835–855.

[10] F. Gavril, The intersection graphs of subtrees in trees are exactly the chordal graphs, *J. Combin. Theory Ser. B* **16** (1974) 47–56.

[11] M. C. Golumbic, *Algorithmic Graph Theory and Perfect Graphs*, 2nd ed., North-Holland, Amsterdam, 2004.

[12] H.-H. Lai, G. J. Chang, and K.-W. Lih, On fully orientability of 2-degenerate graphs, *Inform. Process. Lett.* **105** (2008) 177–181.

[13] H.-H. Lai and K.-W. Lih, On preserving full orientability of graphs, *European J. Combin.* **31** (2010) 598–607.

[14] H.-H. Lai, K.-W. Lih, and L.-D. Tong, Full orientability of graphs with at most one dependent arc, *Discrete Appl. Math.* **157** (2009) 2969–2972.

[15] K.-W. Lih, C.-Y. Lin, and L.-D. Tong, On an interpolation property of outerplanar graphs, *Discrete Appl. Math.* **154** (2006) 166–172.

[16] V. Rödl and L. Thoma, On cover graphs and dependent arcs in acyclic orientations, *Combin. Probab. Comput.* 14 (2005) 585–617.

[17] D. J. Rose, Triangulated graphs and the elimination process, *J. Math. Analys. Appl.* **32** (1970) 597–609.

[18] D. J. Rose, A graph-theoretic study of the numerical solution of sparse positive definite systems of linear equations, in *Graph Theory and Computing*, (R. C. Read, ed.), pp. 183–217, Academic Press, New York, 1972.

[19] R. E. Tarjan and M. Yannakakis, Simple linear-time algorithms to test chordality of graphs, test acyclicity of hypergraphs, and selectively reduce acyclic hypergraphs, *SIAM J. Comput.* **13** (1984) 566–579.

[20] D. B. West, Acyclic orientations of complete bipartite graphs, *Discrete Math.* **138** (1995) 393–396.