FIBONACCI NUMBERS AND POSITIVE BRAIDS

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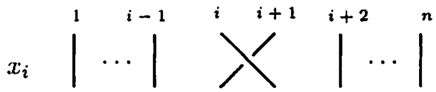
ABSTRACT. The paper contains enumerative combinatorics for positive braids, square free braids, and simple braids, emphasizing connections with classical Fibonacci sequence.

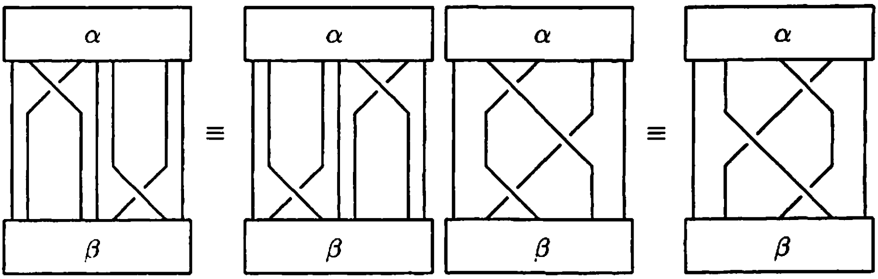
1. Introduction

The classical Fibonacci sequence, $(F_n)_{n\geq 0}$: $0, 1, 1, 2, 3, 5, \ldots$ appears from time to time in enumerative questions related to Artin braids [2], the geometrical analogue of permutations. The *positive n-braids* can be defined as words in the alphabet $\{x_1, x_2, \ldots, x_{n-1}\}$:



in which we identify two words obtained using finitely many changes of type

$$\alpha(x_i x_j)\beta \longleftrightarrow \alpha(x_j x_i)\beta \qquad (\text{for } |i-j|\geq 2)$$
$$\alpha(x_i x_{i+1} x_i)\beta \longleftrightarrow \alpha(x_{i+1} x_i x_{i+1})\beta \qquad (\text{for } i = 1, 2, \ldots, n-2):$$



A central role is played by the *Garside braid* [11]: $\Delta_n = x_1(x_2x_1)..(x_{n-1}..x_1)$.

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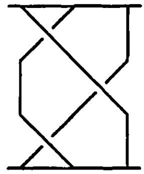
Δ_3 : 

We will denote by $\mathcal{MB}_n$ the set of positive n-braids, by $\mathcal{MB}_n^+$ the set of positive n-braids not containing Δ_n as a subword, and by $\text{Div}(\Delta_n)$ the set of positive n-braids which are subwords of the Garside braid:

$$\text{Div}(\Delta_n) = \{\omega \in \mathcal{MB}_n | \text{ there exist } \alpha, \beta \in \mathcal{MB}_n \text{ such that } \Delta_n = \alpha\omega\beta\}.$$

A square free positive braid is a positive braid not containing a subword of the form x_i^k, where $k \geq 2$: for instance, $\beta = x_1x_2x_1x_2$ is not square free, because $\beta = x_1^2x_2x_1$, but $\Delta_4 = x_1x_2x_1x_3x_2x_1 = (x_2x_1x_3)^2$ is square free. A well known result ([11], Theorem 9, and [9], Lemma 5.4) says that the set of square free positive n-braids is exactly the set $\text{Div}(\Delta_n)$. There are many ways to define the set of *simple n-braids*, $\mathcal{SB}_n \subset \text{Div}(\Delta_n)$ (see e.g. [4]). One definition is:

Definition 1.1. A *simple braid* is a positive braid $\beta \in \mathcal{MB}_n$ which contains a letter x_i at most once.

In our first computations Fibonacci numbers (F_k) appear; b_k and b_k^+ represents the number of braids of length k in $\mathcal{MB}_3$ and $\mathcal{MB}_3^+$ respectively.

Theorem 1.2. *The generating function of $\mathcal{MB}_3$ is*

$$G_{\mathcal{MB}_3}(t) = \sum_{k\geq 0} b_k t^k = 1 + 2t + 4t^2 + 7t^3 + 12t^4 + 20t^5 + \dots$$

where $b_k = F_{k+3} - 1$, $k \geq 0$.

Theorem 1.3. *The generating function of $\mathcal{MB}_3^+$ is*

$$G_{\mathcal{MB}_3^+}(t) = \sum_{k\geq 0} b_k^+ t^k = 1 + 2t + 4t^2 + 6t^3 + 10t^4 + 16t^5 + \dots$$

where $b_k^+ = 2F_{k+1}$, $k \geq 1$.

Theorem 1.4. *The number of simple braids in $\mathcal{SB}_n$ is F_{2n-1}.*

The paper contains some other combinatorial results related to positive braids.

In the next section the proofs of the first two theorems are given.

In the third section the generating polynomial of the square free braids is computed (Proposition 3.1) and the recurrence relation for its coefficients are presented (Proposition 3.2).

A proof of Theorem 1.4, the generating polynomial for simple braids, and some properties of its coefficients (Proposition 4.1) are contained in section 4.

The fifth section contains enumerative results related to the set of conjugacy classes of simple braids (Proposition 5.1).

Connections between multiple Fibonacci-type recurrence [13] and Jones polynomial and Conway-Alexander polynomial for closed braids are presented in [8] and [3]. The close relations between simple braids in $\mathcal{SB}_n$ and the corresponding *simple permutations* in the symmetric group Σ_n and also the simple parts of the Cayley graph and of the permutahedron, the parts corresponding to the simple braids and simple permutations, are studied in [5].

2. Positive braids

The generating function for positive braids was computed by P. Deligne [10] using invariants of Coxeter groups. A direct computation for 3-braids was done by P. Xu [16] and an inductive algorithm for $G_{\mathcal{MB}_n}(t)$ and some generalizations are contained in Z. Iqbal [12]. Using any of these references, we have

Corollary 2.1. ([10], [16], [12]) The generating function for positive 3-braids is given by

$$G_{\mathcal{MB}_3}(t) = \frac{1}{(1-t)(1-t-t^2)}.$$

Proof of Theorem 1.2. The expansion in simple parts $G_{\mathcal{MB}_3}(t) = \frac{2+t}{1-t-t^2} - \frac{1}{1-t}$ and the equality $(1-t-t^2)^{-1} = \sum\limits_{k\geq 0} F_{k+1}t^k$ gives the result:

$$b_k = (2F_{k+1}+F_k)-1 = (F_{k+1}+F_{k+2})-1 = F_{k+3}-1. \qquad \square$$

Proof of Theorem 1.3. Every positive braid β can be written in a unique way as a product $\beta = \Delta_n^k\beta^+$ with $\beta^+ \in \mathcal{MB}_n^+$ (see [11]), therefore the decomposition $\mathcal{MB}_3 = \coprod\limits_{k\geq 0} \Delta_3^k \cdot \mathcal{MB}_3^+$ implies:

$$G_{\mathcal{MB}_3^+}(t) = (1+t^3+t^6+\ldots)^{-1}\cdot G_{\mathcal{MB}_3}(t) = \frac{1+t+t^2}{1-t-t^2} = \sum_{k\geq 1} b_k^+ t^k.$$

Simple computations show that $b_0^+ = 1$, $b_1^+ = 2 = 2F_2$, $b_2^+ = 4 = 2F_3$, and, for $k \geq 3$, $b_k^+ - b_{k-1}^+ - b_{k-2}^+ = 0$, hence the result. $\square$

For a universal upper bound of the growing type of $\mathcal{MB}_n$, see [7].

3. Square free braids

To represent an element of $\mathrm{Div}(\Delta_n)$, i.e. a positive square free braid, we choose the canonical form given by the smallest elements in the length-lexicographic order (see [6], [1]):

$$\beta_{K,J} = \beta_{k_1,j_1}\beta_{k_2,j_2}\dots\beta_{k_s,j_s}$$

where $\beta_{k,j} = x_k x_{k-1}\dots x_{j+1}x_j$, $0 \le s \le n-1$, $1 \le k_1 < k_2 \dots < k_s \le n-1$, and $j_h \le k_h$ for $h = 1, \dots, s$ (the case $s = 0$ corresponds to the unit $\beta = 1$). For simplicity, we will write Div_n for $\mathrm{Div}(\Delta_n)$. Let us denote by $d_{n,i}$ the number of divisors of Δ_n of length i and by $G_{Div_n}(t)$ the generating polynomial of the square free n-braids.

Proposition 3.1. $G_{Div_n}(t) = \sum_{i=0}^{n(n-1)/2} d_{n,i}t^i$
$= (1+t)(1+t+t^2)\dots(1+t+t^2+\dots+t^{n-1}).$

Proof. We start the induction with $n = 2$: $\mathrm{Div}_2 = \{1, x_1\}$ and $G_{Div_2}(t) = 1+t$. The canonical form of square free braids shows that the map

$$f : \mathrm{Div}_{n-1} \times \{1, \beta_{n-1,1}, \beta_{n-1,2}, \dots, \beta_{n-1,n-1}\} \longrightarrow \mathrm{Div}_n,$$

defined by $f(\omega, 1) = w$, $f(\omega, \beta_{n,k}) = w \cdot \beta_{n,k}$, is a bijection. The generating polynomial of the set $\{1, \beta_{n-1,k}\}_{k=1,\dots,n-1}$ is $1+t+\dots+t^{n-1}$, so $G_{Div_n}(t) = G_{Div_{n-1}}(t) \cdot (1+t+\dots+t^{n-1})$. □

Corollary 3.2. *The sequence* $(d_{n,i})_{i=0,\dots,\frac{n(n-1)}{2}}$ *is symmetric and unimodal and satisfies the following recurrence relation:*

a) $d_{1,0} = 1$, $d_{1,i} = 0$ *if* $i \neq 0$;

b) $d_{n+1,i} = d_{n,i} + d_{n,i-1} + \dots + d_{n,i-n}$.

Example 3.3. First values of the sequence $d_{n,i}$ (on the n-th line) are given in the triangle:

					1					
				1		1				
			1	2		2	1			
		1	3	5	6	5	3	1		
1	4	9	15	20	22	20	15	9	4	1

....

4. Simple braids

The canonical form of a simple braid in $\mathcal{SB}_n$ is

$$\beta_{K,J} = \beta_{k_1,j_1}\beta_{k_2,j_2}\dots\beta_{k_s,j_s}$$

where $1 \leq k_1 < k_2 < \ldots k_s \leq n-1$, $j_i \leq k_i$ for all $i = 1, 2, \ldots, s$, and also $j_{i+1} > k_i$ for all $i = 1, 2, \ldots, s-1$ (see [4]). Let us denote by $\mathcal{SB}_n^i$ the subset of simple braids of length i in $\mathcal{SB}_n$ and $\sigma_{n,i}$ its cardinality. The generating polynomial of simple n-braids is denoted by $G_{\mathcal{SB}_n}(t) = \sum_{i=0}^{n-1} \sigma_{n,i} t^i$. We are interested in counting the number of simple braids $G_{\mathcal{SB}_n}(1)$.

Proposition 4.1. *The sequence $(\sigma_{n,i})$ is given by the recurrence:*
a) $\sigma_{1,0} = 1$ and $\sigma_{1,i} = 0$ for $i \neq 0$;
b) $\sigma_{n,i} = \sigma_{n-1,i} + \sigma_{n-1,i-1} + \sigma_{n-2,i-2} + \ldots + \sigma_{n-i,0}$.

Proof. The set $\mathcal{SB}_n^i$ can be decomposed as a disjoint union as follows:
$\mathcal{SB}_n^i = \mathcal{SB}_{n-1}^i \amalg (\mathcal{SB}_{n-1}^{i-1} \times \{x_{n-1}\}) \amalg (\mathcal{SB}_{n-1}^{i-2} \times \{x_{n-1}x_{n-2}\}) \amalg .. \amalg \{x_{n-1}..x_{n-i}\}$.
□

An equivalent recurrence for the sequence $(\sigma_{n,i})$ is given by the next Corollary:

Corollary 4.2. *The sequence $(\sigma_{n,i})$ satisfies also the recurrence:*
a) $\sigma_{1,0} = 1$ and $\sigma_{1,i} = 0$ for $i \neq 0$;
b) $\sigma_{n,i} = 2\sigma_{n-1,i-1} + \sigma_{n-1,i} - \sigma_{n-2,i-1}$.

Example 4.3. First values of $\sigma_{n,i}$ (on the n-th line) are given in the triangle:

$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 1 & & 1 & & & \\ & & 1 & & 2 & & 2 & & \\ & 1 & & 3 & & 5 & & 4 & \\ 1 & & 4 & & 9 & & 12 & & 8 \\ & & & & \ldots\ldots & & & & \end{array}$$

This sequence appears in N. J. A. Sloane list at the position A 160232 (see [14]).

Example 4.4. Starting with $\sigma_{n,0} = 1$, $\sigma_{n,1} = n-1$, and using the recurrence of Proposition 4.1 or the recurrence of Corollary 4.2 we get $\sigma_{n,2} = (n-1)(n+2)/2!$, $\sigma_{n,3} = (n-3)(n+4)(n-1)/3!$ and $\sigma_{n,4} = (n-4)(n+1)(n^2+5n-18)/4!$. Using the same recurrences we find that the last non zero coefficient is $\sigma_{n,n-1} = 2^{n-2}$ (if $n \geq 2$).

Proposition 4.5. *$\sigma_{n,i}$ is a polynomial in n of degree i and its leading coefficient is $1/i!$.*

Proof. The induction by i starts with $\sigma_{n,0} = 1$ and $\sigma_{n,1} = n-1$. Using Proposition 4.1, we have

$$\sigma_{n,i} - \sigma_{n-1,i} = \sigma_{n-1,i-1} + \sigma_{n-2,i-2} + \ldots + \sigma_{n-i,0}$$

where the sum is a polynomial in n of degree $i-1$ and leading coefficient $1/(i-1)!$. This implies that $\sigma_{n,i}$ is a polynomial in n of degree i and leading coefficient is $1/i!$.

□

Proof of Theorem 1.4. By definition $G_{SB_n}(1) = \sigma_{n,0}+\sigma_{n,1}+\sigma_{n,2}+\sigma_{n,n-2}+\ldots+\sigma_{n,n-1}$. Using the recurrence given in Proposition 4.1, we expand $G_{SB_n}(1) = \sum_{i=0}^{n-1} \sigma_{n,i}$ and get

$G_{SB_n}(1) = 2G_{SB_{n-1}}(1)+G_{SB_{n-2}}(1)+G_{SB_{n-3}}(1)+\ldots+G_{SB_2}(1)+G_{SB_1}(1)$. Starting an induction with $G_{SB_1}(1) = 1 = F_1 = F_2$, $G_{SB_2}(1) = 2 = F_3$, $G_{SB_3}(1) = 5 = F_5$, we obtain

$$\begin{aligned} G_{SB_n}(1) &= 2F_{2n-3}+F_{2n-5}+\ldots+F_5+F_3+F_2 \\ &= 2F_{2n-3}+F_{2n-5}+\ldots+F_5+F_4 \\ &= \ldots \\ &= 2F_{2n-3}+F_{2n-5}+F_{2n-6} = 2F_{2n-3}+F_{2n-4} \\ &= F_{2n-2}+F_{2n-3} = F_{2n-1}. \end{aligned}$$

□

5. Conjugacy classes of simple braids

Positive n-braids β and β' are called *conjugate* if there is a positive n-braid α such that $\beta\alpha = \alpha\beta'$ (this is an equivalence relation, see [11]). In [4] it is proved that every conjugacy class of a simple n-braid contains a unique simple braid of the form

$$\beta_A = (x_1x_2\ldots x_{s_1-1})(x_{s_1+1}\ldots x_{s_2-1})\ldots(x_{s_{r-1}+1}\ldots x_{s_r-1}),$$

where $A = (a_1, a_2, \ldots, a_r)$ is a sequence of integers satisfying $a_1 \geq a_2 \geq \ldots \geq a_r \geq 2$ and $s_i = a_1+a_2+\ldots+a_i$.

We denote by $c_{n,i}$ the number of conjugacy classes of positive simple n-braids of length i. A partition of a positive integer m is a representation of m in a form $m = m_1+m_2+\ldots+m_k$ where the integers $m_1, m_2, \ldots, m_k$ satisfy the inequalities $m_1 \geq m_2 \geq \ldots \geq m_k \geq 1$. The number of partitions of m into k parts is denoted by $P(m,k)$ (see [15]). By convention, $P(0,0) = 1$.

Proposition 5.1. *The number of conjugacy classes of simple n-braids of length i is given by*

$$c_{n,i} = P(i+min(i,n-i), min(i,n-i)).$$

Proof. Consider $\beta_A = (x_1x_2\ldots x_{s_1-1})(x_{s_1+1}\ldots x_{s_2-1})\ldots(x_{s_{r-1}+1}\ldots x_{s_r-1})$, the canonical representative of a conjugacy class in SB_n, of length $i = s_r - r$. We associate to the sequence $A = (a_1 \geq a_2 \geq \ldots \geq a_r)$ (here $a_r \geq 2$) the partition of i, $i = (a_1-1)+(a_2-1)+\ldots(a_r-1)$. The condition $s_r = a_1+a_2+\ldots+a_r \leq n$ implies $i+r \leq r$, therefore the number of conjugacy classes of simple braids of length i is given by the number of partitions of i into at most $n-i$ parts:

$$c_{n,i} = P(i,1) + P(i,2) + \ldots + P(i, min(i, n-i)).$$

Using the relation $P(n+k,k) = \sum_{i=1}^{k} P(n,k)$ $(k \leq n)$ (see [15]), we obtain the result. □

Example 5.2. First values of the sequence $c_{n,i}$ (on the n-th line) are given in the triangle:

$$\begin{array}{ccccccccccc} & & & & & 1 & & & & & \\ & & & & 1 & & 1 & & & & \\ & & & 1 & & 1 & & 1 & & & \\ & & 1 & & 1 & & 2 & & 1 & & \\ & 1 & & 1 & & 2 & & 2 & & 1 & \\ 1 & & 1 & & 2 & & 3 & & 3 & & 1 \end{array}$$

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