Upper signed k-domination number in graphs *

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ABSTRACT. A signed k-dominating function of a graph $G = (V, E)$ is a function $f : V \rightarrow \{+1, -1\}$ such that $\sum_{u \in N_G[v]} f(u) \geq k$ for each vertex $v \in V$. A signed k-dominating function f of a graph G is *minimal* if no $g < f$ is also a signed k-dominating function. The weight of a signed k-dominating function is $w(f) = \sum_{v \in V} f(v)$. The *upper signed k-domination number* $\Gamma_{s,k}(G)$ of G is the maximum weight of a minimal signed k-dominating function on G. In this paper, we establish a sharp upper bound on $\Gamma_{s,k}(G)$ for a general graph in terms of its minimum and maximum degree and order, and construct a class of extremal graphs which achieved the upper bound. As immediate consequences of our result, we present sharp upper bounds on $\Gamma_{s,k}(G)$ for regular graphs and nearly regular graphs.

Keywords: Upper bound; Upper signed k-domination number; Regular graph; Nearly regular graph

MSC: 05C69

1 Introduction

All graphs considered in this paper are finite connected simple graphs. Let $G = (V, E)$ be a graph with vertex set V and edge set E. Terminology not defined here will generally conform to that in [1]. For a vertex $v \in V$, the *open neighborhood* of v is $N_G(v) = \{u \in V |\ uv \in E\}$ and the *closed neighborhood* of v is $N_G[v] = \{v\} \cup N_G(v)$. The *degree* of v in G is $d_G(v) = |N_G(v)|$, and the *minimum degree* and *maximum degree* of G are denoted by $\delta(G)$ and $\Delta(G)$, respectively. When no ambiguity can occur, we often

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simply write $d(v)$, δ and Δ instead of $d_G(v)$, $\delta(G)$ and $\Delta(G)$, respectively. If each vertex in G has an even degree, then we call G an *Eulerian* graph. If each vertex in G has an odd degree, then we call G an *odd-degree* graph. A graph G is called *r-regular* if $d(v) = r$ for all $v \in V$. If $d(v) = r + 1$ or r for all $v \in V$, then we call G a *nearly* $(r + 1)$*-regular* graph. For a subset $S \subseteq V$, we let $d_S(v)$ denote the number of vertices in S that are adjacent to v, the closed neighborhood of S is $N[S] = \bigcup_{v \in S} N_G[v]$, and the subgraph of G induced by S is denoted by $G[S]$. For vertex-disjoint subsets $X, Y \subseteq V$, we use $e(X, Y)$ to denote the number of edges between X and Y.

For a positive integer $k \geq 1$, a *signed k-dominating function* (SkDF) of a graph G is a function $f : V \rightarrow \{+1, -1\}$ such that $\sum_{u \in N_G[v]} f(u) \geq k$ for each vertex $v \in V$. The *weight* of f is $w(f) = \sum_{v \in V} f(v)$, and for $S \subseteq V$ we define $f(S) = \sum_{v \in S} f(v)$, so $w(f) = f(V)$. For a vertex $v \in V$, we denote $f(N_G[v])$ by $f[v]$ for notational convenience. An SkDF f of G is minimal if there does not exist an SkDF g, $f \neq g$, for which $g(v) \leq f(v)$ for each vertex $v \in V$. The *upper signed k-domination number* $\Gamma_{s,k}(G)$ of G is the maximum weight of a minimal SkDF on G. In particular, the $\Gamma_{s,1}(G) = \Gamma_s(G)$ corresponds to the well-known *upper signed domination number* (see, [2, 3, 4, 5, 6]). Throughout this paper, we always assume that a graph G has minimum degree $\delta(G) \geq k - 1$ and $k \in \mathbb{N}$. A minimal SkDF of weight $\Gamma_{s,k}(G)$ is called a $\Gamma_{s,k}(G)$-function.

In [4] and [3] Favaron and Henning independently gave the sharp upper bounds on Γ_s of an r-regular graph in terms of its order.

Theorem 1 (Favaron [4] and Henning [3]) *If G is an r-regular graph, $r \geq 1$, of order n, then $\Gamma_s(G) \leq n(r+1)^2/(r^2+4r-1)$ if r is odd, and $\Gamma_s(G) \leq n(r+1)/(r+3)$ if r is even, and these bounds are sharp.*

Further, Wang and Mao [6] established the best possible upper bounds on Γ_s of a nearly $(r + 1)$-regular graph in terms of its order.

Theorem 2 (Wang and Mao [6]) *If G is a nearly $(r + 1)$-regular graph of order n, then $\Gamma_s(G) \leq n(r^2 + 3r + 4)/(r^2 + 5r + 2)$ for r odd, and $\Gamma_s(G) \leq n(r + 2)^2/(r^2 + 6r + 4)$ for r even, and these bounds are sharp.*

In [5] Tang and Chen presented sharp upper bounds on Γ_s of an arbitrary graph in terms of its minimum degree, maximum degree and order.

Theorem 3 (Tang and Chen [5]) *If G is a graph of order n, then $\Gamma_s(G) \leq (\delta\Delta + 3\Delta - \delta + 1)n/(\delta\Delta + 3\Delta + \delta - 1)$ for δ odd, and $\Gamma_s(G) \leq (\delta\Delta + 4\Delta - \delta)n/(\delta\Delta + 4\Delta + \delta)$ for δ even. In particular, if G is an Eulerian graph, then $\Gamma_s(G) \leq (\delta\Delta + 2\Delta - \delta)n/(\delta\Delta + 2\Delta + \delta)$. Furthermore, these bounds are sharp.*

Obviously, Tang and Chen generalized the results in Theorems 1 and 2 to general graph, if $\delta = \Delta = r$ or $\delta = r$ and $\Delta = r+1$ in Theorem 3, then we see that Theorems 1 and 2 are special cases of Theorem 3.

In this paper, we generalize the results Γ_s in Theorems 3 to $\Gamma_{s,k}$ of an arbitrary graph. We establish the upper bound on $\Gamma_{s,k}$ for a general graph in terms of its minimum degree, maximum degree, order and positive integer k, and construct a class of extremal graphs which achieved the upper bound. In particular, if G is an r-regular graph or a nearly $(r+1)$-regular graph, we present sharp upper bound on $\Gamma_{s,k}$ in terms of its degree, order and positive integer k.

2 Main results

Theorem 4 *If G is a graph of order n with minimum degree δ and maximum degree Δ, then*

$$\Gamma_{s,k}(G) \leq \begin{cases} \dfrac{\Delta(\delta+k+2)-(\delta-k)}{\Delta(\delta+k+2)+(\delta-k)}n & \text{for } \delta-k+1 \text{ odd,} \\[2ex] \dfrac{\Delta(\delta+k+3)-(\delta-k+1)}{\Delta(\delta+k+3)+(\delta-k+1)}n & \text{for } \delta-k+1 \text{ even.} \end{cases}$$

In particular, if G is an Eulerian graph and k is odd , or G is an odd-degree graph and k is even, then

$$\Gamma_{s,k}(G) \leq \frac{\Delta(\delta+k+1)-(\delta-k+1)}{\Delta(\delta+k+1)+(\delta-k+1)}n.$$

Furthermore, these bounds are sharp.

Clearly, if $k=1$, then $\big(\Delta(\delta+k+2)-(\delta-k)\big)n/\big(\Delta(\delta+k+2)+(\delta-k)\big) = (\delta\Delta+3\Delta-\delta+1)n/(\delta\Delta+3\Delta+\delta-1)$ for δ being odd, and $\big(\Delta(\delta+k+3)-(\delta-k+1)\big)n/\big(\Delta(\delta+k+3)+(\delta-k+1)\big) = (\delta\Delta+4\Delta-\delta)n/(\delta\Delta+4\Delta+\delta)$ for δ being even. Furthermore, if G is an Eulerian graph, then $\Gamma_s(G) \leq \big(\Delta(\delta+k+1)-(\delta-k+1)\big)n/\big(\Delta(\delta+k+1)+(\delta-k+1)\big) = (\delta\Delta+2\Delta-\delta)n/(\delta\Delta+2\Delta+\delta)$. Thus, we see that Theorem 3 is special case of Theorem 4.

To prove Theorem 4, we shall need the following lemmas.

Lemma 5 ([2]) *If r and n are positive integers with $r < n$ and n is even, then we can construct an r-regular graph on n vertices.*

Lemma 6 *A signed k-dominating function f on a graph G is minimal if and only if for every vertex v of weight $+1$, there exists a vertex $u \in N[v]$ such that $f[u] = k$ or $k+1$.*

The proof of Lemma 6 is straightforward and therefore omitted. Now we can present the proof of Theorem 4.

Proof of Theorem 4. Let f be a $\Gamma_{s,k}(G)$-function of G, and let $P = \{v \in V | \ f(v) = +1\}$ and $M = \{v \in V | \ f(v) = -1\}$. Further, we let $|P| = p$ and $|M| = m$, thus, $w(f) = |P| - |M| = n - 2m$. If $k = \delta$ or $\delta + 1$, then the results are trivial. Hence in what follows we assume $k \leq \delta - 1$.

For each vertex $v \in P$, $f[v] = d_P(v)+1-d_M(v) = d(v)-2d_M(v)+1 \geq k$, and so $d_M(v) \leq \lfloor (d(v) - k + 1)/2 \rfloor$. We write $s_1 = \lfloor (\delta - k + 1)/2 \rfloor$, $t_1 = \lfloor (\Delta - k + 1)/2 \rfloor$. Hence we can partition P into $t_1 + 1$ sets by defining $P_i = \{v \in P | \ d_M(v) = i\}$ and letting $|P_i| = p_i$ for $i = 0, 1, \cdots, t_1$. Then we have

$$n = m + p = m + \sum_{i=0}^{t_1} p_i \tag{1}$$

For any vertex $v \in M$, $f[v] = d_P(v) - 1 - d_M(v) = 2d_P(v) - d(v) - 1 \geq k$, and so $d_P(v) \geq \lceil (d(v)+k+1)/2 \rceil$. We write $s_2 = \lceil (\delta+k+1)/2 \rceil$, $t_2 = \lceil (\Delta + k + 1)/2 \rceil$. We define $M_j = \{v \in M | \ d_P(v) = j\}$ for $j = s_2, s_2 + 1, \cdots, t_2$, and $M' = M - \bigcup_{j=s_2}^{t_2} M_j$. Let $|M_j| = m_j$, and so $|M'| = m - \Sigma_{j=s_2}^{t_2} m_j$. Clearly, the sets $M_{s_2}, M_{s_2+1}, \cdots, M_{t_2}, M'$ form a partition of M. Since each vertex in M' is adjacent to at most Δ vertices of P, we have

$$\sum_{i=1}^{t_1} ip_i = e(P, M) \leq (s_2 m_{s_2} + \cdots + t_2 m_{t_2}) + \Delta[m - (m_{s_2} + \cdots + m_{t_2})].$$

Hence,

$$\sum_{i=1}^{t_1} ip_i \quad \leq \quad \Delta m - \sum_{j=s_2}^{t_2} (\Delta - j\ )m_j. \tag{2}$$

If $P_0 = \emptyset$, then by (1) and (2), we have

$$n = m + \sum_{i=1}^{t_1} p_i \leq m + \sum_{i=1}^{t_1} ip_i \leq (\Delta + 1)m.$$

Solving the above inequality for m, we obtain that $m \geq n/(\Delta + 1)$, and hence $\Gamma_{s,k}(G) = n - 2m \leq (\Delta - 1)n/(\Delta + 1)$. Observing that

$$\frac{\Delta - 1}{\Delta + 1} n < \min \left\{ \frac{\Delta(\delta + k + 2) - (\delta - k)}{\Delta(\delta + k + 2) + (\delta - k)} n, \frac{\Delta(\delta + k + 3) - (\delta - k + 1)}{\Delta(\delta + k + 3) + (\delta - k + 1)} n \right\}.$$

Then we see that the conclusion holds. Thus we may assume that $P_0 \neq \emptyset$.

According to our partition for P and M, we obtain that for any $v \in \bigcup_{i=0}^{t_1} P_i$ and such that $f[v] = d(v) - 2i + 1 \geq k + 2$, we have $i \leq (d(v) - k + 1)/2 - 1$. Hence, when $i \leq \lfloor (\delta - k + 1)/2 \rfloor - 1 = s_1 - 1$, we have $f[v] \geq k + 2$ for any $v \in \bigcup_{i=0}^{s_1-1} P_i$. Similarly, for any $v \in M'$, we have $d_P(v) \geq t_2 + 1 = \lceil (\Delta + k + 1)/2 \rceil + 1$, it is clear that $f[v] = 2d_P(v) - d(v) - 1 \geq k + 2$. So if $f[v] = k$ or $k + 1$ for $v \in V$, then $v \in (\bigcup_{i=s_1}^{t_1} P_i) \cup (\bigcup_{j=s_2}^{t_2} M_j)$.

For any $v \in P_0$, since $f[v] = d(v) + 1 \geq k + 2$ and f is minimal, by Lemma 6, v has at least one neighbor u such that $u \notin P_0$ and $f[u] = k$ or $k + 1$. Let $Q = \{u \in N[P_0] |\ f[u] = k \text{ or } k + 1\}$. Noting that for any $v \in \bigcup_{i=0}^{s_1-1} P_i$, $f[v] \geq k + 2$, we see that $Q \subseteq \bigcup_{i=s_1}^{t_1} P_i$. So

$$p_0 = |P_0| \leq e(P_0, Q) = e(P_0, \bigcup_{i=s_1}^{t_1} (P_i \cap Q)). \tag{3}$$

For any vertex $u \in P_i \cap Q$ ($s_1 \leq i \leq t_1$), by Lemma 6, there must exist a neighbor u' of u such that $f[u'] = k$ or $k+1$. Noting that $u' \in (\bigcup_{i=s_1}^{t_1} P_i) \cup (\bigcup_{j=s_2}^{t_2} M_j)$. If $u' \in \bigcup_{i=s_1}^{t_1} P_i$ and $u' \neq u$, then each u has at most $k + i - 1$ neighbors in P_0; If $u' \in \bigcup_{i=s_1}^{t_1} P_i$ and $u' = u$, then each u has at most $k + i$ neighbors in P_0; If $u' \in \bigcup_{j=s_2}^{t_2} M_j$, then each u has at most $k + i$ neighbors in P_0.

Hence we can write $P_i \cap Q$ ($s_1 \leq i \leq t_1$) as the disjoint union of two sets QP_i' and QP_i'', where $QP_i' = \{u \in P_i \cap Q \mid d_{P_0}(u) = k + i\}$ and $QP_i'' = \{u \in P_i \cap Q \mid d_{P_0}(u) \leq k + i - 1\}$, let $|QP_i'| = p_i'$, $|QP_i''| = |P_i \cap Q| - p_i'$. Thus, by inequality (3), we have

$$\begin{aligned} p_0 &\leq e(P_0, \bigcup_{i=s_1}^{t_1} QP_i') + e(P_0, \bigcup_{i=s_1}^{t_1} QP_i'') \\ &\leq \sum_{i=s_1}^{t_1} (i + k) p_i' + \sum_{i=s_1}^{t_1} (i + k - 1)(|P_i \cap Q| - p_i') \\ &\leq \sum_{i=s_1}^{t_1} (i + k) p_i' + \sum_{i=s_1}^{t_1} (i + k - 1)(p_i - p_i') \\ &= \sum_{i=s_1}^{t_1} (i + k - 1) p_i + \sum_{i=s_1}^{t_1} p_i'. \end{aligned} \tag{4}$$

We now distinguish two possibilities depending on the parity of $\delta - k + 1$.

Case 1. $\delta - k + 1$ is odd.

Then $s_1 = \lfloor(\delta - k + 1)/2\rfloor = (\delta - k)/2$. Noting that when $i \geq s_1 = (\delta - k)/2$, the inequality $(\delta + k + 2)i/(\delta - k) \geq i + k + 1$ holds, then by equality (1), inequalities (2) and (4), we obtain that

$$\begin{aligned}
n &\leq m + \Big(\sum_{i=s_1}^{t_1}(i+k-1)p_i + \sum_{i=s_1}^{t_1} p_i'\Big) + \sum_{i=1}^{t_1} p_i \\
&= m + \sum_{i=s_1}^{t_1}(i+k+1)p_i + \sum_{i=1}^{s_1-1} p_i - \sum_{i=s_1}^{t_1} p_i + \sum_{i=s_1}^{t_1} p_i' \\
&\leq m + \frac{\delta+k+2}{\delta-k}\sum_{i=s_1}^{t_1} ip_i + \sum_{i=1}^{s_1-1} p_i - \sum_{i=s_1}^{t_1} p_i + \sum_{i=s_1}^{t_1} p_i' \\
&\leq m + \frac{\delta+k+2}{\delta-k}\sum_{i=1}^{t_1} ip_i - \Big(\sum_{i=s_1}^{t_1} p_i - \sum_{i=s_1}^{t_1} p_i'\Big) \\
&\leq m + \frac{\delta+k+2}{\delta-k}\sum_{i=1}^{t_1} ip_i \\
&\leq m + \frac{\delta+k+2}{\delta-k} m\Delta - \frac{\delta+k+2}{\delta-k}\sum_{j=s_2}^{t_2}(\Delta - j)m_j \\
&\leq m + \frac{\delta+k+2}{\delta-k} m\Delta.
\end{aligned}$$

Then we have $n \leq m + (\delta + k + 2)m\Delta/(\delta - k)$, which implies that $m \geq (\delta - k)n/(\Delta(\delta + k + 2) + (\delta - k))$, and hence

$$\Gamma_{s,k}(G) = w(f) = n - 2m \leq \frac{\Delta(\delta+k+2) - (\delta-k)}{\Delta(\delta+k+2) + (\delta-k)} n.$$

That the bound is sharp may be seen as follows. For any positive integers $l \geq k$ and $r \geq l + k$ and q, where $2l + k \leq q \leq 2r - 1$, let $F_{l,r}$ be the graph with vertex set $V = X \cup Y \cup Z$ with $|X| = l$, $|Y| = 2r$ and $|Z| = 2r(l + k)$, where X and Y are independent sets of vertices. The edge set of $F_{l,r}$ is constructed as follows: Add $2rl$ edges between X and Y so that $G[X \cup Y]$ forms a complete bipartite graph with partition sets X and Y. Add $2r(l + k)$ edges between Y and Z so that each vertex of Y is precisely adjacent to $l + k$ vertices of Z and each vertex of Z is precisely adjacent to one vertex of Y. Add edges joining vertices of Z so that $G[Z]$ is a q-regular graph (since $q < 2r(l + k) = |Z|$ and $|Z|$ is even, it follows from Lemma 5 that such an addition of edges is possible). The graph $F_{l,r}$ is shown in Fig. 1.

By construction, $F_{l,r}$ is a graph of order $n = l + 2r + 2r(l + k)$ with maximum degree $\Delta = 2r$ and minimum degree $\delta = 2l + k$. Let f be a

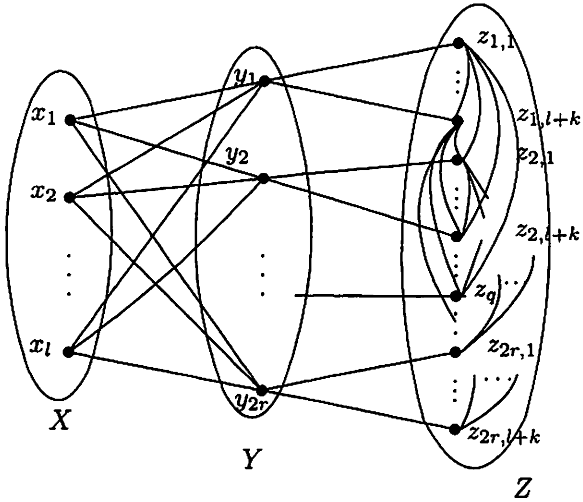

Figure 1: The graph $F_{l,r}$ ($G[Z]$ is a q-regular graph).

function defined on V such that $f(v) = -1$ for $v \in X$ and $f(v) = +1$ for $v \in Y \cup Z$. It is easy to check that f is an SkDF of $F_{l,r}$, and by Lemma 6, f is minimal. Clearly, $w(f) = n - 2|X| = 2r + 2r(l+k) - l$, then it is easy to check that

$$w(f) = 2r + 2r(l+k) - l = \frac{\Delta(\delta+k+2) - (\delta-k)}{\Delta(\delta+k+2) + (\delta-k)} n.$$

Consequently, $\Gamma_{s,k}(F_{l,r}) = (\Delta(\delta+k+2) - (\delta-k))n/(\Delta(\delta+k+2) + (\delta-k))$.

Case 2. $\delta - k + 1$ is even.

Then $s_1 = \lfloor (\delta-k+1)/2 \rfloor = (\delta-k+1)/2$. Noting that when $i \ge s_1 = (\delta-k+1)/2$, the inequality $(\delta+k+3)i/(\delta-k+1) \ge i+k+1$ holds, then by equality (1), inequalities (2) and (4) again, we have

$$\begin{aligned} n &\le m + \Big(\sum_{i=s_1}^{t_1}(i+k-1)p_i + \sum_{i=s_1}^{t_1} p_i'\Big) + \sum_{i=1}^{t_1} p_i \\ &= m + \sum_{i=s_1}^{t_1}(i+k+1)p_i + \sum_{i=1}^{s_1-1} p_i - \sum_{i=s_1}^{t_1} p_i + \sum_{i=s_1}^{t_1} p_i' \\ &\le m + \frac{\delta+k+3}{\delta-k+1}\sum_{i=s_1}^{t_1} ip_i + \sum_{i=1}^{s_1-1} p_i - \sum_{i=s_1}^{t_1} p_i + \sum_{i=s_1}^{t_1} p_i' \\ &\le m + \frac{\delta+k+3}{\delta-k+1}\sum_{i=1}^{t_1} ip_i - \Big(\sum_{i=s_1}^{t_1} p_i - \sum_{i=s_1}^{t_1} p_i'\Big) \end{aligned}$$

$$\begin{aligned}
&\leq m + \frac{\delta+k+3}{\delta-k+1}\sum_{i=1}^{t_1} ip_i \\
&\leq m + \frac{\delta+k+3}{\delta-k+1}m\Delta - \frac{\delta+k+3}{\delta-k+1}\sum_{j=s_2}^{t_2}(\Delta-j)m_j \\
&\leq m + \frac{\delta+k+3}{\delta-k+1}m\Delta.
\end{aligned}$$

Then we have $n \leq m + (\delta+k+3)\Delta m/(\delta-k+1)$, which implies that $m \geq (\delta-k+1)n/(\Delta(\delta+k+3)+(\delta-k+1))$, and hence

$$\Gamma_{s,k}(G) = w(f) = n - 2m \leq \frac{\Delta(\delta+k+3)-(\delta-k+1)}{\Delta(\delta+k+3)+(\delta-k+1)}n.$$

That the bound is sharp may be seen as follows. For any positive integers $l \geq k$ and $r \geq l+k$, let $G_{l,r}$ be the graph with vertex set $V = X \cup Y \cup Z$ with $|X| = l$, $|Y| = 2r$ and $|Z| = 2r(l+k)$, where X is an independent set of vertices. The edge set of $G_{l,r}$ is constructed as follows: Add $2rl$ edges between X and Y so that $G[X \cup Y]$ forms a complete bipartite graph with partition sets X and Y. Add $2r(l+k)$ edges between Y and Z so that each vertex of Y is precisely adjacent to $l+k$ vertices of Z and each vertex of Z is precisely adjacent to one vertex of Y. Add edges joining vertices of Y so that $G[Y]$ is a 1-regular graph. Add edges joining vertices of Z so that $G[Z]$ is a $(2l+k-2)$-regular graph (since $2l+k-2 < 2r(l+k) = |Z|$ and $|Z|$ is even, it follows from Lemma 5 that such an addition of edges is possible). The graph $G_{l,r}$ is shown in Fig. 2.

By construction, $G_{l,r}$ is a graph of order $n = l + 2r + 2r(l+k)$ with maximum degree $\Delta = 2r$ and minimum degree $\delta = 2l + k - 1$. Let f be a function defined on V such that $f(v) = -1$ for $v \in X$ and $f(v) = +1$ for $v \in Y \cup Z$. It is easy to check that f is an SkDF of $G_{l,r}$, and by Lemma 6, f is minimal. Clearly, $w(f) = n - 2|X| = 2r + 2r(l+k) - l$, then it is easy to check that

$$w(f) = 2r + 2r(l+k) - l = \frac{\Delta(\delta+k+3)-(\delta-k+1)}{\Delta(\delta+k+3)+(\delta-k+1)}n.$$

Consequently, $\Gamma_{s,k}(G_{l,r}) = (\Delta(\delta+k+3)-(\delta-k+1))n/(\Delta(\delta+k+3)+(\delta-k+1))$.

In particular, if G is an Eulerian graph, that is, every vertex of G has even degree, then for any vertex $v \in V$, $f[v]$ is odd. Further, if k is odd, then each vertex $u \in P_i \cap Q$ $(s_1 \leq i \leq t_1)$ has at most $i+k-1$ neighbors in P_0, and $f[u] = k$. Similarly, if G is an odd-degree graph, then for any vertex $v \in V$, $f[v]$ is even, further, if k is even, then each vertex $u \in P_i \cap Q$

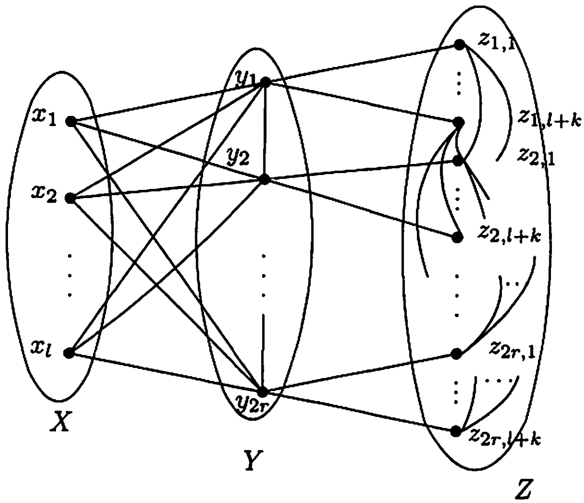

Figure 2: The graph $G_{l,r}$ ($G[Z]$ is a $(2l+k-2)$-regular graph).

$(s_1 \le i \le t_1)$ has at most $i+k-1$ neighbors in P_0, and $f[u] = k$. Thus $\delta - k + 1$ is even, and the inequality (4) can be improved as

$$p_0 \quad \le \quad \sum_{i=s_1}^{t_1} (i+k-1)p_i. \tag{5}$$

Since $(\delta+k+1)i/(\delta-k+1) \ge i+k$ when $i \ge s_1 = \lfloor(\delta-k+1)/2\rfloor = (\delta-k+1)/2$, then by equality (1), inequalities (2) and (5), using similar proof above, we have

$$n \quad \le \quad m + \frac{\delta+k+1}{\delta-k+1} m\Delta,$$

which gives

$$\Gamma_{s,k}(G) = w(f) = n - 2m \le \frac{\Delta(\delta+k+1)-(\delta-k+1)}{\Delta(\delta+k+1)+(\delta-k+1)} n.$$

If G is an Eulerian graph and k is odd, that the bound is sharp may be seen as follows. For any positive integers $l \ge k$ and $r \ge l + (k-1)/2$ and odd number q, where $2l+k \le q < 2r$, let $H_{l,r}$ be the graph with vertex set $V = X \cup Y \cup Z$ with $|X| = 2l^2$, $|Y| = 4lr$ and $|Z| = 4lr(l+k-1)$, where X and Y are independent sets of vertices. The edge set of $H_{l,r}$ is constructed as follows: Add $4l^2r$ edges between X and Y so that each vertex in X has degree $2r$ while each vertex in Y has degree l. Add $4lr(l+k-1)$

edges between Y and Z so that each vertex of Y is precisely adjacent to $l+k-1$ vertices of Z and each vertex of Z is precisely adjacent to one vertex of Y. Add edges joining vertices of Z so that $G[Z]$ is a q-regular graph. The graph $H_{l,r}$ is shown in Fig. 3. By construction, $H_{l,r}$ is a

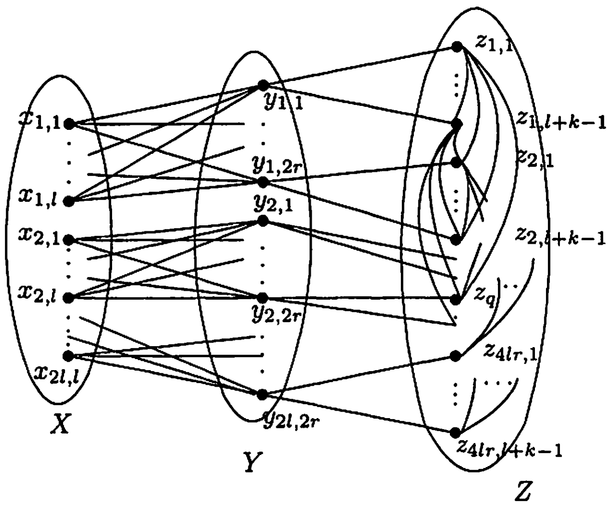

Figure 3: The Eulerian graph $H_{l,r}$(k and q are odd and $2l+k \leq q < 2r$, $G[Z]$ is a q-regular graph).

graph of order $n = 2l^2 + 4lr(l+k)$ with maximum degree $\Delta = 2r$ and minimum degree $\delta = 2l+k-1$. Let f be a function defined on V such that $f(v) = -1$ for $v \in X$ and $f(v) = +1$ for $v \in Y \cup Z$. It is easy to check that f is an SkDF of $H_{l,r}$, and by Lemma 6, f is minimal. Clearly, $w(f) = n - 2|X| = 4lr(l+k) - 2l^2$, then it is easy to check that

$$w(f) = 4lr(l+k) - 2l^2 = \frac{\Delta(\delta+k+1)-(\delta-k+1)}{\Delta(\delta+k+1)+(\delta-k+1)}n.$$

Consequently, $\Gamma_{s,k}(H_{l,r}) = (\Delta(\delta+k+1)-(\delta-k+1))n/(\Delta(\delta+k+1)+(\delta-k+1))$.

If G is an odd-degree graph and k is even, that the bound is sharp may be seen as follows. For any positive integers $l \geq k$ and $r \geq l + k/2$ and even number q, where $2l+k \leq q \leq 2r$, let $I_{l,r}$ be the graph with vertex set $V = X \cup Y \cup Z$ with $|X| = 2l^2$, $|Y| = 2l(2r+1)$ and $|Z| = 2l(2r+1)(l+k-1)$, where X and Y are independent sets of vertices. The edge set of $I_{l,r}$ is constructed as follows: Add $2l^2(2r+1)$ edges between X and Y so that each vertex in X has degree $2r+1$ while each vertex in Y has degree l. Add $2l(2r+1)(l+k-1)$ edges between Y and Z so that each vertex of

Y is precisely adjacent to $l+k-1$ vertices of Z and each vertex of Z is precisely adjacent to one vertex of Y. Add edges joining vertices of Z so that $G[Z]$ is a q-regular graph.

By construction, $I_{l,r}$ is a graph of order $n = 2l^2 + 2l(2r+1)(l+k)$ with maximum degree $\Delta = 2r+1$ and minimum degree $\delta = 2l+k-1$. Let f be a function defined on V such that $f(v) = -1$ for $v \in X$ and $f(v) = +1$ for $v \in Y \cup Z$. It is easy to check that f is an SkDF of $I_{l,r}$, and by Lemma 6, f is minimal. Clearly, $w(f) = n - 2|X| = 2l(2r+1)(l+k) - 2l^2$, then it is easy to check that

$$w(f) = 2l(2r+1)(l+k) - 2l^2 = \frac{\Delta(\delta+k+1) - (\delta-k+1)}{\Delta(\delta+k+1) + (\delta-k+1)} n.$$

Consequently, $\Gamma_{s,k}(I_{l,r}) = (\Delta(\delta+k+1) - (\delta-k+1))n/(\Delta(\delta+k+1) + (\delta-k+1))$. □

As immediate consequences of Theorem 4 when $\Delta = \delta = r$ or $\Delta = r+1$ and $\delta = r$, we have the following results.

Corollary 7 *If $G = (V,E)$ is an r-regular graph with $r \geq 1$ of order n, then*

$$\Gamma_{s,k}(G) \leq \begin{cases} \dfrac{(r+1)(r+k)}{(r+1)^2 + (k+1)(r-1)} n & \text{for } r-k+1 \text{ odd}, \\[2ex] \dfrac{(r+1)(r+k-1)}{(r+1)^2 + k(r-1)} n & \text{for } r-k+1 \text{ even}. \end{cases}$$

Furthermore, these bounds are sharp.

Corollary 8 *If $G = (V,E)$ is a nearly $(r+1)$-regular graph with $r \geq 1$ of order n, then*

$$\Gamma_{s,k}(G) \leq \begin{cases} \dfrac{(r+1)(r+k+1) + (k+1)}{r^2 + r(k+4) + 2} n & \text{for } r-k+1 \text{ odd}, \\[2ex] \dfrac{(r+1)(r+k+2) + k}{r^2 + r(k+5) + 4} n & \text{for } r-k+1 \text{ even}. \end{cases}$$

Furthermore, these bounds are sharp.

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