On total irregularity strength of generalized Halin graph

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Abstract

We investigate two modifications of the well-known irregularity strength of graphs, namely the total edge irregularity strength and the total vertex irregularity strength.

In this paper, we determine the exact value of the total edge (vertex) irregularity strength for Halin graphs.

Keywords : total edge irregularity strength, total vertex irregularity strength, Halin graph, generalized Halin graph.

1 Introduction and Definitions

As a standard notation, assume that $G = (V, E)$ is a finite, simple and undirected graph with $|V(G)|$ vertices and $|E(G)|$ edges. A labeling of a graph is any mapping that sends some set of graph elements to a set of numbers (usually positive integers). If the domain is the vertex-set or the edge-set, the labelings are called respectively vertex-labelings or edge-labelings. If the domain is $V \cup E$ then we call the labeling a total labeling. In many cases it is interesting to consider the sum of all labels associated with a graph element. This will be called the *weight* of the graph element.

An *edge irregular total k-labeling* of a graph $G = (V, E)$ is a labeling $\sigma : V \cup E \to \{1, 2, \ldots, k\}$ such that the total edge-weights $wt(xy) = \sigma(x) + \sigma(xy) + \sigma(y)$ are different for all pairs of distinct edges. Similarly, a *vertex irregular total k-labeling* of a graph $G = (V, E)$ is a labeling of the vertices and edges with integers $1, 2, \ldots, k$ such that the weights of any two different vertices are distinct, where the weight of a vertex is the sum of the label of the vertex itself and the labels of its incident edges. Moreover, the minimum k for which the graph G has an edge irregular total k-labeling is called the *total edge irregularity strength* of G, $tes(G)$; and the minimum k for which the graph G has a vertex irregular total k-labeling is called the *total vertex irregularity strength* of G, $tvs(G)$.

The notions of the total edge irregularity strength and total vertex irregularity strength were first introduced by Bača, Jendroľ, Miller and Ryan in the recent paper [6]. The original motivation for the definition of the total edge (vertex) irregularity strength came from irregular assignments and the irregularity strength of graphs introduced in [9] by Chartrand *et al.* and studied by numerous authors [4, 5, 7, 12, 15]. Finding the irregularity strength of a graph seems to be hard even for graphs with simple structure, see [10, 11] and [18].

A simple lower bound for total edge irregularity strength determined in [6] is given by the following.

Theorem 1 [6] *Let $G = (V, E)$ be a graph with maximum degree $\Delta = \Delta(G)$. Then*

$$tes(G) \geq max\left\{\left\lceil \frac{|E(G)| + 2}{3} \right\rceil, \left\lceil \frac{\Delta + 1}{2} \right\rceil\right\}.$$

Ivančo and Jendroľ [14] conjectured that the bound from Theorem 1 is attained for all graphs except K_5. Brandt, Miškuf and Rautenbach [8] recently proved that this is true for graphs whose size is at least 111000 times their maximum degree.

Bača, Jendroľ, Miller and Ryan proved the following theorem.

Theorem 2 [6] *Let $G = (V, E)$ be a graph with minimum degree $\delta = \delta(G)$ and maximum degree $\Delta = \Delta(G)$. Then*

$$\left\lceil \frac{|V(G)| + \delta}{\Delta + 1} \right\rceil \leq tvs(G) \leq |V(G)| + \Delta - 2\delta + 1.$$

In [6] the authors determined the exact value of the total edge irregularity strength for certain families of graphs, namely paths, cycles, stars, wheels and friendship graphs, and obtained the exact value of the total vertex irregularity strength for stars, complete graphs, cycles and prisms.

Ivančo and Jendroľ [14] determined the total edge irregularity strength for any tree. Jendroľ, Miškuf and Soták [16, 17] showed the exact value of the total edge irregularity strength of complete graphs and complete bipartite graphs. Motivated by the papers [10] and [21] Miškuf and Jendroľ [19] determined the exact value of the total edge irregularity strength of grids. Ahmad and Bača determined the exact value of the total edge irregularity strength of a categorical product of two paths in [1] and of a categorical product of a cycle and a path in [3].

In this paper, we determine the exact value of the total edge (vertex) irregularity strength for Halin graphs.

2 Main Results

A *Halin graph* $H(T)$ (see [13]) is a planar graph constructed from a plane embedding of a tree T with at least four vertices and with no vertices of degree 2, by connecting all the leaves of the tree (the vertices of degree 1) with a cycle C that passes around the tree in the natural cyclic order defined by the embedding of the tree. The tree T is called the characteristic tree of $H(T)$, and C is called the adjoint cycle of $H(T)$. Every wheel W_n is the Halin graph $H(S_n)$.

A 2-connected planar graph $H(G)$ without vertices of degree 2, possessing a cycle C such that

(i) all vertices of C have degree 3 in $H(G)$,

(ii) $H(G) - C = G$ is connected graph

is called a *generalized Halin graph.* The cycle C is called the outer cycle of $H(G)$ and its vertices are called outer vertices. The graph $G = H(G) - C$

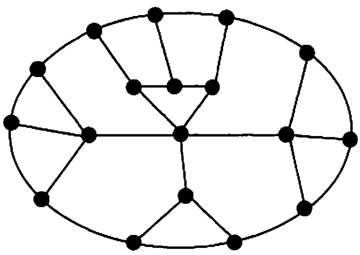

Figure 1: The generalized Halin graph

is called the characteristic graph of $H(G)$ and its vertices are called inner vertices, see Figure 1.

In the next theorem we determine the exact value of the total edge irregularity strength of generalized Halin graph $H(G)$.

Theorem 3 *Let $H(G)$ be a generalized Halin graph of size q with t inner vertices and l outer vertices such that $1 < t \leq l$ and $\lceil \frac{q+2}{3} \rceil - l \leq 1$. Then*

$$tes(H(G)) = \left\lceil \frac{q+2}{3} \right\rceil.$$

Proof. Let $k = \lceil \frac{q+2}{3} \rceil$. According to Theorem 1 we have $tes(H(G)) \geq k$. To show that k is an upper bound for $tes(H(G))$ we construct a total k-labeling for $H(G)$. Let $V(H(G)) = \{v_i : 1 \leq i \leq l\} \cup \{x_j : 1 \leq j \leq t\}$ be the vertex set and $E(H(G)) = \{v_i v_{i+1} : 1 \leq i \leq l-1\} \cup \{v_l v_1\} \cup \{e_i :$ e_i is incident to v_i for $1 \leq i \leq l\} \cup \{f_s : f_s$ is incident to inner vertices for $1 \leq s \leq q - 2l\}$ be the edge set of $H(G)$.

For $l \geq 5$ we define a total k-labeling σ of $H(G)$ as follows:

Case 1. If $k - l \neq \{0, 1\}$, then

$$\sigma(x_i) = k \text{ for } 1 \leq i \leq t,$$

$$\sigma(v_i) = \begin{cases} 1 & \text{for } 1 \leq i \leq k-2 \\ k-2 & \text{for } i = k-1 \\ k & \text{for } k \leq i \leq l-1 \\ k-2 & \text{for } i = l, \end{cases}$$

$$\sigma(v_iv_{i+1}) = \begin{cases} i & \text{for } 1 \le i \le k-3 \\ 1 & \text{for } i = k-2 \\ 5 & \text{for } i = k-1 \\ l+2-i & \text{for } k \le i \le l-2 \\ 4 & \text{for } i = l-1, \end{cases}$$

$$\sigma(v_lv_1) = 2,$$

$$\sigma(e_i) = \begin{cases} i & \text{for } 1 \le i \le k-2 \\ 2 & \text{for } i = k-1 \\ 2l+2-k-i & \text{for } k \le i \le l-1 \\ 3 & \text{for } i = l \end{cases}$$

$$\sigma(f_s) = 2l - 2k + 2 + s \quad \text{for} \quad 1 \le s \le q - 2l.$$

Case 2. If $k - l = 0$, then we use the total k-labeling from *Case 1*, where $\sigma(v_l) = k-2$, $\sigma(v_{l-1}v_l) = 5$ and $\sigma(e_l) = 4$.

Case 3. If $k - l = 1$, then we use the total k-labeling from *Case 1*, where $\sigma(e_l) = 2$.

Observe that if $k - l \neq \{0, 1\}$, then

$$\begin{aligned} wt(v_iv_{i+1}) &= \sigma(v_i) + \sigma(v_{i+1}) + \sigma(v_iv_{i+1}) \\ &= \begin{cases} i+2 & \text{for } 1 \le i \le k-2 \\ k+1 & \text{for } i = l \\ 2k+2 & \text{for } i = l-1 \\ 2k+3 & \text{for } i = k-1 \\ 2k+l+2-i & \text{for } k \le i \le l-2, \end{cases} \end{aligned}$$

$$wt(e_i) = \begin{cases} k+1+i & \text{for } 1 \le i \le k-1 \\ 2k+1 & \text{for } i = l \\ 2l+2+k-i & \text{for } k \le i \le l-1 \end{cases}$$

and

$$wt(f_s) = 2l + 2 + s \quad \text{for} \quad 1 \le s \le q - 2l.$$

If $k - l = 0$, then

$$wt(v_iv_{i+1}) = \begin{cases} i+2 & \text{for } 1 \le i \le k-2 \\ i+1 & \text{for } i = l \\ 2k+1 & \text{for } i = k-1, \end{cases}$$

$$wt(e_i) = \begin{cases} k+1+i & \text{for } 1 \le i \le k-1 \\ 2k+2 & \text{for } i = l = k \end{cases}$$

and

$$wt(f_s) = 2l + 2 + s \text{ for } 1 \leq s \leq q - 2l.$$

If $k - l = 1$, then

$$wt(v_iv_{i+1}) = i + 2 \text{ for } 1 \leq i \leq k - 1,$$

$$wt(e_i) = \begin{cases} k + 1 + i & \text{for } 1 \leq i \leq k - 2 \\ 2k & \text{for } i = k - 1 \end{cases}$$

and

$$wt(f_s) = 2l + 2 + s \text{ for } 1 \leq s \leq q - 2l.$$

To take care of $H(G)$, for $3 \leq l \leq 4$, we give the following special labelings:

For $l = 3, 4$ and $t = 1$ we have wheels and from [6] it follows that $tes(W_n) = \lceil \frac{2n+2}{3} \rceil$.

For $l = 4$ and $t = 2$ we define the total 4-labeling σ as follows $\sigma(v_1) = \sigma(v_2) = 1$, $\sigma(v_3) = \sigma(v_4) = 2$, $\sigma(x_1) = \sigma(x_2) = 4$, $\sigma(v_iv_{i+1}) = 1$ for $1 \leq i \leq 3$, $\sigma(v_4v_1) = 3$, $\sigma(x_1x_2) = 3$, $\sigma(e_1) = 2$, $\sigma(e_2) = \sigma(e_3) = 3$, $\sigma(e_4) = 4$.

For $l = 3$ and $t = 3$ and for $l = 4$ and $t = 4$ we put $\sigma(v_i) = 1$, $\sigma(e_i) = \sigma(v_iv_{i+1}) = \sigma(x_ix_{i+1}) = i$ and $\sigma(x_i) = k$ for every $1 \leq i \leq l$.

For $l = 4$ and $t = 3$ we define the total 5-labeling σ as follows: $\sigma(v_i) = 1$, $\sigma(e_i) = \sigma(v_iv_{i+1}) = i$ for every $1 \leq i \leq 4$. $\sigma(x_j) = 5$ and $\sigma(x_jx_{j+1}) = j$ for every $1 \leq j \leq 3$.

It is easy to check that the weights of the edges of generalized Halin graph $H(G)$ under the labeling σ constitute the set $\{3, 4, \ldots, q + 2\}$ and the function σ is a map from $V(H(G)) \cup E(H(G))$ into $\{1, 2, \ldots, k\}$. Thus

$$tes(H(G)) \leq \left\lceil \frac{q+2}{3} \right\rceil.$$

Combining with the lower bound, we conclude that

$$tes(H(G)) = \left\lceil \frac{q+2}{3} \right\rceil.$$

This completes the proof. □

Nierhoff [20] proved that for all graphs $G = (V, E)$ with no component of order at most 2 and $G \neq K_3$, the irregularity strength of G is at most $|V(G)| - 1$. If we extend an edge labeling (irregular assignment) to the total labeling such that every vertex of graph G receives the value 1 then we obtain a vertex irregular total labeling and

$$tvs(G) \leq |V(G)| - 1. \qquad (1)$$

Ahmad and Bača in [2] showed that for Jahangir graph $J_{n,2}$, $n \geq 4$, $tvs(J_{n,2}) = \lceil \frac{n+1}{2} \rceil$ and for circulant graph $C_n(1,2)$, $n \geq 5$, $tvs(C_n(1,2)) = \lceil \frac{n+4}{5} \rceil$. Halin graph $H(S_n)$ is the wheel W_n. Wijaya and Slamin in [22] showed that $tvs(W_n) = \lceil \frac{n+3}{4} \rceil$.

A tree on p vertices is called a *double star* $S_{m,n}$ if it has exactly two vertices that are not leaves, one of degree m, say the vertex a, and the other, say the vertex b, of degree n with $p = m + n$. Let $H(S_{m,n})$ be Halin graph with double star $S_{m,n}$ as its characteristic tree. The following lemma gives the lower and upper bounds of total vertex irregularity strength for Halin graph $H(S_{m,n})$.

Lemma 1 *Let $H(S_{m,n})$ be Halin graph with the characteristic tree $S_{m,n}$ for $3 \leq m \leq n$. Then*

$$\max\left\{\left\lceil \frac{m+n+1}{4} \right\rceil, \left\lceil \frac{n+m+2}{m+1} \right\rceil, \left\lceil \frac{n+m+3}{n+1} \right\rceil\right\} \leq tvs(H(S_{m,n})) \leq n + m - 1.$$

Proof. Halin graph $H(S_{m,n})$ contains $m + n - 2$ vertices of degree 3, one vertex of degree m and one vertex of degree n. The upper bound of total vertex irregularity strength follows from (1). To prove the lower bound consider the weights of the vertices of $H(S_{m,n})$. The smallest weight among all vertices of $H(S_{m,n})$ is at least 4, so the largest weight of vertex of degree 3 is at least $n + m + 1$. Since the weight of any vertex of degree 3 is the sum of four positive integers, thus at least one label is at least $\lceil \frac{n+m+1}{4} \rceil$.

The largest value among the weights of vertices of degree 3 and m is at least $n + m + 2$ and this weight is the sum of at most $m + 1$ integers. Hence the largest label contributing to this weight must be at least $\left\lceil \frac{n+m+2}{m+1} \right\rceil$.

If we consider all vertices of Halin graph $H(S_{m,n})$ then the lower bound $\left\lceil \frac{n+m+3}{n+1} \right\rceil$ follows from Theorem 2.

This gives $\max\left\{\left\lceil \frac{n+m+1}{4} \right\rceil, \left\lceil \frac{n+m+2}{m+1} \right\rceil, \left\lceil \frac{n+m+3}{n+1} \right\rceil\right\} \leq tvs(H(S_{m,n}))$ and we are done. □

Lemma 2 $tvs(H(S_{3,3})) = tvs(H(S_{3,4})) = 3$.

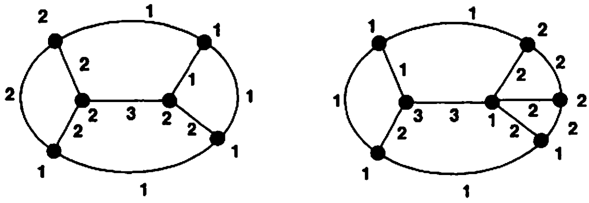

Figure 2: Halin graphs $H(S_{3,3})$ and $H(S_{3,4})$

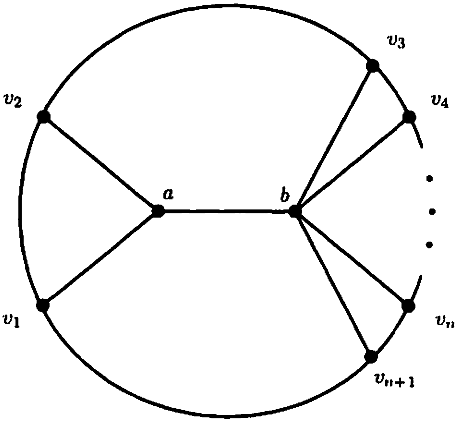

Figure 3: Halin graph $H(S_{3,n})$

Proof. From Lemma 1 it follows that $tvs(H(S_{3,3})) \geq \lceil \frac{9}{4} \rceil = 3$ and $tvs(H(S_{3,4})) \geq \max\{\lceil \frac{9}{4} \rceil, \lceil \frac{10}{5} \rceil\} = 3$. For the converse, we define a suitable vertex irregular total labelings by Figure 2. □

Theorem 4 *Let $H(S_{3,n})$ be Halin graph with the characteristic tree $S_{3,n}$ for $n \geq 5$. Then $tvs(H(S_{3,n})) = \lceil \frac{n+5}{4} \rceil$.*

Proof. Let $V(H(S_{3,n})) = \{v_i : 1 \leq i \leq n+1\} \cup \{a, b\}$ be the vertex set and $E(H(S_{3,n})) = \{v_i v_{i+1} : 1 \leq i \leq n\} \cup \{ab\} \cup \{v_{n+1}v_1\} \cup \{e_i : e_i$ is incident to v_i for $1 \leq i \leq n+1\}$ be the edge set of Halin graph $H(S_{3,n})$, see Figure 3. From Lemma 1 it follows that $tvs(H(S_{3,n})) \geq \max\{\lceil \frac{n+4}{4} \rceil, \lceil \frac{n+5}{4} \rceil, \lceil \frac{n+6}{n+1} \rceil\} = \lceil \frac{n+5}{4} \rceil$. Let $\lceil \frac{n+5}{4} \rceil = k$. To show that k is an upper bound for $tvs(H(S_{3,n}))$ we define a total k-labeling as follows.

$$\sigma(a) = \sigma(ab) = 1 \text{ and } \sigma(b) = k.$$

$$\sigma(v_i) = \begin{cases} \lceil \frac{i+1}{3} \rceil & \text{for } 1 \le i \le n+2-k \\ n+3-i & \text{for } n+4-k \le i \le n+1. \end{cases}$$

When $i = n+3-k$, then

$$\sigma(v_i) = \begin{cases} k & \text{for } n \equiv 2,3 \pmod 4 \\ \lceil \frac{n+4-k}{3} \rceil & \text{for } n \equiv 0,1 \pmod 4), \end{cases}$$

$$\sigma(e_i) = \begin{cases} \lceil \frac{i+1}{3} \rceil & \text{for } 1 \le i \le n+2-k \\ n+3-i & \text{for } n+3-k \le i \le n+1, \end{cases}$$

$$\sigma(v_i v_{i+1}) = \begin{cases} \lceil \frac{i}{3} \rceil + 1 & \text{for } 1 \le i \le n+1-k \\ n+2-i & \text{for } n+3-k \le i \le n. \end{cases}$$

When $i = n+2-k$, then

$$\sigma(v_{n+2-k} v_{n+3-k}) = \begin{cases} \lceil \frac{n+2-k}{3} \rceil + 1 & \text{for } n \equiv 0 \pmod 4) \\ k & \text{otherwise,} \end{cases}$$

and

$$\sigma(v_{n+1} v_1) = 1.$$

Thus, the vertex weights of $H(S_{3,n})$ are as follows:

1. $wt(a) = \sigma(a) + \sigma(e_1) + \sigma(e_2) + \sigma(ab) = 4$,
2. $wt(v_1) = \sigma(v_1 v_{n+1}) + \sigma(v_1) + \sigma(v_1 v_2) + \sigma(e_1) = 5$,
3. for $2 \le i \le n+1-k$

$$\begin{aligned} wt(v_i) &= \sigma(v_{i-1} v_i) + \sigma(v_i) + \sigma(v_i v_{i+1}) + \sigma(e_i) \\ &= 2\lceil (i+1)/3 \rceil + \lceil (i-1)/3 \rceil + \lceil i/3 \rceil + 2, \end{aligned}$$

4. for $n \equiv 0 \pmod 4)$

$$wt(v_{n+2-k}) = \sigma(v_{n+1-k} v_{n+2-k}) + \sigma(v_{n+2-k}) + \sigma(v_{n+2-k} v_{n+3-k}) + \sigma(e_{n+2-k}) = \lceil (n+1-k)/3 \rceil + 2\lceil (n+3-k)/3 \rceil + \lceil (n+2-k)/3 \rceil + 2,$$

5. for $n \equiv 1,\ 2,\ 3 \pmod 4)$

$$wt(v_{n+2-k}) = \sigma(v_{n+1-k} v_{n+2-k}) + \sigma(v_{n+2-k}) + \sigma(v_{n+2-k} v_{n+3-k}) + \sigma(e_{n+2-k}) = \lceil (n+1-k)/3 \rceil + 2\lceil (n+3-k)/3 \rceil + k + 1,$$

6. for $n \equiv 0 \pmod 4$

$$wt(v_{n+3-k}) = \sigma(v_{n+2-k}v_{n+3-k}) + \sigma(v_{n+3-k}) + \sigma(v_{n+3-k}v_{n+4-k}) + \sigma(e_{n+3-k}) = \lceil (n+2-k)/3 \rceil + \lceil (n+4-k)/3 \rceil + 2k,$$

7. for $n \equiv 1 \pmod 4$

$$wt(v_{n+3-k}) = \sigma(v_{n+2-k}v_{n+3-k}) + \sigma(v_{n+3-k}) + \sigma(v_{n+3-k}v_{n+4-k}) + \sigma(e_{n+3-k}) = \lceil (n+4-k)/3 \rceil + 3k - 1,$$

8. for $n \equiv 2,\ 3 \pmod 4$

$$wt(v_{n+3-k}) = \sigma(v_{n+2-k}v_{n+3-k}) + \sigma(v_{n+3-k}) + \sigma(v_{n+3-k}v_{n+4-k}) + \sigma(e_{n+3-k}) = 4k - 1,$$

9. for $n + 4 - k \leq i \leq n$

$$\begin{aligned} wt(v_i) &= \sigma(v_{i-1}v_i) + \sigma(v_i) + \sigma(v_iv_{i+1}) + \sigma(e_i) \\ &= 4n + 11 - 4i, \end{aligned}$$

10. $wt(v_{n+1}) = \sigma(v_nv_{n+1}) + \sigma(v_{n+1}) + \sigma(v_{n+1}v_1) + \sigma(e_{n+1}) = 7$, and

11. the weight of the central vertex b is

$$\begin{aligned} wt(b) &= \sigma(b) + \sigma(ab) + \sum_{i=3}^{n+1} \sigma(e_i) \\ &= k + 1 + \sum_{i=3}^{n+2-k} \lceil (i+1)/3 \rceil + \sum_{i=n+3-k}^{n+1} (n+3-i). \end{aligned}$$

One can easily check that distinct vertices in $H(S_{3,n})$ have different weights and so $tvs(H(S_{3,n})) \leq k$. Combining with the lower bound, we conclude that $tvs(H(S_{3,n})) = k$. This completes the proof. □

Theorem 5 *Let $H(S_{m,n})$ be Halin graph with the characteristic tree $S_{m,n}$ for $4 \leq m \leq n$. Then $tvs(H(S_{m,n})) = \lceil \frac{m+n+1}{4} \rceil$.*

Proof. Let $V(H(S_{m,n})) = \{v_i : 1 \leq i \leq m+n-2\} \cup \{a, b\}$ be the vertex set and $E(H(S_{m,n})) = \{v_iv_{i+1} : 1 \leq i \leq m+n-3\} \cup \{ab\} \cup \{v_{m+n-2}v_1\} \cup \{e_i : e_i$ is incident to v_i for $1 \leq i \leq m+n-2\}$ be the edge set of Halin graph $H(S_{m,n})$, see Figure 4. From Lemma 1 it follows

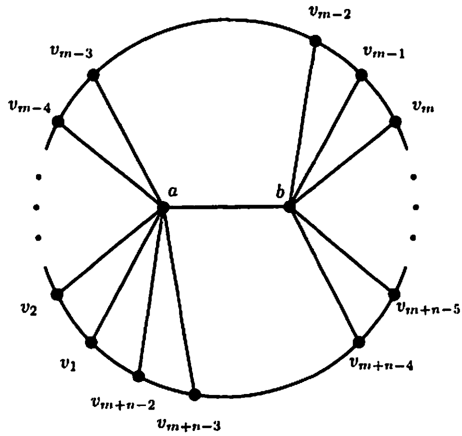

Figure 4: Halin graph $H(S_{m,n})$

that $tvs(H(S_{m,n})) \geq \max\{\lceil\frac{m+n+1}{4}\rceil, \lceil\frac{m+n+2}{m+1}\rceil, \lceil\frac{m+n+3}{n+1}\rceil\} = \lceil\frac{m+n+1}{4}\rceil$ for $3 \leq m \leq n$.

Let $k = \lceil\frac{m+n+1}{4}\rceil$. It is enough to prove that $tvs(H(S_{m,n})) \leq k$. We define a total k-labeling σ in the following way.

$$\sigma(a) = \sigma(b) = \sigma(ab) = \sigma(v_{m+n-3}) = \sigma(v_{m+n-2}) = \sigma(v_{m+n-3}v_{m+n-2}) = k,$$

$$\sigma(v_{m+n-2}v_1) = 1 \quad \text{and} \quad \sigma(v_i) = 1 \quad \text{for every} \quad 1 \leq i \leq k+2.$$

$$\sigma(e_i) = \begin{cases} i & \text{if } 1 \leq i \leq k \\ k & \text{if } k+1 \leq i \leq m+n-2, \end{cases}$$

$$\sigma(v_iv_{i+1}) = \begin{cases} 1 & \text{if } 1 \leq i \leq k \\ \lceil\frac{i-k+3}{3}\rceil & \text{if } k+1 \leq i \leq m+n-4. \end{cases}$$

For $3k-2 \leq m+n-4$

$$\sigma(v_i) = \begin{cases} \lceil\frac{i-k+1}{3}\rceil & \text{if } k+3 \leq i \leq 3k-3 \\ \lceil\frac{i-k+1}{3}\rceil + 1 & \text{if } 3k-2 \leq i \leq m+n-4. \end{cases}$$

For $3k-2 > m+n-4$

$$\sigma(v_i) = \left\lceil\frac{i-k+1}{3}\right\rceil \quad \text{for} \quad k+3 \leq i \leq m+n-4.$$

This labeling gives weight of the vertices as follows:

$$wt(a) = 4k + \sum_{i=1}^{m-3} \sigma(e_i)$$

and

$$wt(b) = 2k + \sum_{i=m-2}^{m+n-4} \sigma(e_i).$$

For $3k - 2 \leq m + n - 4$

$$wt(v_i) = \begin{cases} 3 + i & \text{if } 1 \leq i \leq 3k - 3 \\ 4 + i & \text{if } 3k - 2 \leq i \leq m + n - 4 \\ 3k + \lceil \frac{m+n-k-1}{3} \rceil & \text{if } i = m + n - 3 \\ 3k + 1 & \text{if } i = m + n - 2. \end{cases}$$

For $3k - 2 > m + n - 4$

$$wt(v_i) = \begin{cases} 3 + i & \text{if } 1 \leq i \leq m + n - 4 \\ 3k + \lceil \frac{m+n-k-1}{3} \rceil & \text{if } i = m + n - 3 \\ 3k + 1 & \text{if } i = m + n - 2. \end{cases}$$

It is a matter for routine checking to see that distinct vertices in $H(S_{m,n})$ have different weights. Thus σ is the desired vertex irregular total k-labeling. Combining with the lower bound, we conclude that

$$tvs(H(S_{m,n})) = k = \left\lceil \frac{m + n + 1}{4} \right\rceil.$$

□

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