On Potentially $C_{2,6}$-graphic Sequences *

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Abstract

For given a graph H, a graphic sequence $\pi = (d_1, d_2, \cdots, d_n)$ is said to be potentially H-graphic if there exists a realization of π containing H as a subgraph. In this paper, we characterize the potentially $C_{2,6}$-graphic sequences. This characterization partially answer the problem 6 in Lai and Hu[21].

Key words: graph; degree sequence; potentially $C_{2,6}$-graphic sequences
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1 Introduction

We consider finite simple graphs. Any undefined notation follows that of Bondy and Murty [1]. The set of all non-increasing nonnegative integer sequence $\pi = (d_1, d_2, \cdots, d_n)$ is denoted by NS_n. A sequence $\pi \epsilon NS_n$ is said to be graphic if it is the degree sequence of a simple graph G of order n; such a graph G is referred as a realization of π. The set of all graphic sequence in NS_n is denoted by GS_n. A graphic sequence π is potentially H-graphic if there is a realization of π containing H as a subgraph. Let $\sigma(\pi)$ the sum of all the terms of π, and let $[x]$ be the largest integer less than

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or equal to x. Let $G-H$ denote the graph obtained from G by removing the edges set $E(H)$ where H is a subgraph of G. We denote by $G+H$ the graph with $V(G+H)=V(G)\bigcup V(H)$ and $E(G+H)=E(G)\bigcup E(H)$. The join $G\bigvee H$ of disjoin graphs G and H is the graph obtained from $G+H$ by joining each vertex of G and H. Let K_k denote a complete graph on k vertices. The complement G^c of a simple graph G is simple graph with vertex set V, two vertices being adjacent in G^c if and only if they are not adjacent in G. In the degree sequence, r^t means r repeats t times, that is, in the realization of the sequence there are t vertices of degree r. For $1\leq m\leq \frac{n}{2}$, let $C_{m,n}$ denote the graph $K_m\bigvee(K_m^c+K_{n-2m})$ (See Bondy and Murty[1] P_{58}).

Given a graph H, what is the maximum number of edges of a graph with n vertices not containing H as a subgraph? This number is denoted by $ex(n,H)$, and is known as the Turán number. In terms of graphic sequences, the number $2ex(n,H)+2$ is the minimum even integer l such that every n-term graphical sequence π with $\sigma(\pi)\geq l$ is forcibly H-graphical. Gould, Jacobson and Lehel [8] considered the following variation of the classical Turán-type extremal problems: determine the smallest even integer $\sigma(H,n)$ such that every n-term positive graphic sequence $\pi=(d_1,d_2,\cdots,d_n)$ with $\sigma(\pi)\geq\sigma(H,n)$ has a realization G containing H as a subgraph. They proved that $\sigma(pK_2,n)=(p-1)(2n-p)+2$ for $p\geq 2$; $\sigma(C_4,n)=2[\frac{3n-1}{2}]$ for $n\geq 4$. Erdös, Jacobson and Lehel [4] showed that $\sigma(K_k,n)\geq(k-2)(2n-k+1)+2$ and conjectured that the equality holds. In the same paper, they proved the conjecture is true for $k=3$ and $n\geq 6$. The conjecture is confirmed in [8] and [22,23,24,25]. Ferrara, Gould and Schmitt [6] provided a graph theoretic proof for the value of $\sigma(K_t,n)$. Ferrara, Gould and Schmitt [7] determined $\sigma(F_k,n)$ where F_k denotes the graph of k triangles intersecting at exactly one common vertex. Yin, Chen and Schmitt [35] determined $\sigma(F_{t,r,k},n)$ for $k\geq 2$, $t\geq 3$, $1\leq r\leq t-2$ and n sufficiently large. Recently, Li and Yin [27] further determined $\sigma(K_r,n)$ for $r\geq 7$ and $n\geq 2r+1$. The problem of determining $\sigma(K_r,n)$ is completely solved. Yin et al.[36,37,39,40] determined $\sigma(K_{r,s},n)$ for $s\geq r\geq 1$ and sufficiently large n. Yin, Li, and Mao [41] determined $\sigma(K_{r+1}-e,n)$ for $r\geq 3$ and $r+1\leq n\leq 2r$ and $\sigma(K_5-e,n)$ for $n\geq 5$. Yin and Li[38] gave a good method (Yin-Li method) of determining the values $\sigma(K_{r+1}-e,n)$. After reading[38], using Yin-Li method Yin [42] determined $\sigma(K_{r+1}-K_3,n)$ for $r\geq 3$ and $n\geq 3r+5$. Yin[33] and Lai[18] independently determined $\sigma(K_{1,1,3},n)$.

Lai [16, 17] determined $\sigma(K_5 - C_4, n)$, $\sigma(K_5 - P_3, n)$, $\sigma(K_5 - P_4, n)$ for $n \geq 5$. Determining $\sigma(K_{r+1} - H, n)$, where H is a tree on 4 vertices is more useful than a cycle on 4 vertices (for example, $C_4 \not\subset C_i$, but $P_3 \subset C_i$ for $i \geq 5$). So, after reading [38] and [42], using Yin-Li method Lai and Hu [20] determined $\sigma(K_{r+1} - H, n)$ for $n \geq 4r + 10$, $r \geq 3$, $r + 1 \geq k \geq 4$ and H be a graph on k vertices which containing a tree on 4 vertices but not contain a cycle on 3 vertices and $\sigma(K_{r+1} - P_2, n)$ for $n \geq 4r + 8$, $r \geq 3$. Using Yin-Li method Lai [19] determined $\sigma(K_{r+1} - Z_4, n)$, $\sigma(K_{r+1} - (K_4 - e), n)$, $\sigma(K_{r+1} - K_4, n)$ for $n \geq 5r + 16$, $r \geq 4$ and $\sigma(K_{r+1} - Z, n)$ for $n \geq 5r + 19$, $r + 1 \geq k \geq 5$, $j \geq 5$ where Z is a graph on k vertices and j edges which contains a graph Z_4 but not contain a cycle on 4 vertices.

A harder question is to characterize the potentially H-graphic sequences without zero terms. Luo [29] characterized the potentially C_k-graphic sequences for each $k = 3, 4, 5$. Luo and Warner [30] characterized the potentially K_4-graphic sequences. Eschen and Niu [5] characterized the potentially $K_4 - e$-graphic sequences. Yin and Chen [34] characterized the potentially $K_{r,s}$-graphic sequences for $r = 2, s = 3$ and $r = 2, s = 4$. Yin and Yin [44] characterized the potentially $K_5 - e$, $K_6 - e$ and K_6-graphic sequences. Hu and Lai [9,10,11] characterized the potentially $K_5 - C_4$, $K_5 - P_4$ and $K_5 - E_3$-graphic sequences where E_3 denotes graphs with 5 vertices and 3 edges. Hu and Lai [12,13,14] characterized potentially $K_{3,3}$, $K_6 - C_6$, $K_6 - C_4$ and $K_{2,5}$-graphic sequences. Recently, Xu and Lai [32] characterized potentially $K_6 - C_5$-graphic sequences. Chen [2] characterized potentially $K_6 - 3K_2$-graphic sequences. Yin [43] characterized potentially $K_6 - E(K_3)$-graphic sequences. Yin et al. [45] characterized potentially $K_{1,1,s}$-graphic sequences, for $s = 4$ and 5. Yin, Zhong, and Yang [46] characterized potentially $K_{1,1,6}$-graphic sequences, they also give a simple sufficient condition for a positive graphic sequence $\pi = (d_1, d_2, \cdots, d_n)$ to be potentially $K_{1,1,s}$-graphic for $n \geq s + 2$ and $s \geq 2$. Liu and Lai [28] characterized potentially $K_{1,1,2,2}$-graphic sequences.

In this paper, we characterize potentially $C_{2,6}$-graphic sequences(Diagram of $C_{2,6}$ is shown in Appendix Figure 1). This charaterization partially answer the problem 6 in Lai and Hu[21]. This characterization implies a special situation due to Lai[19].

2 Preparations

Let $\pi = (d_1, \cdots, d_n) \epsilon NS_n, 1 \le k \le n$. Let

$$\pi''_k = \begin{cases} (d_1 - 1, \cdots, d_{k-1} - 1, d_{k+1} - 1, \cdots, d_{d_k+1} - 1, d_{d_k+2}, \cdots, d_n), \\ \text{ if } d_k \ge k, \\ (d_1 - 1, \cdots, d_{d_k} - 1, d_{d_k+1}, \cdots, d_{k-1}, d_{k+1}, \cdots, d_n), \\ \text{if } d_k < k. \end{cases}$$

Denote $\pi'_k = (d'_1, d'_2, \cdots, d'_{n-1})$, where $d'_1 \ge d'_2 \ge \cdots \ge d'_{n-1}$ is a rearrangement of the $n-1$ terms of π''_k. Then π'_k is called the residual sequence obtained by laying off d_k from π. For simplicity, we denote π'_n by π' in this paper.

For a nonincreasing positive integer sequence $\pi = (d_1, d_2, \cdots, d_n)$, we write $m(\pi)$ and $h(\pi)$ to denote the largest positive terms of π and the smallest positive terms of π, respectively. We need the following results.

Theorem 2.1 [8] If $\pi = (d_1, d_2, \cdots, d_n)$ is a graphic sequence with a realization G containing H as a subgraph, then there exists a realization G' of π containing H as a subgraph so that the vertices of H have the largest degrees of π.

Theorem 2.2 [26] If $\pi = (d_1, d_2, \cdots, d_n)$ is a sequence of nonnegative integers with $1 \le m(\pi) \le 2$, $h(\pi) = 1$ and even $\sigma(\pi)$, then π is graphic.

Lemma 2.3 (Kleitman and Wang [15]) π is graphic if and only if π' is graphic.

The following corollary is obvious.

Corollary 2.4 Let H be a simple graph. If π' is potentially H-graphic, then π is potentially H-graphic.

Theorem 2.5[3] Let $\pi = (d_1, \cdots, d_n) \in NS_n$ with even $\sigma(\pi)$. Then $\pi \in GS_n$ if and only if for any t,$1 \le t \le n-1$,

$$\sum_{i=1}^{t} d_i \le t(t-1) + \sum_{j=t+1}^{n} min\{t, d_j\}.$$

Theorem 2.6[19] If $r \ge 4$ and $n \ge r+1$, then $\sigma(K_{r+1} - Z_4, n) \ge \sigma(K_{r+1} - K_4, n)$, and

$$\sigma(K_{r+1} - K_4, n) \ge \begin{cases} (r-1)(2n-r) - 3(n-r) + 1, \\ \text{ if } n - r \text{ is odd} \\ (r-1)(2n-r) - 3(n-r) + 2, \\ \text{if } n - r \text{ is even} \end{cases}$$

Theorem 2.7 [31] Let $\pi = (4^{x_1}, 3^{x_2}, 2^{x_3}, 1^{x_4})$ where $\sigma(\pi)$ is even, $x_1 + x_2 + x_3 + x_4 = n$ and $n \geq 1$. Then $\pi \in GS_n$ if and only if $\pi \notin S$, where $S = \{(2),(2^2),(3,1),(3^2),(3,2,1),(3^2,2),(3^3,1), (3^2,1^2),(4),(4,1^2), (4,2) ,(4,2^2),(4,2^3),(4,2,1^2), (4,3^2), (4,3^2,2), (4,3,1), (4,3,1^3), (4,3^2,1^2), (4,3,2,1), (4^2), (4^2,1^2),(4^2,1^4), (4^2,2,1^2), (4^2,2), (4^2,2^2), (4^2,3^2), (4^2,3,1), (4^2,3,1^3), (4^2,3,2,1), (4^3), (4^3,1^2), (4^3,2,1^2), (4^3,1^4), (4^3,2), (4^3,2^2), (4^3,3,1), (4^4), (4^4,1^2), (4^4,2)\}$.

Before proving the result of Theorem 3.1, we need to develop Lemma 2.8-Lemma 2.13. Let the degree sequence of $C_{2,6}$ is π_1, so $\pi_1 = (d'_1, d'_2, d'_3, d'_4, d'_5, d'_6)= (5^2, 3^2, 2^2)$ and $\pi^{**} =(d_1 - d'_1,\ d_2 - d'_2, d_3 - d'_3, d_4 - d'_4, d_5 - d'_5, d_6 - d'_6, d_7, \cdots, d_n)$. We denote π^* is subsequence of π^{**} without the component 0. Let H be a simple graph, the graphic sequence of H is $\pi_H = (d''_1, d''_2, d''_3, d''_4, d''_5, d''_6)$ and $\pi_H^{**} = \pi - \pi_H=(d_1 - d''_1,\ d_2 - d''_2, d_3 - d''_3, d_4 - d''_4, d_5 - d''_5, d_6 - d''_6, d_7, \cdots, d_n)$. We denote π_H^* is subsequence of π_H^{**} without the component 0.

Lemma 2.8 If $\pi = (d_1, 5^i, 4^{n-1-i}) \in GS_n$, where $1 \leq i \leq 3$ and $n \geq 6$, then π is potentially $C_{2,6}$ graphic.

Proof: Case 1: $i = 1$. $\pi = (d_1, 5, 4^{n-2})$. Let $H = C_{2,6} + \{v_3v_5, v_4v_6, v_5v_6\}$. So $\pi_H = (5^2, 4^4)$. Thus, $\pi_H^* = (d_1 - 5, 4^{n-6})$. Therefore, $(\pi_H^*)' = (4^{(n-6)-(d_1-5)}, 3^{d_1-5})$, where $(\pi_H^*)'$ is the the residual sequence obtained by laying off $d_1 - 5$ from π_H^*.

If $n = 6$, then $\pi = (5^2, 4^4)$. It is easy to verify that π is potentially $C_{2,6}$-graphic.

By Theorem 2.7, if $(\pi_H^*)' \notin S$, then $(\pi_H^*)'$ is graphic. Hence π is potentially H-graphic. Therefore, π is potentially $C_{2,6}$-graphic. By Theorem 2.7, if $(\pi_H^*)' \in S$, then $(\pi_H^*)'=(4),(4^2),(4^3),(4^4),(3^2),(4,3^2)$ or $(4^2,3^2)$. Hence π is one of those sequences: $(5^2,4^5)$, $(5^2,4^6)$,$(5^2,4^7)$,$(5^2,4^8)$,$(7,5,4^6)$,$(7,5,4^7)$, $(7,5,4^8)$. It is easy to verify that all of these are potentially $C_{2,6}$-graphic.

Case 2: $i = 2$. $\pi = (d_1, 5^2, 4^{n-3})$. Let $H = C_{2,6} + \{v_3v_5, v_4v_6, v_5v_6\}$. So $\pi_H = (5^2, 4^4)$. Thus $\pi_H^* = (d_1 - 5, 1, 4^{n-6})$. Therefore, $(\pi_H^*)' = (4^{(n-6)-(d_1-5)}, 3^{d_1-5}, 1)$. By Theorem 2.7, if $(\pi_H^*)' \notin S$, then $(\pi_H^*)'$ is graphic. Hence π is potentially H-graphic. Therefore, π is potentially $C_{2,6}$-graphic. By Theorem 2.7, if $(\pi_H^*)' \in S$, then $(\pi_H^*)' = (3,1), (3^3,1), (4,3,1), (4^2,3,1), (4^3,3,1)$. Hence π is one of those sequences: $(6,5^2,4^4)$,$(8,5^2,4^6)$, $(6,5^2,4^5)$, $(6,5^2,4^6)$, $(6,5^2,4^7)$. It is easy to verify that all of these are potentially $C_{2,6}$-graphic.

Case 3: $i = 3$. $\pi = (d_1, 5^3, 4^{n-4})$. Let $H = C_{2,6}+\{v_3v_5, v_3v_6, v_4v_5, v_4v_6\}$.

So $\pi_H = (5^4, 4^2)$. Thus $\pi_H^* = (d_1 - 5, 4^{n-6})$. Thus, $(\pi_H^*)' = (4^{(n-6)-(d_1-5)}, 3^{d_1-5})$. By Theorem 2.7, if $(\pi_H^*)' \notin S$, then $(\pi_H^*)'$ is graphic. Hence π is potentially H-graphic. Therefore, π is potentially $C_{2,6}$-graphic. By Theorem 2.7, if $(\pi_H^*)' \in S$, then $(\pi_H^*)' = (4), (4^2), (4^3), (4^4), (3^2), (4, 3^2)$ or $(4^2, 3^2)$. Hence π is one of those sequences: $(5^4, 4^3)$,$(5^4, 4^4)$, $(5^4, 4^5)$, $(5^4, 4^6)$, $(7, 5^3, 4^4)$, $(7, 5^3, 4^5)$,$(7, 5^3, 4^6)$. It is easy to verify that all of these are potentially $C_{2,6}$-graphic.

Lemma 2.9 If $\pi = (5^4, 4^i, 3^{n-4-i}) \in GS_n$, where $n \geq 6$ and $n-4-i \geq 1$, then π is potentially $C_{2,6}$ graphic.

Proof: Case 1: $i = 0$. $\pi = (5^4, 3^{n-4})$. If $n = 6$, then $\pi = (5^4, 3^2)$, which contradict $\pi \in GS_n$. So $n \geq 6$, let $H = C_{2,6} + \{v_3v_5, v_4v_6\}$. So $\pi_H = (5^2, 4^2, 3^2)$. Thus $\pi_H^* = (1, 1, 3^{n-6})$. By Theorem 2.7, if $\pi_H^* \notin S$, then π_H^* is graphic. Hence π is potentially H-graphic. Therefore, π is potentially $C_{2,6}$-graphic. By Theorem 2.7, if $\pi_H^* \in S$, then $\pi_H^* = (3^2, 1^2)$. Hence $\pi = (5^4, 3^4)$. It is obvious that π is potentially $C_{2,6}$-graphic.

Case 2: $i = 1$. $\pi = (5^4, 4, 3^{n-5})$. Let $H = C_{2,6} + \{v_3v_5, v_3v_6, v_4v_5\}$. So $\pi_H = (5^3, 4^2, 3)$. Thus $\pi_H^* = (1, 3^{n-6})$. By Theorem 2.7, if $\pi_H^* \notin S$, then π_H^* is graphic. Hence π is potentially H-graphic. Therefore, π is potentially $C_{2,6}$-graphic. By Theorem 2.7, if $\pi_H^* \in S$, then $\pi_H^* = (3, 1), (3^3, 1)$. Hence $\pi = (5^4, 4, 3^2), (5^4, 4, 3^4)$. It is obvious that π is potentially $C_{2,6}$-graphic.

Case 3: $i \geq 2$. $\pi = (5^4, 4^i, 3^{n-4-i})$. Let $H = C_{2,6} + \{v_3v_5, v_3v_6, v_4v_5, v_4v_6\}$. So $\pi_H = (5^4, 4^2)$. Thus $\pi_H^* = (4^{i-2}, 3^{n-4-i})$. By Theorem 2.7, if $\pi_H^* \notin S$, then π_H^* is graphic. Hence π is potentially H-graphic. Therefore, π is potentially $C_{2,6}$-graphic. By Theorem 2.7, if $\pi_H^* \in S$, then $\pi_H^*=$ (3^2),$(4, 3^2)$,$(4^2, 3^2)$. Hence $\pi = (5^4, 4^2, 3^2)$,$(5^4, 4^3,\ 3^2)$,$(5^4, 4^4, 3^2)$. It is obvious that π is potentially $C_{2,6}$-graphic.

Lemma 2.10 Let $\pi = (d_1, 5^i, 4^j, 3^{n-1-i-j}) \in GS_n$, where i = 1,2, $n \geq 6$ and $n - 1 - i - j \geq 1$. Then π is potentially $C_{2,6}$ graphic if and only if $\pi \neq (5^2, 3^6)$.

Proof: Case 1:$i = 1$. $\pi = (d_1, 5, 4^j, 3^{n-2-j})$, where $n - 2 - j \geq 1$.

Case 1.1: $j = 0$. $\pi = (d_1, 5, 3^{n-2})$. Let $H = C_{2,6} + \{v_5v_6\}$. So $\pi_H = (5^2, 3^4)$. Thus $\pi_H^* = (d_1 - 5, 3^{n-6})$ and $(\pi_H^*)' = (3^{(n-6)-(d_1-5)}, 2^{d_1-5})$.

If $n = 6$, then $\pi = (5^2, 3^4)$. It is easy to verify that π is potentially $C_{2,6}$-graphic.

By Theorem 2.7, if $(\pi_H^*)' \notin S$, then $(\pi_H^*)'$ is graphic. Hence π is potentially H-graphic. Therefore, π is potentially $C_{2,6}$-graphic. By Theorem 2.7, if $(\pi_H^*)' \in S$, then $(\pi_H^*)' = (3^2), (3^2, 2), (2), (2^2)$. Hence π is one of those sequences: $(5^2,3^6),(6,5,3^7),(6,5,3^5),(7,5,3^6)$. It is easy to observe that $(5^2,3^6)$ is not potentially $C_{2,6}$-graphic but the others are.

Case 1.2: $j=1$. $\pi=(d_1,5,4,3^{n-3})$. Let $H=C_{2,6}+\{v_5v_6\}$. So $\pi_H=(5^2,3^4)$. Thus $\pi_H^*=(d_1-5,1,3^{n-6})$ and $(\pi_H^*)'=(3^{(n-6)-(d_1-5)}, 2^{d_1-5},1)$. By Theorem 2.7, if $(\pi_H^*)'\notin S$, then $(\pi_H^*)'$ is graphic. Hence π is potentially H-graphic. Therefore, π is potentially $C_{2,6}$-graphic. By Theorem 2.7, if $(\pi_H^*)'\in S$, then $(\pi_H^*)'=(3,1),(3,2,1),(3^3,1)$. Hence π is one of those sequences: $(5^2,4,3^4),(6,5,4,3^5),(5^2,4,3^6)$. It is easy to check that π is potentially $C_{2,6}$-graphic.

Case 1.3: $j=2$. $\pi=(d_1,5,4^2,3^{n-4})$. Let $H=C_{2,6}+\{v_3v_5,v_4v_6\}$. So $\pi_H=(5^2,4^2,3^2)$. Thus $\pi_H^*=(d_1-5,3^{n-6})$ and $(\pi_H^*)'=(3^{(n-6)-(d_1-5)}, 2^{d_1-5})$. By Theorem 2.7, if $(\pi_H^*)'\notin S$, then $(\pi_H^*)'$ is graphic. Hence π is potentially H-graphic. Therefore, π is potentially $C_{2,6}$-graphic. By Theorem 2.7, if $(\pi_H^*)'\in S$, then $(\pi_H^*)'=(3^2),(3^2,2),(2),(2^2)$. Hence π is one of those sequences: $(5^2,4^2,3^4)$,$(6,5,4^2,3^5)$, $(6,5,4^2,3^3)$,$(7,5,4^2,3^4)$. It is easy to check that π is potentially $C_{2,6}$-graphic.

Case 1.4: $j=3$. $\pi=(d_1,5,4^3,3^{n-5})$. Let $H=C_{2,6}+\{v_3v_5,v_4v_5\}$. So $\pi_H=(5^2,4^3,2)$. Thus $\pi_H^*=(d_1-5,1,3^{n-6})$ and $(\pi_H^*)'=(3^{(n-6)-(d_1-5)}, 2^{d_1-5},1)$. By Theorem 2.7, if $(\pi_H^*)'\notin S$, then $(\pi_H^*)'$ is graphic. Hence π is potentially H-graphic. Therefore, π is potentially $C_{2,6}$-graphic. By Theorem 2.7, if $(\pi_H^*)'\in S$, then $(\pi_H^*)'=(3,1),(3,2,1),(3^3,1)$. Hence π is one of those sequences: $(5^2,4^3,3^2)$, $(6,5,4^3,3^3)$,$(5^2,4^3,3^4)$. It is easy to check that π is potentially $C_{2,6}$-graphic.

Case 1.5: $j\geq 4$. $\pi=(d_1,5,4^j,3^{n-2-j})$. Let $H=C_{2,6}+\{v_3v_5,v_4v_6, v_5v_6\}$. So $\pi_H=(5^2,4^4)$. Thus $\pi_H^*=(d_1-5,4^{j-4},3^{n-2-j})$.

Case 1.5.1: $d_1-5\leq j-4$

Let $(\pi_H^*)'=(4^{(j-4)-(d_1-5)},3^{(d_1-5)-(n-2-j)})$. By Theorem 2.7, if $(\pi_H^*)'\notin S$, then $(\pi_H^*)'$ is graphic. Hence π is potentially H-graphic. Therefore, π is potentially $C_{2,6}$-graphic. If $(\pi_H^*)'\in S$, by Theorem 2.7, then $(\pi_H^*)'=(3^2),(4,3^2),(4^2,3^2)$. Hence, π is one of those sequences: $(6,5,4^5,3),(5^2,4^4, 3^2),(6,5,4^7,3),(5^2,4^6,3^2)$. It is easy to check that π is potentially $C_{2,6}$-graphic.

Case 1.5.2: $d_1-5>j-4$

Let $(\pi_H^*)'=(3^{(n-6)+(j-4)-(d_1-5)},2^{(d_1-5)+(j-4)})$. By Theorem 2.7, if $(\pi_H^*)'\notin S$, then $(\pi_H^*)'$ is graphic. Hence π is potentially H-graphic. Therefore, π is potentially $C_{2,6}$-graphic. If $(\pi_H^*)'\in S$, by Theorem 2.7, then $(\pi_H^*)'=(3^2,2),(2),(2^2)$. Hence, π is one of those sequences: $(8,5,4^6,3)$,

$(7,5,4^5,3^2),(6,5,4^4,3),(7,5,4,3^2)$. It is easy to check that π is potentially $C_{2,6}$-graphic.

Case 2: $i=2$. $\pi=(d_1,5^2,4^j,3^{n-3-j})$, where $n-3-j\geq 1$.

Case 2.1: $j=0$. $\pi=(d_1,5^2,3^{n-3})$. Let $H=C_{2,6}+\{v_3v_5,v_3v_6\}$. So $\pi_H=(5^3,3^3)$. Thus $\pi_H^*=(d_1-5,3^{n-6})$ and $(\pi_H^*)'=(3^{(n-6)-(d_1-5)},2^{d_1-5})$. By Theorem 2.7, if $(\pi_H^*)'\notin S$, then $(\pi_H^*)'$ is graphic. Hence π is potentially H-graphic. Therefore, π is potentially $C_{2,6}$-graphic. By Theorem 2.7, if $(\pi_H^*)'\in S$, then $(\pi_H^*)'=(3^2),(3^2,2),(2),(2^2)$. Hence, π is one of those sequences: $(5^3,3^5),(6,5^2,3^6),(6,5^2,3^4),(7,5^2,3^5)$. It is easy to observe that π is potentially $C_{2,6}$-graphic.

Case 2.2: $j=1$. $\pi=(d_1,5^2,4,3^{n-4})$. Let $H=C_{2,6}+\{v_3v_5,v_3v_6\}$. So $\pi_H=(5^3,3^3)$. Thus $\pi_H^*=(d_1-5,1,3^{n-6})$ and $(\pi_H^*)'=(3^{(n-6)-(d_1-5)},2^{d_1-5},1)$. By Theorem 2.7, if $(\pi_H^*)'\notin S$, then $(\pi_H^*)'$ is graphic. Hence π is potentially H-graphic. Therefore, π is potentially $C_{2,6}$-graphic. By Theorem 2.7, if $(\pi_H^*)'\in S$, then $(\pi_H^*)'=(3,1),(3,2,1),(3^3,1)$. Hence, π is one of those sequences: $(5^3,4,3^3),(6,5^2,4^2,3^4),(5^3,4,3^5)$. It is easy to observe that π is potentially $C_{2,6}$-graphic.

Case 2.3: $j=2$. $\pi=(d_1,5^2,4^2,3^{n-5})$. Let $H=C_{2,6}+\{v_3v_5,v_3v_6,v_4v_5\}$. So $\pi_H=(5^3,4^2,3)$. Thus $\pi_H^*=(d_1-5,3^{n-6})$ and $(\pi_H^*)'=(3^{(n-6)-(d_1-5)},2^{d_1-5})$. By Theorem 2.7, if $(\pi_H^*)'\notin S$, then $(\pi_H^*)'$ is graphic. Hence π is potentially H-graphic. Therefore, π is potentially $C_{2,6}$-graphic. By Theorem 2.7, if $(\pi_H^*)'\in S$, then $(\pi_H^*)'=(3^2),(3^2,2),(2),(2^2)$. Hence, π is one of those sequences: $(5^3,4^2,3^3),(6,5^2,4^2,3^4),(6,5^2,4^2,3^2),(7,5^2,4^2,3^3)$. It is easy to observe that π is potentially $C_{2,6}$-graphic.

Case 2.4: $j\geq 3$. $\pi=(d_1,5^2,4^j,3^{n-3-j})$. Let $H=C_{2,6}+\{v_3v_5,v_3v_6,v_4v_5\}$. So $\pi_H=(5^3,4^2,3)$. Thus $\pi_H^*=(d_1-5,1,4^{(j-3)},3^{n-3-j})$.

Case 2.4.1: $d_1-5\leq j-3$. Let $(\pi_H^*)'=(4^{(j-3)-(d_1-5)},3^{(n-3-j)+(d_1-5)},1)$. By Theorem 2.7, if $(\pi_H^*)'\notin S$, then $(\pi_H^*)'$ is graphic. Hence π is potentially H-graphic. Therefore, π is potentially $C_{2,6}$-graphic. By Theorem 2.7, if $(\pi_H^*)'\in S$, then $(\pi_H^*)'=(3,1)$, $(3^3,1)$, $(4,3,1)$, $(4^2,3,1)$, $(4^3,3,1)$. Hence, π is one of those sequences: $(5^3,4^3,3)$, $(7,5^2,4^5,3)$, $(6,5^2,4^4,3^2)$, $(5^3,4^3,3^3)$, $(5^3,4^4,3)$, $(5^3,4^5,3)$, $(5^3,4^6,3)$. It is easy to observe that π is potentially $C_{2,6}$-graphic.

Case 2.4.2: $d_1-5>j-3$.

Let $(\pi_H^*)'=(3^{(n-6)+(j-3)-(d_1-5)},2^{(d_1-5)-(j-3)},1)$. By Theorem 2.7, if $(\pi_H^*)'\notin S$, then $(\pi_H^*)'$ is graphic. Hence π is potentially H-graphic. Therefore, π is potentially $C_{2,6}$-graphic. By Theorem 2.7, if $(\pi_H^*)'\in S$, then $(\pi_H^*)' = (3,2,1)$. Thus, $\pi = (6,5^2,4^3,3^2)$. It is easy to observe that π is potentially $C_{2,6}$-graphic.

Lemma 2.11 If $\pi = (5^3,4^i,3^j,2^{n-3-i-j}) \in GS_n$, where $n-3-i-j \geq 1$, $n \geq 7$ and j is odd, then π is potentially $C_{2,6}$ graphic if and only if $\pi \neq (5^3,3,2^3)$.

Proof: Case 1:$i = 0$. $\pi = (5^3,3^j,2^{n-3-j})$, where $n-3-j \geq 1$.

Case 1.1: $j = 1$. $\pi = (5^3,3,2^{n-4})$. Thus $\pi^* = (2,2^{n-6})$. By Theorem 2.7, if $\pi^* \notin S$, then π is potentially H-graphic. By Theorem 2.7, if $\pi^* \in S$, then $\pi^* = (2,2)$. Hence $\pi = (5^3,3,2^3)$. It is easy to observe that $(5^3,3,2^3)$ is not potentially $C_{2,6}$-graphic.

Case 1.2: $j \geq 3$. $\pi = (5^3,3^j,2^{n-3-j})$. Thus $\pi^* = (2,3^{j-3},2^{n-3-j})$. By Theorem 2.7, if $\pi^* \notin S$, then π is potentially H-graphic. By Theorem 2.7, if $\pi^* \in S$, then $\pi^* = (2,2)$. Hence $\pi = (5^3,3^3,2)$. It is easy to observe that $(5^3,3^3,2)$ is potentially $C_{2,6}$-graphic.

Case 2:$i = 1$. $\pi = (5^3,4,3^j,2^{n-4-j})$, where $n-4-j \geq 1$.

Case 2.1: $j = 1$. $\pi = (5^3,4,3,2^{n-5})$. Let $H = C_{2,6} + \{v_3v_5\}$. So $\pi_H = (5^2,4,3^2,2)$. Thus $\pi_H^* = (1,1,2^{n-6})$. By Theorem 2.7, if $\pi_H^* \notin S$, then π_H^* is graphic. Hence π is potentially H-graphic. Therefore, π is potentially $C_{2,6}$-graphic. By Theorem 2.7, $\pi_H^* \notin S$, so π is potentially $C_{2,6}$-graphic.

Case 2.2: $j \geq 3$. $\pi = (5^3,4,3^j,2^{n-4-j})$. Let $H = C_{2,6} + \{v_3v_5, v_3v_6\}$. So $\pi_H = (5^3,3^3)$. Thus $\pi_H^* = (1,3^{j-2},2^{n-4-j})$. By Theorem 2.7, if $\pi_H^* \notin S$, then π_H^* is graphic. Hence π is potentially H-graphic. Therefore, π is potentially $C_{2,6}$-graphic. By Theorem 2.7, if $\pi_H^* \in S$, then $\pi_H^* = (3,2,1)$. Hence $\pi = (5^3,4,3^3,2)$. It is easy to observe that $(5^3,4,3^3,2)$ is potentially $C_{2,6}$-graphic.

Case 3: $i = 2$. $\pi = (5^3,4^2,3^j,2^{n-5-j})$. Let $H = C_{2,6}+\{v_3v_5, v_3v_6, v_4v_5\}$. So $\pi_H = (5^3,4^2,3)$. Thus $\pi_H^* = (1,1,3^{j-1},2^{n-5-j})$. By Theorem 2.7, if $\pi_H^* \notin S$, then π_H^* is graphic. Hence π is potentially H-graphic. Therefore, π is potentially $C_{2,6}$-graphic. By Theorem 2.7, it is obvious that $\pi_H^* \notin S$, therefor π is potentially $C_{2,6}$-graphic.

Case 4: $i \geq 3$. $\pi = (5^3,4^i, 3^j,2^{n-3-i-j})$. Let $H = C_{2,6} + \{v_3v_5, v_3v_6, v_4v_5\}$. So $\pi_H = (5^3,4^2,3)$. Thus $\pi_H^* = (1,4^{i-3},3^j,2^{n-3-i-j})$. By Theorem 2.7, if $\pi_H^* \notin S$, then π_H^* is graphic. Hence π is potentially H-graphic. Therefore, π is potentially $C_{2,6}$-graphic. If $\pi_H^* \in S$, by Theorem 2.7, then $\pi_H^* = (3,2,1), (4,3,2,1), (4^2,3,2,1)$. Hence, $\pi = (5^3,4^3,3,2)$, $(5^3,4^4,3,2)$, $(5^3,4^5,3,2)$. It is easy to observe that π is potentially $C_{2,6}$-graphic.

Lemma 2.12 If $\pi = (d_1, 5, 4^i, 3^j, 2^{n-2-i-j}) \in GS_n$, where $n \geq 6$ and $n - i - j - 2 \geq 1$, then π is potentially $C_{2,6}$ graphic if and only if $\pi \neq (5^2, 3^2, 2^3)$ and $(5^2, 3^2, 2^4)$.

Proof: Case 1: $i = 0$. $\pi = (d_1, 5, 3^j, 2^{n-2-j})$, where $n - 2 - j \geq 1$.

Case 1.1: $j = 2$. $\pi = (d_1, 5, 3^2, 2^{n-4})$, then $\pi^* = (d_1 - 5, 2^{n-6})$. Let $(\pi^*)' = (2^{(n-6)-(d_1-5)}, 1^{d_1-5})$. By Theorem 2.7, if $(\pi^*)' \in S$, then $(\pi^*)' = (2), (2^2)$. Hence, $\pi = (5^2, 3^2, 2^3), (5^2, 3^2, 2^4)$. It is easy to see that π is not potentially $C_{2,6}$-graphic.

Case 1.2: $j = 3$. $\pi = (d_1, 5, 3^3, 2^{n-5})$, then $\pi^* = (d_1 - 5, 1, 2^{n-6})$. Let $(\pi^*)' = (2^{(n-6)-(d_1-5)}, 1^{d_1-5}, 1)$. By Theorem 2.7, $(\pi^*)' \notin S$, thus π is potentially $C_{2,6}$-graphic.

Case 1.3: $j \geq 4$. $\pi = (d_1, 5, 3^j, 2^{n-2-j})$. Let $H = C_{2,6} + \{v_5v_6\}$. So $\pi_H = (5^2, 3^4)$. Thus $\pi_H^* = (d_1 - 5, 3^{j-4}, 2^{n-2-j})$.

If $d_1 - 5 \leq j - 4$, let $(\pi_H^*)' = (3^{(j-4)-(d_1-5)}, 2^{(n-2-j)+(d_1-5)})$. By Theorem 2.7, if $(\pi_H^*)' \notin S$, then $(\pi_H^*)'$ is graphic. Hence π is potentially H-graphic. Therefore, π is potentially $C_{2,6}$-graphic. If $(\pi_H^*)' \in S$, by Theorem 2.7, then $(\pi_H^*)' = (3^2, 2), (2), (2^2)$. Hence $\pi = (5^2, 3^6, 2)$, $(5^2, 3^4, 2)$, $(6, 5, 3^5, 2)$, $(5^2, 3^4, 2^2)$. It is easy to observe that π is potentially $C_{2,6}$-graphic.

If $j - 4 < d_1 - 5 \leq n - 6$, let $(\pi_H^*)' = (2^{(n-6)+(j-4)-(d_1-5)}, 1^{(d_1-5)-(j-4)})$. By Theorem 2.7, if $(\pi_H^*)' \notin S$, then $(\pi_H^*)'$ is graphic. Hence π is potentially H-graphic. Therefore, π is potentially $C_{2,6}$-graphic. By Theorem 2.7, $(\pi_H^*)' \notin S$. Hence π is potentially $C_{2,6}$-graphic.

Case 2: $i = 1$. $\pi = (d_1, 5, 4, 3^j, 2^{n-3-j})$, where $n - 3 - j \geq 1$.

Case 2.1: $j = 1$. $\pi = (d_1, 5, 4, 3, 2^{n-4})$. Then $\pi^* = (d_1 - 5, 1, 2^{n-6})$ and $(\pi^*)' = (2^{(n-6)-(d_1-5)}, 1^{d_1-5}, 1)$. By Theorem 2.7, $(\pi^*)' \notin S$, then π is potentially $C_{2,6}$-graphic.

Case 2.2: $j = 2$. $\pi = (d_1, 5, 4, 3^2, 2^{n-5})$. Let $H = C_{2,6} + \{v_3v_5\}$. So $\pi_H = (5^2, 4, 3^2, 2)$. Thus $\pi_H^* = (d_1 - 5, 2^{n-6})$ and $(\pi_H^*)' = (2^{(n-6)-(d_1-5)}, 1^{d_1-5})$. By Theorem 2.7, if $(\pi_H^*)' \notin S$, then $(\pi_H^*)'$ is graphic. Hence π is potentially H-graphic. Therefore, π is potentially $C_{2,6}$-graphic. If $(\pi_H^*)' \in S$, by Theorem 2.7, then $(\pi_H^*)' = (2), (2^2)$. Hence $\pi = (5^2, 4, 3^2, 2^2)$, $(5^2, 4, 3^2, 2^3)$. It is easy to observe that π is potentially $C_{2,6}$-graphic.

Case 2.3: $j \geq 3$. $\pi = (d_1, 5, 4, 3^j, 2^{n-3-j})$. Let $H = C_{2,6} + \{v_5v_6\}$. So $\pi_H = (5^2, 3^4)$. Thus $\pi_H^* = (d_1 - 5, 1, 2^{n-3-j})$ and $(\pi_H^*)' = (2^{(n-3-j)-(d_1-5)}, 1^{d_1-5}, 1)$. By Theorem 2.7, if $(\pi_H^*)' \notin S$, then $(\pi_H^*)'$ is graphic. Hence π is potentially H-graphic. Therefore, π is potentially $C_{2,6}$-graphic. By

Theorem 2.7, it is clearly that $(\pi_H^*)' \notin S$, so π is potentially $C_{2,6}$-graphic.

Case 3: $i = 2$. $\pi = (d_1, 5, 4^2, 3^j, 2^{n-j-4})$, where $n - j - 4 \geq 1$.

Case 3.1: $j = 0$. $\pi = (d_1, 5, 4^2, 2^{n-4})$, Let $H = C_{2,6} + \{v_3v_4\}$. So $\pi_H = (5^2, 4^2, 2^2)$. Thus $\pi_H^* = (d_1 - 5, 2^{n-6})$. Let $(\pi_H^*)' = (2^{(n-6)-(d_1-5)}, 1^{d_1-5})$. By Theorem 2.7, if $(\pi_H^*)' \notin S$, then $(\pi_H^*)'$ is graphic. Hence π is potentially H-graphic. Therefore, π is potentially $C_{2,6}$-graphic. By Theorem 2.7, $(\pi_H^*)' \notin S$, then π is potentially $C_{2,6}$-graphic.

Case 3.2: $j = 1$. $\pi = (d_1, 5, 4^2, 3, 2^{n-5})$. Let $H = C_{2,6} + \{v_3v_5\}$. So $\pi_H = (5^2, 4, 3^2, 2)$. Thus $\pi_H^* = (d_1-5, 1, 2^{n-6})$. Let $(\pi_H^*)' = (2^{(n-6)-(d_1-5)}, 1^{d_1-5}, 1)$. By Theorem 2.7, if $(\pi_H^*)' \notin S$, then $(\pi_H^*)'$ is graphic. Hence π is potentially H-graphic. Therefore, π is potentially $C_{2,6}$-graphic. By Theorem 2.7, $(\pi_H^*)' \notin S$, then π is potentially $C_{2,6}$-graphic.

Case 3.3: $j \geq 2$. $\pi = (d_1, 5, 4^2, 3^j, 2^{n-j-4})$, where $n - j - 4 \geq 1$. Let $H = C_{2,6} + \{v_3v_5, v_4v_6\}$. So $\pi_H = (5^2, 4^2, 3^2)$. Thus $\pi_H^* = (d_1 - 5, 3^{j-2}, 2^{n-j-4})$.

If $d_1 - 5 \leq j - 2$, let $(\pi_H^*)' = (3^{(j-2)-(d_1-5)}, 2^{(n-j-4)+(d_1-5)})$. By Theorem 2.7, if $(\pi_H^*)' \notin S$, then $(\pi_H^*)'$ is graphic. Hence π is potentially H-graphic. Therefore, π is potentially $C_{2,6}$-graphic. If $(\pi_H^*)' \in S$, by Theorem 2.7, then $(\pi_H^*)' = (2), (2^2), (3^2, 2)$. Hence $\pi = (5^2, 4^2, 3^2, 2)$, $(5^2, 4^2, 3^2, 2^2)$, $(6, 5, 4^2, 3^3, 2)$,$(5^2, 4^2, 3^4, 2)$. It is easy to observe that π is potentially $C_{2,6}$-graphic.

If $d_1 - 5 > j - 2$, let $(\pi_H^*)' = (2^{(n-6)+(j-2)-(d_1-5)}, 1^{(d_1-5)-(j-2)})$. By Theorem 2.7, if $(\pi_H^*)' \notin S$, then $(\pi_H^*)'$ is graphic. Hence π is potentially H-graphic. Therefore, π is potentially $C_{2,6}$-graphic. By Theorem 2.7, $(\pi_H^*)' \notin S$, therefore π is potentially $C_{2,6}$-graphic.

Case 4: $i = 3$. $\pi = (d_1, 5, 4^3, 3^j, 2^{n-5-j})$, where $n - 5 - j \geq 1$.

Case 4.1: $j = 0$. $\pi = (d_1, 5, 4^3, 2^{n-5})$. Let $H = C_{2,6} + \{v_3v_5, v_4v_5\}$. So $\pi_H = (5^2, 4^3, 2)$. Thus $\pi_H^* = (d_1 - 5, 2^{n-6})$ and $(\pi_H^*)' = (2^{(n-6)-(d_1-5)}, 1^{d_1-5})$. By Theorem 2.7, if $(\pi_H^*)' \notin S$, then $(\pi_H^*)'$ is graphic. Hence π is potentially H-graphic. Therefore, π is potentially $C_{2,6}$-graphic. If $(\pi_H^*)' \in S$, by Theorem 2.7, then $(\pi_H^*)' = (2), (2^2)$. Hence $\pi = (5^2, 4^3, 2^2), (5^2, 4^3, 2^3)$. It is easy to observe that π is potentially $C_{2,6}$-graphic.

Case 4.2: $j \geq 1$. $\pi = (d_1, 5, 4^3, 3^j, 2^{n-5-j})$. Let $H = C_{2,6}+\{v_3v_5, v_4v_5\}$. So $\pi_H = (5^2, 4^3, 2)$. Thus $\pi_H^* = (d_1 - 5, 1, 3^{j-1}, 2^{n-5-j})$.

If $d_1 - 5 < j - 1$, let $(\pi_H^*)' = (3^{(j-1)-(d_1-5)}, 2^{(n-5-j)+(d_1-5)}, 1)$. By Theorem 2.7, if $(\pi_H^*)' \notin S$, then $(\pi_H^*)'$ is graphic. Hence π is potentially H-graphic. Therefore, π is potentially $C_{2,6}$-graphic. If $(\pi_H^*)' \in S$, by

Theorem 2.7, then $(\pi_H^*)' = (3, 2, 1)$. Hence, $\pi = (5^2, 4^3, 3^2, 2)$. It is easy to observe that π is potentially $C_{2,6}$-graphic.

If $j-1 \le d_1 - 5 \le n-6$, let $(\pi_H^*)' = (1, 2^{(n-6)+(j-1)-(d_1-5)}, 1^{(d_1-5)-(j-1)})$. By Theorem 2.7, if $(\pi_H^*)' \notin S$, then $(\pi_H^*)'$ is graphic. Hence π is potentially H-graphic. Therefore, π is potentially $C_{2,6}$-graphic. By Theorem 2.7, $(\pi_H^*)' \notin S$, therefore, π is potentially $C_{2,6}$-graphic.

Case 4.3: $i \ge 4$. $\pi = (d_1, 5, 4^i, 3^j, 2^{n-i-j-2})$. Let $H = C_{2,6} + \{v_3v_5, v_4v_6, v_5v_6\}$. So $\pi_H = (5^2, 4^4)$. Thus $\pi_H^* = (d_1-5, 4^{i-4}, 3^j, 2^{n-i-j-2})$.

Case 4.3.1: $d_1 - 5 \le i - 4$.

Let $(\pi_H^*)' = (4^{(i-4)-(d_1-5)}, 3^{j+(d_1-5)}, 2^{n-i-j-2})$. By Theorem 2.5, if $(\pi_H^*)' \notin S$, then $(\pi_H^*)'$ is graphic. Hence π is potentially H-graphic. Therefore, π is potentially $C_{2,6}$-graphic. If $(\pi_H^*)' \in S$, by Theorem 2.7, then $(\pi_H^*)' = (4,2)$,$(4,2^2)$,$(4,2^3)$,$(4,3^2,2)$,$(4^2,2)$, $(4^2,2^2)$,$(4^3,2)$, $(4^3,2^2)$,$(4^4,2)$, (2), (2^2), $(3^2,2)$. Hence, $\pi = (5^2,4^5,2)$,$(5^2,4^5,2^2)$,$(5^2,4^5,2^3)$,$(5^2,4^6,2)$, $(5^2, 4^5,3^2,2)$, $(6,5,4^6,3,2)$, $(7,5,4^7,2)$,$(5^2,4^6,2^2)$, $(5^2,4^7,2)$,$(5^2,4^7,2^2)$, $(5^2,4^8,2)$, $(5^2,4^4, 2)$,$(5^2,4^4,2^2)$, $(7,5,4^6,2)$,$(6,5,4^5,3,2)$,$(5^2,4^4,3^2,2)$. It is easy to observe that π is potentially $C_{2,6}$-graphic.

Case 4.3.2: $d_1 - 5 > i - 4$.

Let $(\pi_H^*)' = (3^{(n-6)-(n-i-j-2)-(d_1-5)+(i-4)}, 2^{(n-i-j-2)+(d_1-5)-(i-4)})$. By Theorem 2.7, if $(\pi_H^*)' \notin S$, then $(\pi_H^*)'$ is graphic. Hence π is potentially H-graphic. Therefore, π is potentially $C_{2,6}$-graphic. Since $(n-i-j-2)+ (d_1-5)-(i-4) \ge 2$, by Theorem 2.7, if $(\pi_H^*)' \in S$, then $(\pi_H^*)' = (2^2)$. Hence, $\pi = (7,5,4^5,2)$,$(6,5,4^4,3,2)$. It is easy to observe that π is potentially $C_{2,6}$-graphic.

Case 4.3.3: $(i-4)+j \le d_1 - 5 \le n-6$.

Let $(\pi_H^*)' = (3^{i-4}, 2^{(n-6)-(d_1-5)+j}, 1^{(d_1-5)-(i-4)-j})$. By Theorem 2.7, if $(\pi_H^*)' \notin S$, then $(\pi_H^*)'$ is graphic. Hence π is potentially H-graphic. Therefore, π is potentially $C_{2,6}$-graphic. If $(\pi_H^*)' \in S$, by Theorem 2.7, then $(\pi_H^*)' = (2)$,(2^2),$(3,1)$,(3^2),$(3,2)$,$(3^2,2)$, $(3^3,1)$,$(3^2,1^2)$. Hence, $\pi = (5^2,4^4,2)$,$(5^2, 4^4,2^2)$,$(6,5,4^4,3,2)$,$(7,5,4^5,2)$, $(7,5,4^5,2^2)$,$(8,5, 4^5,3,2)$, $(7,5,4^6,2)$,$(9,5,4^7,2)$,$(9,5,4^6,2^2)$. It is easy to observe that π is potentially $C_{2,6}$-graphic.

Lemma 2.13 If $\pi = (5^2, 4^i, 3^j, 2^k, 1^{n-i-j-k-2}) \in GS_n$, where $n \ge 7$ $i+j \ge 2$, $i+j+k \ge 4$ and $n-i-j-k-2 \ge 1$. Then π is potentially $C_{2,6}$ graphic.

Case 1: $i = 0$. $\pi = (5^2, 3^j, 2^k, 1^{n-j-k-2})$, where $n-j-k-2 \ge 1$.

Case 1.1: $j=2$. $\pi=(5^2,3^2,2^k,1^{n-k-4})$, then $\pi^*=(2^{k-2},\ 1^{n-k-4})$. By Theorem 2.7, $\pi^*\notin S$, thus π is potentially $C_{2,6}$-graphic.

Case 1.2: $j=3$. $\pi=(5^2,3^3,2^k,1^{n-k-5})$, then $\pi^*=(1,2^{k-1},\ 1^{n-k-5})$. By Theorem 2.7, $\pi^*\notin S$, thus π is potentially $C_{2,6}$-graphic.

Case 1.3: $j\geq 4$. $\pi=(5^2,3^j,2^k,1^{n-2-j-k})$. Let $H=C_{2,6}+\{v_5v_6\}$. So $\pi_H=(5^2,3^4)$. Thus $\pi_H^*=(3^{j-4},2^k,\ 1^{n-j-k-2})$. By Theorem 2.7, if $\pi_H^*\notin S$, then π_H^* is graphic. Hence π is potentially H-graphic. Therefore, π is potentially $C_{2,6}$-graphic. If $\pi_H^*\in S$, by Theorem 2.7, then $\pi_H^*=(3,1),(3^3,1),(3^2,1^2),(3,2,1)$. Hence, $\pi=(5^2,3^5,1),(5^2,3^7,1),(5^2,3^6,1^2),(5^2,3^5,2,1)$. Obviously π is potentially $C_{2,6}$-graphic.

Case 2: $i=1$. $\pi=(5^2,4,3^j,2^k,1^{n-j-k-3})$, where $n-i-j-k-3\geq 1$.

Case 2.1: $j=1$. $\pi=(5^2,4,3,2^k,1^{n-k-4})$, then $\pi^*=(1,2^{k-2},\ 1^{n-k-4})$. By Theorem 2.7, $\pi^*\notin S$, thus π is potentially $C_{2,6}$-graphic.

Case 2.2: $j=2$. $\pi=(5^2,4,3^2,2^k,1^{n-k-5})$, then $\pi^*=(1,1,\ 2^{k-1},\ 1^{n-k-5})$. By Theorem 2.7, $\pi^*\notin S$, thus π is potentially $C_{2,6}$-graphic.

Case 2.3: $j\geq 3$. $\pi=(5^2,4,3^j,2^k,1^{n-j-k-3})$.

Let $H=C_{2,6}+\{v_5v_6\}$. So $\pi_H=(5^2,3^4)$. Thus $\pi_H^*=(1,3^{j-3},2^k,\ 1^{n-j-k-3})$. By Theorem 2.7, if $\pi_H^*\notin S$, then π_H^* is graphic. Hence π is potentially H-graphic. Therefore, π is potentially $C_{2,6}$-graphic. If $\pi_H^*\in S$, by Theorem 2.7, then $\pi_H^*=(3^2,1^2)$. Hence, $\pi=(5^2,4,3^5,1)$. Obviously π is potentially $C_{2,6}$-graphic.

Case 3: $i=2$. $\pi=(5^2,4^2,3^j,2^k,1^{n-k-j-4})$, where $n-i-j-k-4\geq 1$.

Case 3.1: $j=0$. $\pi=(5^2,4^2,2^k,1^{n-k-4})$, then $\pi^*=(1,1,2^{k-2},\ 1^{n-k-4})$. By Theorem 2.7, $\pi^*\notin S$, thus π is potentially $C_{2,6}$-graphic.

Case 3.2: $j=1$. $\pi=(5^2,4^2,3,2^k,1^{n-k-5})$. Let $H=C_{2,6}+\{v_3v_5\}$. So $\pi_H=(5^2,4,3^2,2)$. Thus $\pi_H^*=(1,2^k,\ 1^{n-k-5})$. By Theorem 2.7, if $\pi_H^*\notin S$, then π_H^* is graphic. Hence π is potentially H-graphic. Therefore, π is potentially $C_{2,6}$-graphic. By Theorem 2.7, $\pi_H^*\notin S$, thus π is potentially $C_{2,6}$-graphic.

Case 3.3: $j\geq 2$. $\pi=(5^2,4^2,3^j,2^k,1^{n-j-k-4})$. Let $H=C_{2,6}+\{v_3v_5,v_4v_6\}$. So $\pi_H=(5^2,4^2,3^2)$. Thus $\pi_H^*=(3^{j-2},\ 2^k,\ 1^{n-j-k-4})$. By Theorem 2.7, if $\pi_H^*\notin S$, then π_H^* is graphic. Hence π is potentially H-graphic. Therefore, π is potentially $C_{2,6}$-graphic. If $\pi_H^*\in S$, by Theorem 2.7, then $\pi_H^*=(3,1),(3,2,1),(3^3,1),(3^2,1^2)$. Hence, $\pi=(5^2,4^2,3^3,1),(5^2,4^2,3^3,2,1),(5^2,4^2,3^5,1),(5^2,4^2,3^4,1^2)$. Obviously π is potentially $C_{2,6}$-graphic.

Case 4: $i=3$. $\pi=(5^2,4^3,3^j,2^k,1^{n-k-j-5})$, where $n-i-j-k-5\geq 1$.

Case 4.1: $j=0$. $\pi=(5^2,4^3,2^k,1^{n-k-5})$. Let $H=C_{2,6}+\{v_3v_5,v_4v_5\}$.

So $\pi_H = (5^2, 4^3, 2)$. Thus $\pi_H^* = (2^{k-1},\ 1^{n-k-5})$. By Theorem 2.7, if $\pi_H^* \notin S$, then π_H^* is graphic. Hence π is potentially H-graphic. Therefore, π is potentially $C_{2,6}$-graphic. By Theorem 2.7, $\pi_H^* \notin S$, thus π is potentially $C_{2,6}$-graphic.

Case 4.2:$j \geq 1$. $\pi = (5^2, 4^3, 3^j, 2^k, 1^{n-j-k-5})$, Let $H = C_{2,6} + \{v_3v_5, v_4v_5\}$. So $\pi_H = (5^2, 4^3, 2)$. Thus $\pi_H^* = (1, 3^{j-1}, 2^k, 1^{n-j-k-5})$. By Theorem 2.7, if $\pi_H^* \notin S$, then π_H^* is graphic. Hence π is potentially H-graphic. Therefore, π is potentially $C_{2,6}$-graphic. If $\pi_H^* \in S$, by Theorem 2.7, then $\pi_H^* = (3^2, 1^2)$. Hence, $\pi = (5^2, 4^3, 3^3, 1)$. It is easy to observe that π is potentially $C_{2,6}$-graphic.

Case 5:$i \geq 4$. $\pi = (5^2, 4^i, 3^j, 2^k, 1^{n-i-j-k-2})$.Let $H = C_{2,6} + \{v_3v_5, v_4v_6, v_5v_6\}$. So $\pi_H = (5^2, 4^4)$. Thus $\pi_H^* = (4^{i-4},\ 3^j,\ 2^k,\ 1^{n-i-j-k-2})$. By Theorem 2.7, if $\pi_H^* \notin S$, then π is potentially H-graphic. Therefore, π is potentially $C_{2,6}$-graphic. If $\pi_H^* \in S$, by Theorem 2.7, then $\pi_H^* =(3,1)$,$(3,2,1)$,$(3^3,1)$, $(3^2,1^2)$,$(4,1^2)$, $(4,2,1^2)$, $(4,3,1)$, $(4,3,1^3)$, $(4,3^2,1^2)$, $(4,3,2,1)$,$(4^2,1^2)$, $(4^2,1^4)$, $(4^2,2,1^2)$, $(4^2,3,1)$, $(4^2,3,1^3)$, $(4^2,3,2,1)$, $(4^3,1^2)$, $(4^3,2,1^2)$, $(4^3,1^4)$, $(4^3,3,1)$, $(4^4,1^2)$. Hence, $\pi = (5^2,4^4,3,1)$,$(5^2,4^4,3,2,1)$,$(5^2,4^4,3^3,1)$,$(5^2,4^4,3^2,1^2)$,$(5^2,4^5,1^2)$, $(5^2,4^5,2,1^2)$, $(5^2,4^5,3,1)$, $(5^2,4^5,3,1^3)$, $(5^2,4^5,3^2,1^2)$, $(5^2,4^5,3,2,1)$,$(5^2,4^6,1^2)$,$(5^2,4^6,1^4)$, $(5^2,4^6,2,1^2)$, $(5^2,4^6,3,1)$, $(5^2,4^6,3,1^3)$, $(5^2,4^6,3,2,1)$, $(5^2,4^7,1^2)$, $(5^2,4^7,2,1^2)$, $(5^2,4^7,1^4)$, $(5^2,4^7,3,1)$, $(5^2,4^8,1^2)$. It is easy to check that π is potentially $C_{2,6}$-graphic.

3 Main Theorems

Theorem 3.1 Let $\pi = (d_1, d_2, \cdots, d_n) \in GS_n$ with $n \geq 6$. Then π is potentially $C_{2,6}$ graphic if and only if the following conditions hold:

(1) $d_2 \geq 5$; $d_4 \geq 3$; $d_6 \geq 2$;

(2) $\pi \neq (5^3, 3, 2^3)$, $(5^2, 3^2, 2^3)$, $(5^2, 3^2, 2^4)$, $(5^2, 3^6)$.

Proof: First we show the conditions (1)-(2) are necessary conditions for π to be potentially $C_{2,6}$ graphic. It is easy to check that $(5^3, 3, 2^3)$, $(5^2, 3^2, 2^3)$, $(5^2, 3^2, 2^4)$ and $(5^2, 3^6)$ are not potentially $C_{2,6}$-graphic. (2) holds. (1) is obvious.

Now we prove the sufficient conditions. Suppose the graphic sequence π satisfies the conditions (1)-(2). Our proof is by induction on n. We first prove the base case where $n = 6$. π is one of the following: (5^6), $(5^4, 4^2)$, $(5^3, 4^2, 3)$, $(5^3, 3^3)$, $(5^2, 4^4)$,$(5^2, 4^2, 3^2)$, $(5^2, 4^3, 2)$, $(5^2, 3^4)$, $(5^2, 4, 3^2, 2)$. It is easy to check that all of these are potentially $C_{2,6}$-graphic. Now suppose that the sufficiency holds for $n-1 (n \geq 7)$, we will show that π is potentially $C_{2,6}$-graphic in terms of the following cases:

Case 1: $d_n \geq 6$. Consider $\pi' = (d'_1, d'_2, \cdots, d'_{n-1})$ where $d'_{n-1} \geq 5$. It is easy to check that π' satisfies (1) and (2). By the induction hypothesis, π' is potentially $C_{2,6}$-graphic.

Case 2: $d_n = 5$. Consider $\pi' = (d'_1, d'_2, \cdots, d'_{n-1})$ where $d'_1 \geq 5$ and $d'_{n-1} \geq 4$. It is clearly that π' satisfies (2). If π' satisfies (1), then by the induction hypothesis, π' is potentially $C_{2,6}$-graphic, and hence so is π.

If π' does not satisfy (1), i.e., $d'_2 = 4$, then $\pi = (5^7)$, a contradiction.

Case 3: $d_n = 4$. Consider $\pi' = (d'_1, d'_2, \cdots, d'_{n-1})$ where $d'_4 \geq 4$ and $d'_{n-1} \geq 4$.

Case 3.1: $d_2 \geq 6$. It is clearly that $d'_2 \geq 5$, $d'_4 \geq 4$ and $d'_{n-1} \geq 3$. It is easy to check that π' satisfies (1) and (2). By the induction hypothesis, π' is potentially $C_{2,6}$-graphic, and hence so is π.

Case 3.2: $d_2 = 5$

Case 3.2.1: $d_6 = 5$. It is clearly that $d'_2 \geq 5$ and $d'_6 \geq 4$. It is easy to check that π' satisfies (1) and (2). By the induction hypothesis, π' is potentially $C_{2,6}$-graphic, and hence so is π.

Case 3.2.2: $d_6 = 4$.

Case 3.2.2.1: $d_5 = 5$, then d_1 is even, so $d_1 \geq 6$. It is clearly that $d'_2 \geq 5$ and $d'_6 \geq 4$. It is easy to check that π' satisfies (1) and (2). By the induction hypothesis, π' is potentially $C_{2,6}$-graphic, and hence so is π.

Case 3.2.2.2: $d_5 = 4$, then $\pi = (d_1, 5^i, 4^{n-1-i})$, where $1 \leq i \leq 3$ and $n - 1 - i \geq 1$. By Lemma 2.8, if π satisfies (1) and (2), then π is potentially $C_{2,6}$-graphic.

Case 4: $d_n = 3$. Consider $\pi' = (d'_1, d'_2, \cdots, d'_{n-1})$ where $d'_{n-1} \geq 2$.

Case 4.1: $d_2 \geq 6$, so $d'_2 \geq 5$, $d'_5 \geq 3$ and $d'_{n-1} \geq 2$. It is clearly that π' satisfies (1). If π' satisfies (2), then by the induction hypothesis, π' is potentially $C_{2,6}$-graphic.

If π' does not satisfy (2), $\pi' = (5^2, 3^6)$, then $\pi = (6^2, 4, 3^6)$. It is easy to check that π is potentially $C_{2,6}$-graphic.

Case 4.2: $d_2 = 5$

Case 4.2.1: $d_5 = 5$, so $d'_2 \geq 5$, $d'_5 \geq 4$, $d'_6 \geq 3$ and $d'_{n-1} \geq 2$. It is easy to check that π' satisfies (1) and (2). By the induction hypothesis, π' is potentially $C_{2,6}$-graphic, and hence so is π.

Case 4.2.2: $d_5 = 4$ or 3

Case 4.2.2.1: $d_4 = 5$

Case 4.2.2.1.1: $d_1 \geq 6$. It is clearly than $d'_2 \geq 5$, $d'_4 \geq 4$ and $d'_{n-1} \geq 3$. It is easy to check that π' satisfies (1) and (2). By the induction hypothesis, π' is potentially $C_{2,6}$-graphic, and hence so is π.

Case 4.2.2.1.2: $d_1 = 5$, then $\pi = (5^4, 4^i, 3^{n-4-i})$, where $n-4-i \geq 1$. By Lemma 2.9, if π satisfies (1) and (2), then π is potentially $C_{2,6}$-graphic.

Case 4.2.2.2: $d_4 = 4$ or $d_4 = 3$, then $\pi = (d_1, 5^i, 4^j, 3^{n-1-i-j})$, where $i = 1$ or 2 and $n-1-i-j \geq 1$. By Lemma 2.10, if π satisfies (1) and (2), then π is potentially $C_{2,6}$-graphic.

Case 5: $d_n = 2$. Consider $\pi' = (d'_1, d'_2, \cdots, d'_{n-1})$ where $d'_{n-1} \geq 2$.

Case 5.1: $d_2 \geq 6$, so $d'_2 \geq 5$, $d'_4 \geq 3$ and $d'_{n-1} \geq 2$. It is clearly that π' satisfies (1). If π' satisfies (2), then by the induction hypothesis, π' is potentially $C_{2,6}$-graphic.

If π' does not satisfy (2), then π' =$(5^3, 3, 2^3)$,$(5^2, 3^2, 2^3)$,$(5^2, 3^2, 2^4)$, $(5^2, 3^6)$, then π = $(6^2, 5, 3, 2^4)$,$(6^2, 3^2, 2^4)$,$(6^2, 3^2, 2^5)$,$(6^2, 3^6, 2)$ It is easy to check that π is potentially $C_{2,6}$-graphic.

Case 5.2: $d_2 = 5$

Case 5.2.1: $d_4 = 5$, then $d'_2 \geq 5$, $d'_4 \geq 4$ and $d'_{n-1} \geq 2$. It is easy to check that π' satisfies (1) and (2). By the induction hypothesis, π' is potentially $C_{2,6}$-graphic, and hence so is π.

Case 5.2.2: $d_4 = 4$ or 3

Case 5.2.2.1: $d_3 = 5$

Case 5.2.2.1.1: $d_1 \geq 6$, then $d'_2 \geq 5$, $d'_3 = 4$, $d'_4 \geq 3$ and $d'_{n-1} \geq 2$. It is easy to check that π' satisfies (1) and (2). By the induction hypothesis, π' is potentially $C_{2,6}$-graphic, and hence so is π.

Case 5.2.2.1.2: $d_1 = 5$, then $\pi = (5^3, 4^i, 3^j, 2^{n-3-i-j})$, where $n-3-i-j \geq 1$ and j is odd. By Lemma 2.11, if π satisfies (1) and (2), then π is potentially $C_{2,6}$-graphic.

Case 5.2.2.2: $d_3 = 4$ or 3, then $\pi = (d_1, 5, 4^i, 3^j, 2^{n-2-i-j})$, where $n-2-i-j \geq 1$. By Lemma 2.12, if π satisfies (1) and (2), then π is potentially $C_{2,6}$-graphic.

Case 6: $d_n = 1$. Consider $\pi' = (d'_1, d'_2, \cdots, d'_{n-1})$.

Case 6.1: $d_1 \geq 6$, so $d'_2 \geq 5$, $d'_4 \geq 3$, $d'_6 \geq 2$ and $d'_{n-1} \geq 1$. It is clearly that π' satisfies (1). If π' satisfies (2), then by the induction hypothesis, π' is potentially $C_{2,6}$-graphic.

If π' does not satisfy (2), π' =$(5^3, 3, 2^3)$,$(5^2, 3^2, 2^3)$,$(5^2, 3^2, 2^4)$, $(5^2, 3^6)$, then π = $(6, 5^2, 3, 2^3, 1)$,$(6, 5, 3^2, 2^3, 1)$,$(6, 5, 3^2, 2^4, 1)$,$(6, 5, 3^6, 1)$. It is easy to check that π is potentially $C_{2,6}$-graphic.

Case 6.2: $d_1 = 5$

Case 6.2.1: $d_3 = 5$, then $d_1' = d_2' = 5$, $d_3' \geq 4$, $d_4' \geq 3$, $d_6' \geq 2$ and $d_{n-1}' \geq 1$. It is easy to check that π' satisfies (1) and (2). By the induction hypothesis, π' is potentially $C_{2,6}$-graphic, and hence so is π.

Case 6.2.2: $d_3 = 4$ or 3, then $\pi = (5^2, 4^i, 3^j, 2^k, 1^{n-i-j-k-2})$, where $i+j+k \geq 4$ and $n-i-j-k-2 \geq 1$. By Lemma 2.13, if π satisfies (1) and (2), then π is potentially $C_{2,6}$-graphic.

Theorem 3.1 partially answer the problem 6 in Lai and Hu[21]: Characterize potentially $K_{r+1} - G$-graphic sequences for the remaining G.

4 Application

In the remaining of this section, we will use the above theorems to find exact values of $\sigma(C_{2,6}, n)$.

Theorem 4.1 (Lai[19]) If $r \geq 4$ and $n \geq 5r + 16$, then

$$\sigma(K_{r+1} - K_4, n) = \sigma(K_{r+1} - (K_4 - e), n) =$$

$$\sigma(K_{r+1} - Z_4, n) = \begin{cases} (r-1)(2n-r) - 3(n-r) + 1, \\ \text{if } n-r \text{ is odd} \\ (r-1)(2n-r) - 3(n-r) + 2, \\ \text{if } n-r \text{ is even} \end{cases}$$

Corollary 4.2 For $n \geq 6$, $\sigma(C_{2,6}, n) = \begin{cases} 5n-4, \text{if } n-5 \text{ is odd} \\ 5n-3, \text{if } n-5 \text{ is even} \end{cases}$.

When $n \geq 41$, Corollary 4.2 is a special case for Theorem 4.1($r = 5$). Owing to $C_{2,6}$ which is just the gragh $K_6 - (K_4 - e)$, we note that the value of $\sigma(C_{2,6}, n)$ was determined by Lai in [19]. Corollary 4.2 can be derived form Theorem 2.6.

Proof: First we claim $\sigma(C_{2,6}, n) \geq \begin{cases} 5n-4, \text{if } n-5 \text{ is odd} \\ 5n-3, \text{if } n-5 \text{ is even} \end{cases}$, for $n \geq 6$.

Since $C_{2,6}$ is just the graph $K_6 - (K_4 - e)$ and $\sigma(K_6 - (K_4 - e)) \geq \sigma(K_6 - K_4)$. Consequently, when $r = 5$, by Theorem 2.6, for $n \geq 6$,

$$\sigma(C_{2,6}, n) \geq \begin{cases} 5n-4, \text{if } n-5 \text{ is odd} \\ 5n-3, \text{if } n-5 \text{ is even} \end{cases}.$$

Now we show if π is an n-term ($n \geq 6$) graphic sequence with $\sigma(\pi)$ satisfies $\sigma(\pi) \geq 5n-4$, then there exists a realization of π containing a $C_{2,6}$.

Case 1: $n-5$ is odd.

If $d_2 \leq 4$, then $\sigma(\pi) \leq (n-1)+4(n-1) = 5n-5<5n-4$, a contradiction. Hence, $d_2 \geq 5$.

If $d_4 \leq 2$, by Theorem 2.5, then $\sigma(\pi) \leq d_1 + d_2 + d_3 + 2(n-3) \leq 3 \times (3-1) + \sum_{j=4}^{n} \min\{3, d_j\} + 2(n-3) = 2n + 2(n-3) = 4n-6 < 5n-4$, a contradiction. Hence, $d_4 \geq 3$.

If $d_6 = 1$, by Theorem 2.5, then $\sigma(\pi) \leq d_1+d_2+d_3+d_4+d_5+(n-5) \leq 5\times(5-1)+\sum_{j=6}^{n} \min\{5, d_j\}+(n-5) = (n+15)+(n-5) = 2n+10 < 5n-4$, a contradiction.

If $\pi = (5^2, 3^2, 2^4)$, then $\sigma(\pi) = 24 < 5 \times 8 - 4 = 36$, hence $\pi \neq (5^2, 3^2, 2^4)$.

If $\pi = (5^2, 3^6)$, then $\sigma(\pi) = 28 < 5 \times 8 - 4 = 36$, hence $\pi \neq (5^2, 3^6)$.

Thus, π satisfies the conditions (1) and (2) in Theorem 3.1. Therefore π is potentially $C_{2,6}$-graphic.

Case 2: $n-5$ is even.

If $d_2 \leq 4$, then $\sigma(\pi) \leq (n-1)+4(n-1) = 5n-5<5n-3$, a contradiction. Hence, $d_2 \geq 5$.

If $d_4 \leq 2$, by Theorem 2.5, then $\sigma(\pi) \leq d_1 + d_2 + d_3 + 2(n-3) \leq 3 \times (3-1) + \sum_{j=4}^{n} \min\{3, d_j\} + 2(n-3) = 2n + 2(n-3) = 4n-6 < 5n-3$, a contradiction.Hence, $d_4 \geq 3$.

If $d_6 = 1$, by Theorem 2.5, then $\sigma(\pi) \leq d_1+d_2+d_3+d_4+d_5+(n-5) \leq 5\times(5-1)+\sum_{j=6}^{n} \min\{5, d_j\}+(n-5) = (n+15)+(n-5) = 2n+10 < 5n-3$, a contradiction.

If $\pi = (5^3, 3, 2^3)$, then $\sigma(\pi) = 24 < 5 \times 8 - 3 = 37$, hence $\pi \neq (5^3, 3, 2^3)$.

If $\pi = (5^2, 3^2, 2^3)$, then $\sigma(\pi) = 22 < 5 \times 8 - 3 = 37$, hence $\pi \neq (5^2, 3^2, 2^3)$.

Thus, π satisfies the conditions (1) and (2) in Theorem 3.1. Therefore, $\sigma(C_{2,6}, n) = \begin{cases} 5n-4, \text{if } n-5 \text{ is odd} \\ 5n-3, \text{if } n-5 \text{ is even} \end{cases}$ and hence π is potentially $C_{2,6}$-graphic.

References

[1] J.A. Bondy, U.S.R. Murty, Graph Theory with Applications, The Macmillan Press Ltd., 1976.

[2] G.Chen, On potentially $K_6 - 3K_2$-graphic sequences, accepted by Ars Combinatoria

[3] P.Erdös, T.Gallai, Graphs with given degrees of vertices, Math. Lapok,11(1960),264-274.

[4] P.Erdös, M.S. Jacobson and J. Lehel, Graphs realizing the same degree sequences and their respective clique numbers, in Graph Theory, Combinatorics and Application, Vol. 1(Y. Alavi et al., eds.), John Wiley and Sons, Inc., New York, 1991, 439-449.

[5] E.M. Eschen and J.B.Niu, On potentially $K_4 - e$-graphic sequences, Australasian Journal of Combinatorics, 29(2004), 59-65.

[6] M.Ferrara, R.J.Gould and J.Schmitt, Potentially K_s^t-graphic degree sequences, submitted.

[7] M.Ferrara, R.J.Gould and J.Schmitt, Graphic sequences with a realization containing a friendship graph, Ars Combinatoria, 85(2007), 161-171.

[8] R.J. Gould, M.S. Jacobson and J. Lehel, Potentially G-graphic degree sequences,in Combinatorics, Graph Theory and Algorithms,Vol. 2 (Y. Alavi et al.,eds.), New Issues Press, Kalamazoo, MI, 1999, 451-460.

[9] L.L.Hu and C.H.Lai , on potentially $K_5 - C_4$-graphic sequences, accepted by Ars Combinatoria.

[10] L.L.Hu and C.H.Lai, on potentially $K_5 - H$-graphic sequences, Czechoslovak Mathematical Journal,59(1)(2009), 137-182.

[11] L.L.Hu and C.H.Lai , on potentially $K_5 - E_3$-graphic sequences, accepted by Ars Combinatoria.

[12] L.L.Hu and C.H.Lai, On Potentially 3-regular graph graphic Sequences, Utilitas Mathematica, 80 (2009), 33 - 51.

[13] L.L.Hu and C.H.Lai, A Characterization On Potentially $K_6 - C_4$-graphic Sequences, accepted by Ars Combinatoria.

[14] L.L.Hu and C.H.Lai , A Characterization On Potentially $K_{2,5}$-graphic Sequences, accepted by Ars Combinatoria.

[15] D.J. Kleitman and D.L. Wang , Algorithm for constructing graphs and digraphs with given valences and factors,Discrete Math., 6(1973),79-88.

[16] C.H.Lai, An extremal problem on potentially $K_m - C_4$-graphic sequences, Journal of Combinatorial Mathematics and Combinatorial Computing, 61 (2007), 59-63.

[17] C.H.Lai, An extremal problem on potentially $K_m - P_k$-graphic sequences, accepted by International Journal of Pure and Applied Mathematics.

[18] C.H.Lai, An extremal problem on potentially $K_{p,1,1}$-graphic sequences, Discrete Mathematics and Theoretical Computer Science 7(2005), 75-81.

[19] C.H.Lai, The smallest degree sum that yields potentially $K_{r+1} - Z$-graphical Sequences, accepted by Ars Combinatoria.

[20] C.H.Lai and L.L Hu, An extremal problem on potentially $K_{r+1} - H$-graphic sequences, Ars Combinatoria, 94 (2010), 289-298.

[21] C.H.Lai and L.L Hu, Potentially $K_m - G$-graphical Sequences: A Survey, Czechoslovak Mathematical Journal, 59(4)(2009),1059-1075.

[22] J.S.Li and Z.X.Song, The smallest degree sum that yields potentially P_k-graphical sequences, J. Graph Theory, 29(1998), 63-72.

[23] J.S.Li and Z.X.Song, on the potentially P_k-graphic sequences, Discrete Math., 195(1999), 255-262.

[24] J.S.Li and Z.X.Song, An extremal problem on the potentially P_k-graphic sequences, Discrete Math., 212(2000), 223-231.

[25] J.S.Li, Z.X.Song and R.Luo, The Erdös-Jacobson-Lehel conjecture on potentially P_k-graphic sequence is true, Science in China(Series A), 41(5)(1998), 510-520.

[26] J.S.Li and J.H.Yin, A variation of an extremal theorem due to Woodall, Southeast Asian Bulletin of Math., 25(2001), 427-434.

[27] J.S.Li and J.H.Yin, The threshold for the Erdös, Jacobson and Lehel conjecture to be true, Acta Math. Sin. (Engl. Ser.), 22(2006), 1133-1138.

[28] M.J.Liu and Chunhui Lai, On potentially $K_{1,1,2,2}$-graphic sequences, accepted by Utilitas Mathematica.

[29] R.Luo, On potentially C_k-graphic sequences, Ars Combinatoria 64(2002), 301-318.

[30] R.Luo, M.Warner, On potentially K_k-graphic sequences, Ars Combinatoria. 75(2005), 233-239.

[31] Z.H.Xu, Characterizations On Potentially $K_6 - H$- graphic Sequences, MS Dissertation, Zhangzhou Teachers College, Zhangzhou, 2009.

[32] Z.H.Xu and C.H.Lai, On Potentially $K_6 - C_5$-graphic Sequences, accepted by Utilitas Mathematica.

[33] J.H.Yin, The smallest degree sum that yields potentially $K_{1,1,3}$-graphic sequences, J. HaiNan University, 22(3)(2004), 200-204.

[34] J.H.Yin and G.Chen, On potentially $K_{r_1,r_2,\cdots,r_m}$-graphic sequences, Utilitas Mathematica, 72(2007), 149-161.

[35] J.H.Yin, G.Chen and J.R.Schmitt, Graphic Sequences with a realization containing a generalized Friendship Graph, Discrete Math, 308(2008), 6226-6232

[36] J.H.Yin and J.S.Li, The smallest degree sum that yields potentially $K_{r,r}$-graphic sequences, Sci. China Ser. A, 45(2002), 694-705.

[37] J.H.Yin and J.S.Li, An extremal problem on the potentially $K_{r,s}$-graphic sequences, Discrete Math., 26(2003), 295-305.

[38] J.H.Yin and J.S.Li, Two sufficient conditions for a graphic sequence to have a realization with prescribed clique size, Discrete Math.,301(2005) 218-227.

[39] J.H.Yin, J.S.Li and G.L.Chen, A variation of a classical *Turán*-type extremal problem, European J.Combin., 25(2004), 989-1002.

[40] J.H.Yin, J.S.Li and G.L.Chen, The smallest degree sum that yields potentially $K_{2,s}$-graphic sequences, Ars Combinatoria, 74(2005), 213-222.

[41] J.H.Yin, J.S. Li and R.Mao, An extremal problem on the potentially $K_{r+1} - e$-graphic sequences, Ars Combinatoria 74(2005), 151-159.

[42] M.X.Yin, The smallest degree sum that yields potentially $K_{r+1} - K_3$-graphic sequences, Acta Math. Appl. Sin. Engl. Ser. 22(3)(2006), no. 3, 451-456.

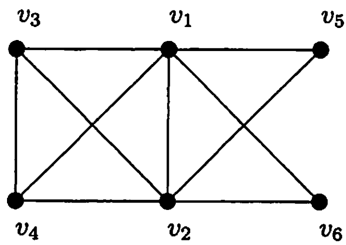
[43] M.X.Yin, A characterization on potentially $K_6 - E(K_3)$-graphic sequences, accepted by Ars Combinatoria

[44] M.X.Yin and J.H.Yin, On potentially H-graphic sequences, Czechoslovak Mathematical Journal, 57(2)(2007),705-724.

[45] M.X.Yin, J.H.Yin, C.Zhong and F.Yang, On the characterization of potentially $K_{1,1,s}$-graphic sequences, accepted by Utilitas Mathematica

[46] M.X.Yin, C.Zhong and F.Yang, A note on the characterization of potentially $K_{1,1,s}$-graphic sequences, Ars Combinatoria, 93(2009),275-287.

Appendix



$C_{2,6}$

Figure 1