Several identities involving q-harmonic numbers by q-Chu-Vandermonde convolution formula

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Abstract

The purpose of this paper is to establish several identities involving q-harmonic numbers by the q-Chu-Vandermonde convolution formula and obtain some q-analogues of several known identities.

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1 Introduction

Harmonic numbers play important roles in number theory, analysis algorithms and special function. For $\alpha \in \mathbb{N}$, the generalized harmonic numbers are defined by

$$H_0^{(\alpha)} = 0 \quad \text{and} \quad H_n^{(\alpha)} = \sum_{i=1}^{n} \frac{1}{i^\alpha}, \quad \text{for } n \in \mathbb{N},$$

when $\alpha = 1$, they reduce to the well known harmonic numbers

$$H_0 = 0 \quad \text{and} \quad H_n = \sum_{i=1}^{n} \frac{1}{i}, \quad \text{for } n \in \mathbb{N}.$$

Many identities involve harmonic numbers(see [2, 5, 7]). In fact, the harmonic numbers are generalized to many forms(see[1, 3, 8]). In this paper, we will establish some identities involving q-harmonic numbers.

For $\alpha \in \mathbb{N}$, the generalized q-harmonic numbers can be defined by

$$H_0^{(\alpha)}(q) = 0 \quad \text{and} \quad H_n^{(\alpha)}(q) = \sum_{i=1}^{n} \left(\frac{q^i}{1-q^i}\right)^{\alpha}, \quad \text{for } n \in \mathbb{N},$$

when $\alpha = 1$, the q-harmonic numbers can be defined by

$$H_0(q) = 0 \quad \text{and} \quad H_n(q) = \sum_{i=1}^{n} \frac{q^i}{1-q^i}, \quad \text{for } n \in \mathbb{N}.$$

It is easy to see that

$$\lim_{q \to 1^-} (1-q)^{\alpha} H_n^{(\alpha)}(q) = H_n^{(\alpha)}, \quad \text{for } n \in \mathbb{N}_0.$$

Recently, Wei, Gong and Wang[6] get some identities involving harmonic numbers by two derivative operators. In this paper, we will apply these derivative operators to q-Chu-Vandermonde convolution to get some identities involving q-harmonic numbers which are q-analogues of several known identities.

First we give some definitions and formulae which will be useful throughout this paper.

For two differentiable functions $f(x)$ and $g(x, y)$, the derivative operator

$\mathcal{D}_x$ and $\mathcal{D}_{xy}^2$ can be defined by

$$\mathcal{D}_x f(x) = \frac{d}{dx} f(x)\Big|_{x=0}, \quad \mathcal{D}_{xy}^2 g(x,y) = \frac{\partial^2}{\partial x \partial y} g(x,y)\Big|_{x=y=0}.$$

The q-Gamma function is defined by

$$\Gamma_q(x) = \frac{(q;q)_\infty}{(q^x;q)_\infty}(1-q)^{1-x} \quad (0 < q < 1).$$

The q-binomial coefficients are defined by

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{\Gamma_q(x+1)}{\Gamma_q(y+1)\Gamma_q(x-y+1)}.$$

It is easy to see

$$\begin{bmatrix} n+x \\ m \end{bmatrix} = \frac{(q^{n-m+1+x};q)_m}{(q;q)_m(1-q)^m}$$

and

$$\mathcal{D}_x \begin{bmatrix} n+x \\ m \end{bmatrix} = \begin{bmatrix} n \\ m \end{bmatrix} (H_{n-m}(q) - H_n(q)) \ln q.$$

The well known q-Chu-Vandermonde convolution is

$$\begin{bmatrix} x+y \\ n \end{bmatrix} = \sum_{k=0}^{n} \begin{bmatrix} x \\ k \end{bmatrix} \begin{bmatrix} y \\ n-k \end{bmatrix} q^{(x-k)(n-k)}. \tag{1}$$

2 Main results and their proofs

Theorem 2.1 For $m, p, l \in \mathbb{N}_0$, $|q| < 1$, there holds

$$\begin{aligned} &\sum_{k=l}^{n} \begin{bmatrix} m+k \\ k-l \end{bmatrix} \begin{bmatrix} p+n-k \\ p \end{bmatrix} H_{m+k}(q) q^{k(p+1)} \\ = \; &\begin{bmatrix} m+p+n+1 \\ n-l \end{bmatrix} (H_{m+p+n+1}(q) - H_{m+p+l+1}(q) + H_{m+l}(q)) q^{l(p+1)}. \end{aligned}$$

Proof. Taking $x \to -m-1-x, y \to -p-1-y$ in (1), we have

$$\sum_{k=0}^{n} \begin{bmatrix} m+k+x \\ k \end{bmatrix} \begin{bmatrix} p+n-k+y \\ n-k \end{bmatrix} q^{k(p+1+y)} = \begin{bmatrix} m+p+n+1+x+y \\ n \end{bmatrix}. \quad (2)$$

Applying the derivative operator $\mathcal{D}_x$ to (2) and then letting $y=0$, we have

$$\begin{aligned} & \sum_{k=0}^{n} \begin{bmatrix} m+k \\ k \end{bmatrix} \begin{bmatrix} p+n-k \\ p \end{bmatrix} H_{m+k}(q) q^{k(p+1)} \\ = & \begin{bmatrix} m+p+n+1 \\ n \end{bmatrix} (H_{m+p+n+1}(q) - H_{m+p+1}(q) + H_m(q)). \quad (3) \end{aligned}$$

Taking $k \to k-l, n \to n-l, m \to m+l$ in (3), we complete the proof of the theorem.

Theorem 2.1 can be written as

$$\begin{aligned} & \sum_{k=l}^{n} \begin{bmatrix} m+k \\ k \end{bmatrix} \begin{bmatrix} p+n-k \\ p \end{bmatrix} \begin{bmatrix} k \\ l \end{bmatrix} H_{m+k}(q) q^{k(p+1)} \\ = & \begin{bmatrix} m+p+n+1 \\ n-l \end{bmatrix} \begin{bmatrix} m+l \\ l \end{bmatrix} \\ & \times (H_{m+p+n+1}(q) - H_{m+p+l+1}(q) + H_{m+l}(q)) q^{l(p+1)}. \quad (4) \end{aligned}$$

Taking $m=0$ in (4), we can obtain the following result.

Corollary 2.2 For $p, l \in \mathbb{N}_0, |q|<1$, there holds

$$\begin{aligned} & \sum_{k=l}^{n} \begin{bmatrix} p+n-k \\ p \end{bmatrix} \begin{bmatrix} k \\ l \end{bmatrix} H_k(q) q^{k(p+1)} \\ = & \begin{bmatrix} p+n+1 \\ n-l \end{bmatrix} (H_{p+n+1}(q) - H_{p+l+1}(q) + H_l(q)) q^{l(p+1)}. \end{aligned}$$

Taking $p=l=0$ in Corollary 2.2, we can obtain the following result.

Corollary 2.3 For $|q|<1$, there holds

$$\sum_{k=0}^{n} H_k(q) q^k = \frac{1-q^{n+1}}{1-q} \left(H_{n+1}(q) - \frac{q}{1-q} \right).$$

Corollary 2.3 is a q-analogue of the following result[2, Equation(2.1)]:

$$\sum_{k=0}^{n} H_k = (n+1)(H_{n+1}-1). \tag{5}$$

Taking $l = 1$ in (4), we have

Corollary 2.4 For $m, p \in \mathbb{N}_0, |q| < 1$, there holds

$$\sum_{k=1}^{n} \begin{bmatrix} m+k \\ k \end{bmatrix} \begin{bmatrix} p+n-k \\ p \end{bmatrix} (1-q^k) H_{m+k}(q) q^{k(p+1)} = \begin{bmatrix} m+p+n+1 \\ n-1 \end{bmatrix}$$
$$\times (1-q^{m+1})(H_{m+p+n+1}(q) - H_{m+p+2}(q) + H_{m+1}(q)) q^{p+1}.$$

Taking $q \to 1^-$ in Corollary 2.4, we have the following result [6, Equation(4)]:

Corollary 2.5 For $m, p \in \mathbb{N}_0, |q| < 1$, there holds

$$\begin{aligned} &\sum_{k=1}^{n} \binom{m+k}{k} \binom{p+n-k}{p} k H_{m+k} \\ = \quad & (m+1) \binom{m+p+n+1}{n-1} (H_{m+p+n+1} - H_{m+p+2} + H_{m+1}). \end{aligned}$$

Taking $m = p = 0$ in Corollary 2.4, we have

Corollary 2.6 For $|q| < 1$, there holds

$$\sum_{k=1}^{n} (1-q^k) H_k(q) q^k = \frac{q(1-q^n)(1-q^{n+1})}{1-q^2} \left(H_{n+1}(q) - \frac{q^2}{1-q^2} \right).$$

Corollary 2.6 can be written as follows:

$$\begin{aligned} &\sum_{k=1}^{n} (1-q^k) H_k(q) q^k \\ = \quad & \frac{q(1-q^n)(1-q^{n+1})}{1-q^2} H_n(q) - \frac{q^3(1-q^{n-1})(1-q^n)}{(1-q^2)^2}, \end{aligned} \tag{6}$$

which is a q-analogue of the following result[2, Equation(2.2)].

$$\sum_{k=1}^{n} kH_k = \frac{n(n+1)}{2}H_n - \frac{(n-1)n}{4}.$$

Taking $m = p = 0$ in (4), we have

$$\sum_{k=l}^{n}(q^{k+1-l};q)_l H_k(q)q^k = \frac{(q^{n+1-l};q)_{l+1}}{1-q^{l+1}}\left(H_{n+1}(q) - \frac{q^{l+1}}{1-q^{l+1}}\right)q^l.$$

Letting

$$f(l) = \sum_{k=l}^{n}(q^{k+1-l};q)_l H_k(q)q^k,$$

and by

$$\sum_{k=0}^{n} H_k(q)q^{2k} = f(0) - f(1),$$

we have

Corollary 2.7 For $|q| < 1$, there holds

$$\sum_{k=0}^{n} H_k(q)q^{2k} = \frac{1-q^{2n+2}}{1-q^2}H_{n+1}(q) - \frac{(1-q^{n+1})(q+2q^2+q^{n+3})}{(1-q^2)^2}.$$

For

$$\sum_{k=1}^{n}(1-q^k)^2 H_k(q)q^k = (1-q)f(1) + qf(2),$$

we have

Corollary 2.8 For $|q| < 1$, there holds

$$\begin{aligned}&\sum_{k=1}^{n}(1-q^k)^2 H_k(q)q^k \\ =\ & \frac{(1-q^n)(1-q^{n+1})(q+q^3-q^{n+2}-q^{n+3})}{(1-q^3)(1+q)}H_{n+1}(q)\end{aligned}$$

$$-\frac{(1-q^n)(1-q^{n+1})(q^3+q^4+q^5+q^7-q^{n+5}-2q^{n+6}-q^{n+7})}{(1-q^3)^2(1+q)^2}.$$

Multiplying both sides of Corollary 2.8 by $(1-q)^{-1}$ and taking $q \to 1^-$, we get

$$\sum_{k=1}^{n} k^2 H_k = \frac{n(n+1)(2n+1)}{6} H_{n+1} - \frac{n(n+1)(4n+5)}{36},$$

which is equivalent to the following result[2, Equation(2.3)].

$$\sum_{k=1}^{n} k^2 H_k = \frac{n(n+1)(2n+1)}{6} H_n - \frac{(n-1)n(4n+1)}{36}.$$

Applying the derivative operator $\mathcal{D}_y$ to (2) and then letting $x = 0, k \to k-l, n \to n-l, m \to m+l$, we have

Theorem 2.9 For $m, p, l \in \mathbb{N}_0, |q| < 1$, there holds

$$\begin{aligned}
&\sum_{k=l}^{n} \begin{bmatrix} m+k \\ k-l \end{bmatrix} \begin{bmatrix} p+n-k \\ p \end{bmatrix} (H_{p+n-k}(q) - k) q^{k(p+1)} \\
= &\begin{bmatrix} m+p+n+1 \\ n-l \end{bmatrix} (H_{m+p+n+1}(q) - H_{m+p+l+1}(q) + H_p(q) - l) q^{l(p+1)}.
\end{aligned}$$

Taking $p = 0$ in Theorem 2.9, we have

Corollary 2.10 For $l \in \mathbb{N}_0, |q| < 1$, there holds

$$\begin{aligned}
&\sum_{k=l}^{n} \begin{bmatrix} m+k \\ k-l \end{bmatrix} (H_{n-k}(q) - k) q^k \\
= &\begin{bmatrix} m+n+1 \\ n-l \end{bmatrix} (H_{m+n+1}(q) - H_{m+l+1}(q) - l) q^l.
\end{aligned}$$

Taking $m = p = 0$ in Theorem 2.1 and Theorem 2.9, we can obtain

Corollary 2.11 For $l \in \mathbb{N}_0, |q| < 1$, there holds

$$\sum_{k=l}^{n} \begin{bmatrix} k \\ l \end{bmatrix} (H_k(q) - H_{n-k}(q) + k)q^k = \begin{bmatrix} n+1 \\ l+1 \end{bmatrix} (H_l(q) + l)q^l.$$

Multiplying both sides of Corollary 2.11 by $1 - q$ and taking $q \to 1^-$, we have

Corollary 2.12 For $l \in \mathbb{N}_0$, there holds

$$\sum_{k=l}^{n} \binom{k}{l} (H_k - H_{n-k}) = \binom{n+1}{l+1} H_l.$$

Corollary 2.13 For $|q| < 1$, there holds

$$\sum_{k=0}^{n} H_k(q)q^{-k} = \frac{q^{-n} - q}{1-q} H_{n+1}(q) - \frac{(n+1)q}{1-q}.$$

Proof. Taking $l = 0$ in Corollary 2.11, we have

$$\sum_{k=0}^{n} H_{n-k}(q)q^k = \sum_{k=0}^{n} H_k(q)q^k + \sum_{k=0}^{n} kq^k.$$

By

$$\sum_{k=0}^{n} H_{n-k}(q)q^k = q^n \sum_{k=0}^{n} H_k(q)q^{-k},$$

$$\sum_{k=0}^{n} H_k(q)q^k = \frac{1 - q^{n+1}}{1-q} H_{n+1}(q) - \frac{q(1 - q^{n+1})}{(1-q)^2},$$

$$\sum_{k=0}^{n} kq^k = \frac{q[1 - (n+1)q^n + nq^{n+1}]}{(1-q)^2},$$

we get the corollary.

Taking $l = 1$ in Corollary 2.11 and using the same method as Corollary 2.13, we can obtain the following corollary.

Corollary 2.14 For $|q| < 1$, there holds

$$\sum_{k=0}^{n} H_k(q)q^{-2k} = \frac{q^{-2n} - q^2}{1 - q^2} H_{n+1}(q) - \frac{q^{1-n} + q^{2-n} + nq^2 - q^3 - (n+1)q^4}{(1-q^2)^2}.$$

It is easy to see that Corollaries 2.7, 2.13 and 2.14 are q-analogues of (5), too.

Applying the derivative operator $\mathcal{D}_{xy}$ to (2) and then taking $k \to k - l, n \to n - l, m \to m + l$, we have

Theorem 2.15 For $m, p, l \in \mathbb{N}_0, |q| < 1$, there holds

$$\begin{aligned}
&\sum_{k=l}^{n} \begin{bmatrix} m+k \\ k-l \end{bmatrix} \begin{bmatrix} p+n-k \\ p \end{bmatrix} H_{m+k}(q)(H_{p+n-k}(q) - k)q^{k(p+1)} \\
= &\begin{bmatrix} m+p+n+1 \\ n-l \end{bmatrix} [(H_{m+p+n+1}(q) - H_{m+p+l+1}(q) + H_p(q) - l) \\
&\times (H_{m+p+n+1}(q) - H_{m+p+l+1}(q) + H_{m+l}(q)) + H^{(2)}_{m+p+l+1}(q) \\
&-H^{(2)}_{m+n+p+1}(q) + H_{m+p+l+1}(q) - H_{m+n+p+1}(q)\Big] q^{l(p+1)}.
\end{aligned}$$

Taking $m = p = 0$ in Theorem 2.15, we have

Corollary 2.16 For $l \in \mathbb{N}_0, |q| < 1$, there holds

$$\begin{aligned}
&\sum_{k=l}^{n} \begin{bmatrix} k \\ l \end{bmatrix} H_k(q)(H_{n-k}(q) - k)q^k \\
= &\begin{bmatrix} n+1 \\ l+1 \end{bmatrix} \Big[(H_{n+1}(q) - H_{l+1}(q) - l)\left(H_{n+1}(q) - \frac{q^{l+1}}{1-q^{l+1}}\right) \\
&+ \left(H^{(2)}_{l+1}(q) - H^{(2)}_{n+1}(q)\right) + (H_{l+1}(q) - H_{n+1}(q))\Big] q^l.
\end{aligned}$$

Taking $l = 0$ in Corollary 2.16, we have

Corollary 2.17 For $|q| < 1$, there holds

$$\sum_{k=0}^{n} H_k(q)(H_{n-k}(q) - k)q^k = \frac{1-q^{n+1}}{1-q}\left[(H_{n+1}(q) - H_1(q))^2 \right.$$
$$\left. +(H_1^{(2)}(q) - H_{n+1}^{(2)}(q)) + (H_1(q) - H_{n+1}(q))\right].$$

Noticing

$$\sum_{k=0}^{n} H_{n-k}(q)(H_k(q) + k)q^{n-k} = \sum_{k=0}^{n} H_k(q)(H_{n-k}(q) - k)q^k + n\sum_{k=0}^{n} H_k(q)q^k,$$

we have

Corollary 2.18 For $|q| < 1$, there holds

$$\sum_{k=0}^{n} H_{n-k}(q)(H_k(q) + k)q^{-k} = \frac{q^{-n}-q}{1-q}\left[(H_{n+1}(q) - H_1(q))^2 \right.$$
$$\left. +(H_1^{(2)}(q) - H_{n+1}^{(2)}(q)) + (n-1)(H_{n+1}(q) - H_1(q))\right].$$

Let

$$g(l) = \sum_{k=l}^{n} (q^{k+1-l};q)_l H_k(q)(H_{n-k}(q) - k)q^k.$$

Noticing

$$\sum_{k=0}^{n} H_k(q)(H_{n-k}(q) - k)q^{2k} = g(0) - g(1),$$

and by Corollary 2.16, we have

Corollary 2.19 For $|q| < 1$, there holds

$$\sum_{k=0}^{n} H_k(q)(H_{n-k}(q) - k)q^{2k} = \frac{1-q^{2n+2}}{1-q^2}\left[(H_{n+1}(q) - H_1(q))^2\right.$$

$$+(H_1^{(2)}(q) - H_{n+1}^{(2)}(q)) + (H_1(q) - H_{n+1}(q))\Big]$$
$$+\frac{q(1-q^n)(1-q^{n+1})}{1-q^2}\left[\frac{H_{n+1}(q) - H_1(q)}{1+q} + \frac{q}{(1+q)(1-q^2)}\right].$$

Using the same method as Corollary 2.18, we have

Corollary 2.20 For $|q| < 1$, there holds

$$\sum_{k=0}^{n} H_{n-k}(q)(H_k(q) + k)q^{-2k} = \frac{q^{-2n} - q^2}{1-q^2}\Big[(H_{n+1}(q) - H_1(q))^2$$
$$+(H_1^{(2)}(q) - H_{n+1}^{(2)}(q)) + (n-1)(H_{n+1}(q)) - H_1(q)\Big]$$
$$+\frac{q(q^{-n}-1)(q^{-n}-q)}{1-q^2}\left[\frac{H_{n+1}(q) - H_1(q)}{1+q} + \frac{(1-n)q - nq^2}{(1+q)(1-q^2)}\right].$$

Multiplying both sides of Corollaries 2.17, 2.18, 2.19 or 2.20 by $(1-q)^2$ and taking $q \to 1^-$, we obtain the same identity [6, Theorem 1]:

$$\sum_{k=0}^{n} H_k H_{n-k} = (n+1)\left[(H_{n+1} - 1)^2 + (1 - H_{n+1}^{(2)})\right].$$

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