SOME PLETHYSTIC IDENTITES AND KOSTKA-FOULKES POLYNOMIALS.

MAHIR BILEN CAN

1. INTRODUCTION.

Symmetric functions $\{E_{n,k}(X)\}_{k=1}^{n}$, defined by the Newton interpolation

$$e_n[X\frac{1-z}{1-q}] = \sum_{k=1}^{n}(z;q)_k \frac{E_{n,k}(X)}{(q;q)_k}$$

plays an important role in the Garsia-Haglund proof of the q,t-Catalan conjecture, [2].

Let $\Lambda^n_{\mathbb{Q}(q,t)}$ be the space of symmetric functions of degree n, over the field of rational functions $\mathbb{Q}(q,t)$, and let $\nabla : \Lambda^n_{\mathbb{Q}(q,t)} \to \Lambda^n_{\mathbb{Q}(q,t)}$ be the Garsia-Bergeron operator.

By studying recursions, Garsia and Haglund show that the coefficient of the elementary symmetric function $e_n(X)$ in the image $\nabla(E_{n,k}(X))$ of $E_{n,k}(X)$ is equal to the following combinatorial summation

$$\langle \nabla(E_{n,k}(X)), e_n(X)\rangle = \sum_{\pi\in D_{n,k}} q^{area(\pi)} t^{bounce(\pi)}, \tag{1.1}$$

where $D_{n,k}$ is the set of all Dyck paths with initial k North steps followed by an East step. Here $area(\pi)$ and $bounce(\pi)$ are two numbers associated with a Dyck path π. It is conjectured in [4], more generally, that the $\nabla E_{n,k}(X)$ are "Schur positive."

In [1], using (1.1), Can and Loehr prove the q,t-Square conjecture of the Loehr and Warrington [7].

The aim of this article is to understand the functions $\{E_{n,k}(X)\}_{k=1}^{n}$ better. We prove that the vector subspace generated by the set $\{E_{n,k}(X)\}_{k=1}^{n}$ of the space $\Lambda^n_{\mathbb{Q}(q)}$ of degree n symmetric functions over the field $\mathbb{Q}(q)$, is equal to the subspace generated by

$$\{s_{(k,1^{n-k})}[X/(1-q)]\}_{k=1}^{n},$$

Schur functions of hook shape, plethystically evaluated at $X/(1-q)$.

In particular, we determine explicitly the transition matrix and its inverse from $\{E_{n,k}(X)\}_{k=1}^{n}$ to $\{s_{(k,1^{n-k})}[X/(1-q)]\}_{k=1}^{n}$. The entries of the matrix turns out to be cocharge Kostka-Foulkes polynomials.

We find the expansion of $E_{n,k}(X)$ into the Hall-Littlewood basis, and as a corollary we recover a closed formula for the cocharge Kostka-Foulkes polynomials $\widetilde{K}_{\lambda,\mu}(q)$ when λ is a hook shape;

$$\widetilde{K}_{(n-k,1^k)\mu}(q) = (-1)^k \sum_{i=0}^{k} (-1)^i q^{\binom{i}{2}} \begin{bmatrix} r \\ i \end{bmatrix}.$$

Here, μ is a partition of n whose first column is of height r.

2. Background.

2.0.1. *Notation.* A partition μ of $n \in \mathbb{Z}_{>0}$, denoted $\mu \vdash n$, is a nonincreasing sequence $\mu_1 \geq \mu_2 \geq \ldots \geq \mu_k > 0$ of numbers such that $\sum \mu_i = n$. The conjugate partition $\mu' = \mu'_1 \geq \ldots \geq \mu'_s > 0$ is defined by setting $\mu'_i = |\{\mu_r : \mu_r \geq i\}|$.

$Par(n,r)$ denotes the set of all partitions $\mu \vdash n$ whose biggest part is equal to $\mu_1 = r$.

We identify a partition μ with its Ferrers diagram, in French notation. Thus, if the parts of μ are $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_k > 0$, then the corresponding Ferrers diagram have μ_i lattice cells in the i^{th} row (counting from bottom to up).

Following Macdonald, [8] the arm, leg, coarm and coleg of a lattice square s are the parameters $a_\mu(s), l_\mu(s), a'_\mu(s)$ and $l'_\mu(s)$ giving the number of cells of μ that are respectively strictly EAST, NORTH, WEST and SOUTH of s in μ.

Given a partition $\mu = (\mu_1, \mu_2, \ldots, \mu_k)$, we set

$$n(\mu) = \sum_{i=1}^{k} (i-1)\mu_i = \sum_{s\in\mu} l_\mu(s). \tag{2.1}$$

We also set

(2.2)

$$\widetilde{h}_\mu(q,t) = \prod_{s\in\mu} (q^{a_\mu(s)} - t^{l_\mu(s)+1}) \quad \text{and} \quad \widetilde{h'}_\mu(q,t) = \prod_{s\in\mu} (t^{l_\mu(s)} - q^{a_\mu(s)+1}).$$

Let $\mathbb{F}$ be a field, and let $X = \{x_1, x_2, \ldots\}$ be an alphabet (a set of indeterminates). The algebra of symmetric functions over $\mathbb{F}$ with the variable set X is denoted by $\Lambda_{\mathbb{F}}(X)$.

If $\mathbb{Q} \subseteq \mathbb{F}$, it is well known that $\Lambda_{\mathbb{F}}(X)$ is freely generated by the set of power-sum symmetric functions

$$\{p_r(X) : r = 1, 2, \ldots \text{ and, } p_r(X) = x_1^r + x_2^r + \cdots\}.$$

The algebra, $\Lambda_{\mathbb{F}}(X)$ has a natural grading (by degree).

$$\Lambda_{\mathbb{F}}(X) = \bigoplus_{n\geq 0} \Lambda_{\mathbb{F}}^n(X),$$

where $\Lambda_{\mathbb{F}}^n(X)$ is the space of homogenous symmetric functions of degree n.

A basis for the vector space $\Lambda^n_{\mathbb{F}}(X)$ is give by the set $\{p_\mu\}_{\mu\vdash n}$,

$$p_\mu(X) = \prod_{i=1}^{k} p_{\mu_i}(X), \text{ where } \mu = \sum_{i=1}^{k} \mu_i. \tag{2.3}$$

Another basis for $\Lambda^n_{\mathbb{F}}(X)$ is given by the Schur functions $\{s_\mu(X)\}_{\mu\vdash n}$, where $s_\mu(X)$ is defined as follows. Let

$$e_n(X) = \sum_{1\le i_1<\cdots<i_n} x_{i_1}x_{i_2}\cdots x_{i_n} \tag{2.4}$$

be the n'th elementary symmetric function. If $\mu = \sum_{i=1}^k \mu_i$, then

$$s_\mu(X) = \det(e_{\mu'_i-i+j}(X))_{1\le i,j\le m}, \tag{2.5}$$

where μ'_i is the i'th part of the conjugate partition $\mu' = (\mu'_1, ..., \mu'_l)$ and $m \ge l$.

2.0.2. *Plethysm.* For the purposes of this section, we represent an alphabet $X = \{x_1, x_2, ...\}$ as a formal sum $X = \sum x_i$. Thus, if $Y = \sum y_i$ is another alphabet, then

$$XY = (\sum x_i)(\sum y_i) = \sum_{i,j} x_i y_j = \{x_i y_j\}_{i,j\ge 1}, \tag{2.6}$$

and

$$X + Y = (\sum_i x_i) + (\sum_j y_j) = \{x_i, y_j\}_{i,j\ge 1}. \tag{2.7}$$

The formal additive inverse, denoted $-X$, of an alphabet $X = \sum x_i$ is defined so that $-X + X = 0$.

In this vein, if $p_k(X) = \sum_{k\ge 1} x_i^k$ is a power sum symmetric function, we define

$$p_k[XY] = p_k[X]p_k[Y] \tag{2.8}$$

$$p_k[X+Y] = p_k[X] + p_k[Y] \tag{2.9}$$

$$p_k[-X] = -p_k[X]. \tag{2.10}$$

This operation is called *plethysm.* Since $\Lambda_{\mathbb{F}}$ is freely generated by the power sums, the plethysm operator can be extended to the other symmetric functions. In fact, using plethysm, one defines the following bases for $\Lambda^n_{\mathbb{Q}(q)}$ and $\Lambda^n_{\mathbb{Q}(q,t)}$, respectively.

Theorem-Definition 1. (*cocharge Hall-Littlewood polynomials*)
There exists a basis $\{\widetilde{H}_\mu(X;q)\}_{\mu\vdash n}$ for the vector space $\Lambda^n_{\mathbb{Q}(q)}$, which is uniquely characterized by the properties

(1) $\widetilde{H}_\mu(X;q) \in \mathbb{Z}[q]\{s_\lambda : \lambda \ge \mu\}$,
(2) $\widetilde{H}_\mu[(1-q)X;q] \in \mathbb{Z}[q]\{s_\lambda : \lambda \ge \mu'\}$,
(3) $\langle \widetilde{H}_\mu(X;q), s_{(n)}\rangle = 1$.

Theorem-Definition 2. (*Modified Macdonald polynomials*)
There exists a basis $\{\widetilde{H}_\mu(X;q,t)\}_{\mu\vdash n}$ for the vector space $\Lambda^n_{\mathbb{Q}(q,t)}$, which is uniquely characterized by the properties

(1) $\widetilde{H}_\mu(X;q,t) \in \mathbb{Z}[q]\{s_\lambda : \lambda \geq \mu\}$,
(2) $\widetilde{H}_\mu[(1-q)X;q,t] \in \mathbb{Z}[q]\{s_\lambda : \lambda \geq \mu'\}$,
(3) $\langle \widetilde{H}_\mu[X(1-t);q,t], s_{(n)}\rangle = 1$.

It follows from these Theorem-Definitions that

$$\widetilde{H}_\mu(X;0,t) = \widetilde{H}_\mu(X;t), \tag{2.11}$$

$$\widetilde{H}_\mu(X;q,t) = \widetilde{H}_{\mu'}(X;t,q). \tag{2.12}$$

2.0.3. *Kostka-Foulkes and Kostka-Macdonald polynomials.* Let

$$\widetilde{H}_\mu(X;q) = \sum_\lambda \widetilde{K}_{\lambda\mu}(q)s_\lambda, \text{ and } \widetilde{H}_\mu(X;q,t) = \sum_\lambda \widetilde{K}_{\lambda\mu}(q,t)s_\lambda$$

be, respectively, the Schur basis expansions of the Hall-Littlewood and Macdonald symmetric functions. The coefficients of the Schur functions are called, respectively, the *cocharge Kostka-Foulkes polynomials*, and the *modified Kostka-Macdonald polynomials*. It is known that $\widetilde{K}_{\lambda\mu}(q,t), \widetilde{K}_{\lambda\mu}(q) \in \mathbb{N}[q,t]$.

It follows from equation (2.11) and the Schur basis expansions that

$$\widetilde{K}_{\lambda\mu}(0,t) = \widetilde{K}_{\lambda\mu}(t). \tag{2.13}$$

2.0.4. *Cauchy Identities.* Let $X = \sum x_i$ be an alphabet, and let

$$\Omega[X] = \exp(\sum_{k=1}^{\infty} p_k(X)/k).$$

Then,

$$\Omega[X] = \prod_i \frac{1}{1-x_i} = \sum_{n=0}^{\infty} s_n(X), \tag{2.14}$$

$$\Omega[X] = \prod_i (1-x_i) = \sum_{n=0}^{\infty} s_{1^n}(X). \tag{2.15}$$

If $Y = \sum y_i$ is another alphabet, then

$$e_n[XY] = \sum_{\mu\vdash n} s_\mu[X]s_{\mu'}[Y], \tag{2.16}$$

$$e_n[XY] = \sum_{\mu\vdash n} \frac{\widetilde{H}_\mu[X;q,t]\widetilde{H}_\mu[Y;q,t]}{\widetilde{h}_\mu(q,t)\widetilde{h'}_\mu(q,t)}, \tag{2.17}$$

where $\widetilde{h}_\mu(q,t)$ and $\widetilde{h'}_\mu(q,t)$ are as in (2.2).

$$(2.18)\qquad s_\mu[1-z]=\begin{cases}(-z)^k(1-z) & \text{if } \mu=(n-k,1^k),\\ 0 & \text{otherwise.}\end{cases}$$

2.0.5. *Cauchy's q-binomial theorem.* Let $(z;q)_k = (1-z)(1-qz)\cdots(1-q^{k-1}z)$, and let

$$\begin{bmatrix}k\\r\end{bmatrix}=\frac{(q;q)_k}{(q;q)_r(q;q)_{k-r}}.$$

Then, the Cauchy q-binomial theorem states that

$$(2.19)\qquad (z;q)_k=\sum_{r=0}^{k} z^r(-1)^r e_r[1,q,\dots,q^{k-1}]=\sum_{r=0}^{k} z^r q^{\binom{r}{2}}(-1)^r\begin{bmatrix}k\\r\end{bmatrix}.$$

3. Symmetric functions $E_{n,k}(X)$.

The family $\{E_{n,k}(X)\}_{k=1}^{n}$ of symmetric functions are defined by the plethystic identity

$$(3.1)\qquad e_n[X\frac{1-z}{1-q}]=\sum_{k=1}^{n}\frac{(z;q)_k}{(q;q)_k}E_{n,k}(X).$$

Let $0\le k\le r$, and let

$$(3.2)\qquad T_{k+1,r}=(-1)^k\sum_{i=0}^{k}(-1)^i q^{\binom{i}{2}}\begin{bmatrix}r\\i\end{bmatrix}.$$

Proposition 3.1. *For* $k=0,\dots,n-1$,

$$(3.3)\qquad s_{(k+1,1^{n-k-1})}[X/(1-q)]=\sum_{r=k+1}^{n} T_{k+1,r}\frac{E_{n,r}(X)}{(q;q)_r}.$$

Proof. Using the Cauchy q-binomial theorem, we see that the coefficient of $(-z)^k$ on the right hand side of (3.1) is

$$(3.4)\qquad q^{\binom{k}{2}}\sum_{i=0}^{n-k}\begin{bmatrix}k+i\\k\end{bmatrix}\frac{E_{n,k+i}}{(q;q)_{k+i}}.$$

On the other hand, by the identities (2.16) and (2.18),

$$\begin{aligned}e_n[X\frac{1-z}{1-q}] &= \sum_{\lambda} s_\lambda[\frac{X}{1-q}]s_{\lambda'}[1-z]\\ &= \sum_{\lambda'=(n-r,1^r)} s_\lambda[\frac{X}{1-q}](-z)^r(1-z)\\ &= \sum_{r=0}^{n-1} s_{(r+1,1^{n-r-1})}[\frac{X}{1-q}](-z)^r(1-z),\end{aligned}$$

which is equal to

$$s_{1^n}[\frac{X}{1-q}] + (-z)(s_{1^n}[\frac{X}{1-q}] + s_{2,1^{n-2}}[\frac{X}{1-q}]) + \cdots + (-z)^n s_n[\frac{X}{1-q}].$$

Comparing the coefficient of $(-z)^k$ gives, for $k \geq 1$,

$$q^{\binom{k}{2}} \sum_{i=0}^{n-k} \begin{bmatrix} k+i \\ k \end{bmatrix}_q \frac{E_{n,k+i}}{(q;q)_{k+i}} = s_{k,1^{n-k}}[\frac{X}{1-q}] + s_{k+1,1^{n-k-1}}[\frac{X}{1-q}], \tag{3.5}$$

and

$$\sum_{i=1}^{n} \frac{E_{n,i}}{(q;q)_i} = s_{1^n}[\frac{X}{1-q}]. \tag{3.6}$$

We take the alternating sums of the equations (3.5) and (3.6) to get

$$\begin{aligned} s_{k+1,1^{n-k-1}}[\frac{X}{(1-q)}] &= (-1)^k \left(\sum_{j=1}^{n} \frac{E_{n,j}}{(q;q)_j} \right) \\ &+ \sum_{i=1}^{k} (-1)^{k+i} \left(q^{\binom{i}{2}} \sum_{j=0}^{n-i} \begin{bmatrix} i+j \\ i \end{bmatrix} \frac{E_{n,i+j}}{(q;q)_{i+j}} \right). \end{aligned}$$

By collecting $E_{n,k}(X)$'s, and using (2.19) we obtain

$$s_{(k+1,1^{n-k-1})}[X/(1-q)] = \sum_{r=k+1}^{n} T_{k+1,r} \frac{E_{n,r}(X)}{(q;q)_r}.$$

□

Let S and E be the matrices

$$S = \begin{pmatrix} s_{1^n}[X/(1-q)] \\ s_{2,1^{n-1}}[X/(1-q)] \\ \vdots \\ s_n[X/(1-q)] \end{pmatrix} \text{ and } E = \begin{pmatrix} \frac{E_{n,1}}{(q;q)_1} \\ \frac{E_{n,2}}{(q;q)_2} \\ \vdots \\ \frac{E_{n,n}}{(q;q)_n} \end{pmatrix},$$

respectively, and let T be the transition matrix from E to S, so that $S = TE$. Then, T is an upper triangular matrix with the $k+1, r$'th entry

$$T_{k+1,r} = (-1)^k \sum_{i=0}^{k} (-1)^i q^{\binom{i}{2}} \begin{bmatrix} r \\ i \end{bmatrix}.$$

For example, when $n = 5$,

$$T = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & q & (q+1)\,q & (q^2+q+1)\,q & (q^3+q^2+q+1)\,q \\ 0 & 0 & q^3 & q^3\,(q^2+q+1) & q^3\,(q^4+q^3+2\,q^2+q+1) \\ 0 & 0 & 0 & q^6 & q^6\,(q^3+q^2+q+1) \\ 0 & 0 & 0 & 0 & q^{10} \end{pmatrix}.$$

Then,

$$T^{-1} = \begin{pmatrix} 1 & -q^{-1} & q^{-2} & -q^{-3} & q^{-4} \\ 0 & q^{-1} & -\frac{q+1}{q^3} & \frac{q^2+q+1}{q^5} & -\frac{q^3+q^2+q+1}{q^7} \\ 0 & 0 & q^{-3} & -\frac{q^2+q+1}{q^6} & \frac{q^4+q^3+2q^2+q+1}{q^9} \\ 0 & 0 & 0 & q^{-6} & -\frac{q^3+q^2+q+1}{q^{10}} \\ 0 & 0 & 0 & 0 & q^{-10} \end{pmatrix}.$$

Proposition 3.2. *T^{-1} is (necessarily) upper triangular, and its $k+1, r$'th entry is equal to*

$$(T^{-1})_{k+1,r} = (-1)^{r-k} q^{-r(k+1)} T_{k+1,r}. \tag{3.7}$$

Proof. Let L be the upper triangular matrix with the $k+1, r$'th entry

$$L_{k+1,r} = (-1)^{r-k} q^{-r(k+1)} T_{k+1,r} \quad \text{for } r > k.$$

Clearly, TL is an upper triangular matrix, and the $i+1, j$'th entry of TL is

$$(TL)_{i+1,j} = \sum_{k=1}^{n} T_{i+1,k} L_{k,j}. \tag{3.8}$$

It is straightforward to check that $(TL)_{i+1,i+1} = 1$. We use induction on j to prove that for all $i+1 < j$, $(TL)_{i+1,j} = 0$. So, we assume that for all $i+1 < j$, $(TL)_{i+1,j} = 0$, and we are going prove that for all $i+1 < j+1$, $(TL)_{i+1,j+1} = 0$.

First of all, using the q-binomial identity

$$\begin{bmatrix} r \\ m \end{bmatrix} = \begin{bmatrix} r-1 \\ m \end{bmatrix} + \begin{bmatrix} r-1 \\ m-1 \end{bmatrix} q^{r-m}, \text{ for } m \geq 0, \tag{3.9}$$

it is easy to show that

$$T_{i+1,k} = T_{i+1,k-1} + q^i T_{i,k-1}. \tag{3.10}$$

It follows that

$$L_{k+1,j+1} = -q^{-(k+1)} L_{k+1,j} + q^{-k} L_{k,j-1}. \tag{3.11}$$

Therefore,

$$
\begin{aligned}
\sum_{k=i+1}^{j+1} T_{i+1,k}L_{k,j+1} &= \sum_{k=i+1}^{j+1} T_{i+1,k}(-q^{-(k+1)}L_{k,j} + q^{-k}L_{k-1,j-1}) \\
&= \sum_{k=i+1}^{j+1} -q^{-k}T_{i+1,k}L_{k,j} + \sum_{k=i+1}^{j+1} q^{-(k-1)}T_{i+1,k}L_{k-1,j}.
\end{aligned}
$$

Using (3.10) in the last summation, we have

$$
\begin{aligned}
\sum_{k=i+1}^{j+1} T_{i+1,k}L_{k,j+1} &= \sum_{k=i+1}^{j+1} -q^{-k}T_{i+1,k}L_{k,j} + \sum_{k=i+1}^{j+1} q^{-(k-1)}T_{i+1,k-1}L_{k-1,j} \\
&+ \sum_{k=i+1}^{j+1} qT_{i+1,k-1}L_{k-1,j}.
\end{aligned}
$$

After rearranging the indices, and using the induction hypotheses, the right hand side of the equation simplifies to 0. Therefore, the proof is complete.

□

Corollary 3.3. *Let $A \subseteq \Lambda^n_{\mathbb{Q}(q)}(X)$ be the n-dimensional subspace generated by the set $\{E_{n,k}(X)\}_{k=1}^n$, and let $B \subseteq \Lambda^n_{\mathbb{Q}(q)}(X)$ be the n-dimensional subspace generated by $\{s_{k,1^{n-k}}[\frac{X}{1-q}]\}_{k=1}^n$. Then, $A = B$.*

Proof. It is clear by Proposition 3.2 that $A = B$. The dimension claim follows from Proposition 4.1 below. □

The expression $(-1)^n p_n = \sum_{k=0}^{n-1}(-1)^k s_{k+1,1^{n-k-1}}$ is the bridge between Schur functions of hook type with the power sum symmetric functions. By the linearity of plethysm we have

$$
(-1)^n p_n[X]/(1-q^n) = (-1)^n p_n[X/(1-q)] = \sum_{k=0}^{n-1}(-1)^k s_{k+1,1^{n-k-1}}[X/(1-q)],
$$

and therefore

$$
(-1)^n p_n = (1-q^n)\sum_{k=0}^{n-1}(-1)^k s_{k+1,1^{n-k-1}}[X/(1-q)]. \tag{3.12}
$$

Corollary 3.4. *For all $n \geq 1$,*

$$
(-1)^n p_n = \sum_{r=1}^{n} \frac{1-q^n}{1-q^r} E_{n,r}. \tag{3.13}
$$

Proof. By Proposition 3.1 and (3.12) we get

$$(3.14) \qquad (-1)^n p_n = (1-q^n)\sum_{k=0}^{n-1}\sum_{r=k+1}^{n}(-1)^k T_{k+1,r}\frac{E_{n,r}(X)}{(q;q)_r}.$$

By rearranging the summations and using the Cauchy's q-binomial theorem once more, we finish the proof. □

4. HALL-LITTLEWOOD EXPANSION.

Proposition 4.1. *For $k = 1, \ldots, n$,*

$$\frac{E_{n,k}(X)}{(q;q)_k} = \sum_{\mu\in Par(n,k)} \frac{\widetilde{H}_{\mu'}(X;q)}{\widetilde{h}_\mu(q,0)\widetilde{h'}_\mu(q,0)}$$

$$= \sum_{\mu\in Par(n,k)} \frac{\widetilde{H}_{\mu'}(X;q)}{(-q)^n q^{2n(\mu')}\prod_{s\in\mu, l_\mu(s)=0}(1-q^{-a_\mu(s)-1})}.$$

Proof. Let $Y = (1-t)(1-z)$. Then, by the Cauchy identity (2.17), we have

$$(4.1) \qquad \sum_{k=1}^{n}(z;q)_k\frac{E_{n,k}(X)}{(q;q)_k} = \sum_{\mu\vdash n}\frac{\widetilde{H}_\mu[X;q,t]\widetilde{H}_\mu[(1-t)(1-z);q,t]}{\widetilde{h}_\mu(q,t)\widetilde{h'}_\mu(q,t)}.$$

The left hand side of the equation (4.1) is independent of the variable t. Since $\widetilde{h}_\mu(q,0) \neq 0$, and since $\widetilde{h'}_\mu(q,0) \neq 0$, we are allowed to make the substitution $t = 0$ on both sides of the equation.

Note that

$$\widetilde{h}_\mu(q,0)\widetilde{h'}_\mu(q,0) = \prod_{s\in\mu} q^{a_\mu(s)} \prod_{s\in\mu, l_\mu(s)\neq 0}(-q^{a_\mu(s)+1}) \prod_{s\in\mu, l_\mu(s)=0}(1-q^{a_\mu(s)+1})$$

$$= (-q)^n \prod_{s\in\mu} q^{2a_\mu(s)} \prod_{s\in\mu, l_\mu(s)=0}\frac{1-q^{a_\mu(s)+1}}{-q^{a_\mu(s)+1}}$$

$$= (-q)^n \prod_{s\in\mu} q^{2a_\mu(s)} \prod_{s\in\mu, l_\mu(s)=0}(1-q^{-a_\mu(s)-1})$$

$$(4.2) \qquad = (-q)^n q^{2n(\mu')} \prod_{s\in\mu, l_\mu(s)=0}(1-q^{-a_\mu(s)-1}).$$

The equality (4.2) follows from (2.1).

Using the Schur expansion $\widetilde{H}_\mu(X;q,t) = \sum_\lambda \widetilde{K}_{\lambda\mu}(q,t)s_\lambda$, we see that the plethystic substitution $X \to (1-z)$, followed by the evaluation at $t = 0$ is the same as the evaluation $\widetilde{H}_\mu(X,q,0)$ at $t = 0$, followed by the plethystic substitution $X \to (1-z)$. Also, by Corollary 3.5.20 of [6], we know that

$$\widetilde{H}_\mu[1-z;q,t] = \Omega[-zB_\mu], \text{ where } B_\mu = \sum_{i\geq 1} t^{i-1}\frac{1-q^{\mu_i}}{1-q}.$$

Therefore,

$$\widetilde{H}_\mu[1-z;q,0] = \Omega[-zB_\mu]|_{t=0} \tag{4.3}$$
$$= \Omega[-z(1+q+\cdots+q^{\mu_1-1})] \tag{4.4}$$
$$= \prod_{i=0}^{\mu_1-1}(1-zq^i) \tag{4.5}$$
$$= (z;q)_{\mu_1}. \tag{4.6}$$

It follows from (2.11) and (2.12) that

$$\widetilde{H}_\mu(X;q,0) = \widetilde{H}_{\mu'}(X;q). \tag{4.7}$$

By combining (4.1), (4.2), (4.6) and (4.7), we get

$$\sum_{k=1}^{n}(z;q)_k\frac{E_{n,k}(X)}{(q;q)_k} = \frac{1}{(-q)^n}\sum_{\mu\vdash n}(z;q)_{\mu_1}\frac{\widetilde{H}_{\mu'}(X;q)}{q^{2n(\mu')}\prod_{s\in\mu,l_\mu(s)=0}(1-q^{-a_\mu(s)-1})}. \tag{4.8}$$

By comparing the coefficient of $(z;q)_k$ in (4.8), we find that

$$\frac{E_{n,k}(X)}{(q;q)_k} = \sum_{\mu\in Par(n,k)}\frac{\widetilde{H}_{\mu'}(X;q)}{(-q)^n q^{2n(\mu')}\prod_{s\in\mu,l_\mu(s)=0}(1-q^{-a_\mu(s)-1})}. \tag{4.9}$$

Hence, the proof is complete.

□

Lemma 4.2. *Let $\lambda \vdash n$ be a partition of n. Then,*

$$s_\lambda[\frac{X}{(1-q)(1-t)}] = \sum_{\mu\vdash n}\frac{\widetilde{K}_{\lambda'\mu}(q,t)\widetilde{H}_\mu(X;q,t)}{\widetilde{h}_\mu(q,t)\widetilde{h'}_\mu(q,t)}. \tag{4.10}$$

Proof. This follows from Theorem 1.3 of [3]. □

Corollary 4.3. *Let $\lambda \vdash n$ be a partition. Then,*

$$s_\lambda[\frac{X}{1-q}] = \sum_{\mu}\frac{\widetilde{K}_{\lambda'\mu'}(q)\widetilde{H}_{\mu'}(X;q)}{\widetilde{h}_\mu(q,0)\widetilde{h'}_\mu(q,0)} \tag{4.11}$$
$$= \sum_{\mu}\frac{\widetilde{K}_{\lambda'\mu'}(q)\widetilde{H}_{\mu'}(X;q)}{(-q)^n q^{2n(\mu')}\prod_{s\in\mu,l_\mu(s)=0}(1-q^{-a_\mu(s)-1})}.$$

Proof. It follows from (2.12) and (2.13) that $\widetilde{K}_{\lambda'\mu}(q,0) = \widetilde{K}_{\lambda'\mu'}(0,q) = \widetilde{K}_{\lambda'\mu'}(q)$. Since,

$$\widetilde{h}_\mu(q,0)\widetilde{h'}_\mu(q,0) = (-q)^n q^{2n(\mu')}\prod_{s\in\mu,l_\mu(s)=0}(1-q^{-a_\mu(s)-1}),$$

and $\widetilde{H}_\mu(X;q,0) = \widetilde{H}_{\mu'}(X;q)$, the proof follows from Lemma 4.2. □

Theorem 4.4. *Let* $1 \leq k \leq r \leq n$*, and let* $\mu \in Par(n,r)$*. Then,*

$$T_{k,r} = \widetilde{K}_{(n-k+1,1^{k-1})\mu'}(q).$$

Proof. Recall that

$$s_{k+1,1^{n-k-1}}[\frac{X}{1-q}] = \sum_{r=k+1}^{n} T_{k+1,r}\frac{E_{n,r}(X)}{(q;q)_r},$$

where

$$T_{k+1,r} = (-1)^k \sum_{i=0}^{k} (-1)^i q^{\binom{i}{2}} \begin{bmatrix} r \\ i \end{bmatrix}.$$

Therefore, by Corollary 4.3 and Proposition 4.1 we have

$$\begin{aligned}
\sum_{\mu} \frac{\widetilde{K}_{(n-k+1,1^{k-1})\mu'}(q)\widetilde{H}_{\mu'}(X;q)}{\widetilde{h}_\mu(q,0)\widetilde{h'}_\mu(q,0)} &= s_{k,1^{n-k}}[\frac{X}{1-q}] \\
&= \sum_{r=k}^{n} T_{k,r} \sum_{\mu \in Par(n,r)} \frac{\widetilde{H}_{\mu'}(X;q)}{\widetilde{h}_\mu(q,0)\widetilde{h'}_\mu(q,0)} \\
&= \sum_{\mu \in \bigcup_{r=k}^{n} Par(n,r)} \frac{T_{k,r}\widetilde{H}_{\mu'}(X;q)}{\widetilde{h}_\mu(q,0)\widetilde{h'}_\mu(q,0)}.
\end{aligned}$$

The theorem follows from comparison of the coefficients of $\widetilde{H}_{\mu'}(X;q)$. □

REFERENCES

[1] M.B. Can, N. Loehr, *A proof of the q,t-Square conjecture.* J. Combin. Theory Ser. A. 113 (2006), no. 7, 1419–1434.

[2] A.M. Garsia, J. Haglund, *A proof of the q,t-Catalan positivity conjecture.* Discrete Mathematics, **256** (2002), 677–717.

[3] A.M. Garsia, M. Haiman, G. Tesler *Explicit plethysic formulas for Macdonald q,t-Kostka coefficients.*
The Andrews Festschrift. Seminaire Lotharingien 42 (1999).

[4] J. Haglund, *The q, t-Catalan numbers and the space of diagonal harmonoics.* AMS University Lecture Series (in press).

[5] J. Haglund, M. Haiman, N. Loehr, J. B. Remmel, and A. Ulyanov, *A combinatorial formula for the character of the diagonal coinvariants.* Duke Math. J. **126** (2005) no. 2, 195–232.

[6] M. Haiman, *Combinatorics, symmetric functions and Hilbert schemes.* Current Developments in Mathematics 2002, no. 1 (2002), 39–111.

[7] N. Loehr, G. Warrington, *Square q, t-lattice paths and* $\nabla(p_n)$. Trans. Amer. Math. Soc. 359 no. 2, (2007), 649–669.

[8] I.G. Macdonald, *Symmetric functions and Hall polynomials.* Second Ed., Oxford University Press, New York, (1995).